COLLOQUIUM MATHEMATICUM

VOL. XXXVI 1976 FASC. 1

LOEWNER-TYPE APPROXIMATIONS FOR CONVEX FUNCTIONS

BY

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1. Introductory remarks. We denote by $\mathscr{P}$ the class of functions $P(z)$ regular in the open unit disk Δ, $\Delta = \{z: |z| < 1\}$, and satisfying the conditions $P(0) = 1$ and $\mathrm{Re}P(z) > 0$ for z in Δ. Then we use the symbol $\mathscr{K}$ to represent the class of functions $f(z) = z + a_2 z^2 + \ldots$ regular and univalent in Δ and for which $f[\Delta]$, the image of Δ under $f(z)$, is a convex domain; in other words, $\mathscr{K}$ is the family of normalized convex conformal maps of the disk. It is well-known that $f(z)$ is in $\mathscr{K}$ if and only if $1 + zf''(z)/f'(z)$ is in $\mathscr{P}$ (see [7]).

If $f(z)$ is in $\mathscr{K}$, then the character of the domain $f[\Delta]$ suggests that for non-negative real numbers t the function

$$f(z, t) = f(z) + tzf'(z) \tag{1.1}$$

is also univalent and that $f(z)$ is subordinate to $f(z, t)$ (cf. [5]), i.e., $f(z) \prec f(z, t)$. From the latter we infer the existence of a univalent function $w(z, t)$ satisfying the conditions of Schwarz's Lemma, namely $|w(z, t)| \leqslant |z|$ for z in Δ, such that

$$f\big(w(z, t)\big) + tw(z, t)f'\big(w(z, t)\big) = f(z). \tag{1.2}$$

The class of all functions $w(z, t)$ satisfying (1.2) for any admissible t and $f(z)$ is denoted by $\mathscr{F}$; the members of $\mathscr{F}$ are all univalent and of bound one in Δ.

The purpose of this paper is to study some properties of the functions in $\mathscr{F}$. In the next section, it is shown that a function is in $\mathscr{F}$ if and only if it satisfies a Loewner type of differential equation (cf. [7] and [4]); then, in a subsequent section, this differential equation is exploited to find some growth properties and coefficient estimates of functions in $\mathscr{F}$.

Similar ideas have been introduced and studied extensively by Dziubiński and his co-workers (see [1] for further references) in their study of quasi-starlike functions. It is the case that the notion of a quasi-

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convex function has been introduced by Taładaj [8]; but that treatment is a consequence of restricting the definition of quasi-starlikeness to the case where a starlike function happens to be convex. The present development makes explicit use of convexity and it follows that our results do not, in general, hold for the starlike case.

2. A Loewner-type equation for $\mathscr{F}$. We begin by taking a closer look at the solutions of (1.2).

THEOREM 1. *If $f(z)$ is in $\mathscr{K}$ and $t \geqslant 0$, then there is a unique solution*

$$w(z,t) = w(z;\, t;\, f) = \frac{1}{t+1}z + \ldots \tag{2.1}$$

of (1.2) *which is univalent and bounded by one in Δ and a member of $\mathscr{F}$.*

For each value of t, (1.1) defines a holomorphic function such that

$$\frac{f'(z,t)}{f'(z)} = 1 + t\left(1 + \frac{zf''(z)}{f'(z)}\right). \tag{2.2}$$

Because $f(z)$ is in $\mathscr{K}$, it follows that (2.2) has a positive real part in Δ; consequently, $f(z,t)$ is univalent and close-to-convex in Δ (cf. [3]).

Now, if we treat t as an independent variable, differentiation with respect to z and t gives

$$\operatorname{Re}\left\{\frac{zf_1(z,t)}{f_2(z,t)}\right\} = \operatorname{Re}\left\{1 + t\left(1 + \frac{zf''(z)}{f'(z)}\right)\right\} > 0 \tag{2.3}$$

for z in Δ and $t \geqslant 0$. It follows from the theorem of Bielecki and Lewandowski [2] (see also Pommerenke [6]) that

$$f[\Delta, t_1] \subset f[\Delta, t_2] \quad \text{for } 0 \leqslant t_1 \leqslant t_2 \tag{2.4}$$

and, in particular, that

$$f(z) \prec f(z,t) \quad \text{for } t \geqslant 0. \tag{2.5}$$

The last relation ensures the existence of a function given as in (2.1) such that

$$f(z) = f\big(w(z,t),\, t\big) \tag{2.6}$$

for each non-negative t, with $w(z,0) \equiv z$. The univalence of $f(z,t)$ guarantees that $w(z,t) = f\big(f(z),t\big)$ is likewise univalent.

