COLLOQUIUM MATHEMATICUM

VOL. XXIX 1974 FASC. 2

SHAPES OF SATURATED SUBSETS OF COMPACTA

BY

B. J. BALL (ATHENS, GEORGIA)

The fact that two compacta X and Y have the same shape implies, of course, very little about the possible shapes of arbitrary subsets of X relative to those of Y. If