THEOREM 2. *$w(z,t)$ is in $\mathscr{F}$ if and only if it satisfies the Loewner-type equation*

$$\frac{\partial w(z,t)}{\partial t} = \frac{-w(z,t)}{1 + tP\big(w(z,t)\big)}, \quad w(z,0) = z, \tag{2.7}$$

for $t \geqslant 0$ and $P(z)$ in $\mathscr{P}$.

Suppose $w(z, t)$ is in $\mathscr{F}$; then (1.2) assumes the form

$$f(w(z,t)) + tw(z,t)f'(w(z,t)) = f(z) \tag{2.8}$$

with the boundary condition of (2.7) being satisfied. Differentiating (2.8) with respect to t gives

$$\frac{\partial w(z,t)}{\partial t} = \frac{-w(z,t)}{1+t\left[1+w(z,t)\dfrac{f''(w(z,t))}{f'(w(z,t))}\right]} \tag{2.9}$$

and, because $f(z)$ is in $\mathscr{K}$, the expression in brackets may be written as $P(w(z,t))$ for suitable $P(z)$ in $\mathscr{P}$.

Conversely, suppose that (2.7) is satisfied with the given conditions. It follows from a result of Kufarev [4] that $w(z, t)$ is holomorphic and univalent in Δ. Now, since $P(z)$ is in $\mathscr{P}$, there is a function $f(z)$ in $\mathscr{K}$ defined by the condition $P(z) = 1+zf''(z)/f'(z)$ such that, for $t \geqslant 0$,

$$\frac{\partial w(z,t)}{\partial t}\left[f'(w(z,t)) + tf'(w(z,t)) + tw(z,t)f''(w(z,t))\right] + \\ + w(z,t)f'(w(z,t)) = 0. \tag{2.10}$$

It follows from (2.9) that there is a regular function $g(z)$, independent of t, such that, for $t \geqslant 0$,

$$f(w(z,t)) + tw(z,t)f'(w(z,t)) = g(z). \tag{2.11}$$

However, the boundary conditions of (2.7) imply that $g(z) \equiv f(z)$; this gives (1.2), hence the theorem is proved.

To conclude this section, we make an observation about the relationship between classes $\mathscr{K}$ and $\mathscr{F}$. If $w(z, t)$ in $\mathscr{F}$ and $f(z)$ in $\mathscr{K}$ are related (2.7) or (1.2), then from (2.7) we see that

$$\lim_{t\to\infty} w(z,t) = 0$$

locally uniformly for z in Δ and, consequently, from (2.1) that

$$\lim_{t\to\infty} tw(z,t) = f(z).$$

3. Extremal properties of $\mathscr{F}$. In this section, we make use of the differential equation of Theorem 2 to study some extremal problems within the family $\mathscr{F}$.

THEOREM 3. *If $w(z, t)$ is in $\mathscr{F}$, then for z in Δ*

$$\frac{2A-\tau+\sqrt{\tau(\tau-4A)+4A}}{2(1-A)} \leqslant |w(z,t)| \leqslant \frac{2B+\tau-\sqrt{\tau(\tau+4B)-4B}}{2(1+B)}, \tag{3.1}$$

where

$$(3.2)\qquad A = |z|(1+|z|)^{-1} \quad \textit{and} \quad B = |z|(1-|z|)^{-1}, \qquad \tau = t+1,$$

and both bounds are sharp.

From Theorem 2 we get

$$(3.3)\qquad \frac{\partial}{\partial s}\log|w(z,s)| = -\mathrm{Re}\left\{\frac{1}{1+sp\big(w(z,s)\big)}\right\} < 0.$$

Consequently, we observe that for each fixed z in Δ, $z \neq 0$, $|w(z,s)|$ is strictly decreasing in s. Hence, as s assumes values from 0 to $\tau-1 > 0$, $|w(z,s)|$ assumes corresponding values from $|z|$ to $|w(z,t)|$ and, therefore, s may be regarded as a function of $|w|$.

Since $P(z)$ appearing in (3.3) is in $\mathscr{P}$, there is a function $\eta(z)$ satisfying Schwarz's Lemma such that

$$(3.4)\qquad P(w) = \frac{1+\eta(w)}{1-\eta(w)} \quad \text{for } w \text{ in } \Delta.$$

Now, if we let $w = w(z,s)$, then a straightforward computation yields

$$(3.5)\qquad \frac{1-|w|}{1-|w|+s(1+|w|)} \leqslant \mathrm{Re}\left\{\frac{1}{1+sP(w)}\right\} \leqslant \frac{1+|w|}{1+|w|-s(1-|w|)},$$

which, in conjunction with (3.3), gives

$$(3.6)\qquad \frac{1}{|w|}+\frac{1-|w|}{|w|(1+|w|)}s \leqslant -\frac{ds}{d|w|} \leqslant \frac{1}{|w|}+\frac{1+|w|}{|w|(1-|w|)}s.$$

(3.6) is a linear differential inequality which may be solved by techniques similar to those used for solving linear differential equations; for example, it is possible to separate variables in the two left-most members of (3.6) by multiplying the resulting inequality by

$$\exp\int\frac{1-|w|}{|w|(1+|w|)}\,d|w|.$$

Then, integrating over the interval $[0, \tau-1]$, the resulting inequality gives

$$(3.7)\qquad \tau-1 \leqslant \frac{\big(1+|w(z,t)|\big)^2}{|w(z,t)|}\left(\frac{|z|}{1+|z|}-\frac{|w(z,t)|}{1+|w(z,t)|}\right).$$

In the calculations of (3.7), it is essential to keep in mind the functional relationship between s and $|w|$. Expression (3.7) is equivalent to the left-hand side of (3.1). The upper bound given in (3.1) is obtained in a similar way. The lower bound is rendered sharp by the function $w(z, t)$ given by the equation

$$\frac{\tau w + w^2}{(1+w)^2} = \frac{z}{1+z}, \tag{3.8}$$

and the upper bound by the function defined by

$$\frac{\tau w - w^2}{(1-w)^2} = \frac{z}{1-z}. \tag{3.9}$$

This completes the proof.

The left-hand side of (3.1) gives the following covering property:

COROLLARY 3.1. *The image domain of Δ under each member of $w(z, t)$ of $\mathcal{F}$ always includes the disk*

$$|W| < \frac{1}{(\tau-1)+\sqrt{(\tau-1)^2+1}} = \sqrt{t^2+1}-t; \tag{3.10}$$

this estimate is the best possible.

THEOREM 4. *If $w(z, t)$ is in $\mathcal{F}$, then for z in Δ*

$$\left|\arg\frac{w(z,t)}{z}\right| \leqslant 2\left(1-\frac{1}{\tau}\right)\left[F(\sin^{-1}|z|, k) - F\left(\sin^{-1}|w(z,t)|, k\right)\right], \tag{3.11}$$

where

$$F(\zeta, k) = \int_0^{\zeta}(1-k^2\sin^2 x)^{-1/2}dx, \qquad k = |1-2\tau^{-1}|, \ \tau = t+1. \tag{3.12}$$

The bound in (3.11) follows by separating variables in (2.6) and writing

$$d\arg w = -s\frac{\operatorname{Im}\{P(w)\}}{\operatorname{Re}\{1+sP(w)\}}\frac{d|w|}{|w|} = S(w, s)\frac{d|w|}{|w|}, \tag{3.13}$$

with $w = w(z, s)$ and $P(z)$ in $\mathcal{P}$. Now suppose $\eta(w)$ is chosen as in (3.4). Then

$$S(w, s) = -\frac{s\operatorname{Im}\left\{\left(1+\eta(w)\right)\left(1-\overline{\eta(w)}\right)\right\}}{\operatorname{Re}\left\{\left[1-\eta(w)+s\left(1+\eta(w)\right)\right]\left(1-\overline{\eta(w)}\right)\right\}}. \tag{3.14}$$

Maximizing the right-hand side of (3.14) over all functions $\eta(z)$ which satisfy Schwarz's Lemma and making use of the relation $\left|\eta\left(w(z, s)\right)\right| \leqslant |w(z, s)| = |w|$, we obtain

$$S(w, t) \leqslant \frac{2|w|}{\sqrt{1-|w|^2}}\,\frac{s}{\sqrt{1-|w|^2+2(1+|w|^2)s+(1+|w|^2)s^2}}. \tag{3.15}$$

Maximizing the last bound as a function of s over the interval $[0, t-1]$, we find that the maximum occurs at the right end point of the interval, i.e., for $s = t-1$. This estimate for (3.15) together with (3.13) give

$$d \arg w \leqslant \frac{d|w|}{\sqrt{(1-|w|^2)[t^2-(t-1)^2|w|^2]}}. \tag{3.16}$$

The theorem now follows by integration of (3.16).

We now turn to an examination of the coefficients of functions in $\mathscr{F}$ and for this purpose we require a different form of the differential equation (2.6).

Let $w(z, s)$ be a solution of equation (2.6) and let

$$\varphi(z, s) = w(w(z, s), s_0), \quad 0 \leqslant s \leqslant s_0. \tag{3.17}$$

It follows from the boundary conditions that

$$\varphi(z, s_0) = z, \quad \varphi(z, 0) = w(z, s_0), \tag{3.18}$$

and from (2.6) that

$$\frac{\partial \varphi(z, s)}{\partial s} = z \frac{\partial \varphi(z, s)}{\partial z} \frac{1}{1+sP(z)}, \quad 0 \leqslant s \leqslant s_0. \tag{3.19}$$

Assume now that

$$\varphi(z, s) = \sum_{k=1}^{\infty} a_k(s) z^k, \quad z \text{ in } \Delta, \tag{3.20}$$

$$P(z) = 1 + \sum_{k=1}^{\infty} c_k z^k, \quad z \text{ in } \Delta, \tag{3.21}$$

and

$$\begin{aligned} \frac{1}{1+sP(z)} &= \frac{1}{1+s} - \frac{c_1}{(1+s)^2} + \left(\frac{s^2 c_1^2}{(1+s)^3} - \frac{s c_2}{(1+s)^2}\right) z^2 + \ldots \\ &= \frac{1}{1+s} + \sum_{k=1}^{\infty} b_k(s) z^k \quad \text{for } z \text{ in } \Delta. \end{aligned} \tag{3.22}$$

Using representations (3.20), (3.21), and (3.22), comparison of coefficients in (3.19) gives the differential equations

$$a_1'(s) = \frac{a_1(s)}{1+s} \tag{3.23}$$

and

$$a_k'(s) = \frac{k a_k(s)}{1+s} + \sum_{j=1}^{k-1} j a_j(s) b_{k-j}(s) \tag{3.24}$$

for $k = 2, 3, \ldots$ and $0 \leqslant s \leqslant s_0$. With these differential equations, we are able to prove the following result:

THEOREM 5. *If* $t \geqslant 1$ *and*

$$w(z, t) = \frac{1}{t} z + \sum_{k=2}^{\infty} \gamma(t) z^k \tag{3.25}$$

is in $\mathcal{F}$, *then*

$$|\gamma_2(t)| \leqslant \frac{(t-1)^2}{t^3} \tag{3.26}$$

and

$$|\gamma_3(t)| \leqslant \begin{cases} \dfrac{(t-1)^2(t+2)}{3t^4} & \text{for } 1 \leqslant t \leqslant 3, \\[2ex] \dfrac{2(t-1)^3(t+3)}{3t^5} + \dfrac{(t-1)^2(t+2)}{3t^4} & \text{for } 3 \leqslant t \end{cases} \tag{3.27}$$

and both these bounds are sharp.

(For ease of calculation the values of t were chosen in the range $t \geqslant 1$, which differs slightly from (1.1).)

We begin the proof by rewriting (3.24) as

$$a_k'(s) = \frac{k}{1+s} a_k(s) + R_k(s), \tag{3.28}$$

and here we restrict the variable s to the interval $[0, T]$, where $t-1 = T$. The boundary data give the conditions $a_k(T) = 0$ and $a_k(0) = \gamma_k(T)$ for $k \geqslant 2$ and $a_1(T) = 1$. Then the solution of (3.28) is given by

$$a_k(s) = C(1+s)^k + (1+s)^k \int \frac{R_k(s)}{(1+s)^k} ds \tag{3.29}$$

for C a constant.

Using the boundary data again gives

$$\gamma_k(t) = A_k(0) - A_k(T), \quad k = 2, 3, \ldots, \tag{3.30}$$

where

$$A_k(s) = \int \frac{R_k(s)}{(1+s)^k} ds \tag{3.31}$$

And, in the case $k = 1$, (3.23) gives

$$a_1(s) = \frac{1+s}{t}, \tag{3.32}$$

which corresponds to $\gamma_1(0) = 1/t$ in (3.25). Then from (3.31), (3.29), and (3.28), we have

$$A_2(s) = \frac{-C_1}{t}\int \frac{s}{(1+s)^3}\,ds \tag{3.33}$$

and, consequently,

$$|\gamma_2(T)| = \left|\frac{C_1}{2(1+T)}\left(\frac{T}{1+T}\right)^2\right| \leqslant \frac{1}{t}\left(1-\frac{1}{t}\right)^2 \tag{3.34}$$

which is equivalent to (3.26).

Similar calculations give

$$A_3(s) = \frac{C_1^2}{t}\left[\frac{1}{2}u^2 - \frac{1}{3}u^3 + \frac{1-2t}{2t}\left(u-\frac{1}{2}u^2\right)\right] + C_2\left(\frac{1}{2}u - \frac{1}{3}u^2\right), \tag{3.35}$$

with the representation $u = (1+s)^{-1}$, and this leads to

$$\gamma_3(t) = C_1^2\frac{(t-1)^3(t-3)}{6t^5} + C_2\frac{(t-1)^2(t+2)}{6t}. \tag{3.36}$$

We now maximize (3.36) using known bounds on the coefficients of $P(z)$ in (3.21).

For $t \geqslant 3$, $\gamma_3(t)$ is a linear combination of the C_1^2 and C_2 with non-negative coefficients, consequently,

$$|\gamma_3(t)| \leqslant \frac{2(t-1)^3(t-3)}{3t^5} + \frac{(t-1)^2(t+2)}{3t^4}. \tag{3.37}$$

On the other hand, for $1 \leqslant t \leqslant 3$, we have

$$\gamma_3(t) = \frac{(t-1)^2(t+2)}{6t^4}\left[C_2 - \frac{(3-t)(t-1)}{t(t+2)}C_1^2\right]. \tag{3.38}$$

The last expression may be maximized by using the following result due to Ziegler [9]:

If $P(z)$ is in $\mathscr{P}$ and has representation (3.21), then

$$|C_2 - \mu C_1^2| \leqslant 2\max[1, |1-2\mu|]. \tag{3.39}$$

Formula (3.39) is obtained by writing $P(z)$ in form (3.4) and then maximizing over the coefficients of $\eta(z)$. The bound is rendered sharp by $P(z) = (1+z^2)/(1-z^2)$ if $|1-2\mu| \leqslant 1$ and by $P(z) = (1+z)/(1-z)$ if $|1-2\mu| \geqslant 1$.

In the case under consideration $\mu = [(3-t)(t-1)]/[t(t+2)]$ and it follows that $1 \leqslant t \leqslant 3$, $0 \leqslant 1-2\mu \leqslant 1$. Therefore the corresponding bound in (3.39) is equal to 2. This, in (3.38), together with (3.37) gives the bounds of Theorem 5.

The bounds of the theorem are sharp for all admissible t and $w(z, t)$ in $\mathscr{F}$ chosen as solutions of (1.2) for appropriate $f(z)$ in $\mathscr{K}$. If $f(z) = z/(1-z)$, then the corresponding values of $w(z, t)$ give the sharp bounds for $\gamma_2(t)$ in (3.26) and also for $\gamma_3(t)$ in (3.27) when $t \geqslant 3$. Letting

$$f(z) = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right)$$

we get the bound on $\gamma_3(t)$ in (3.27) for $1 \leqslant t \leqslant 3$.

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Reçu par la Rédaction le 15. 6. 1975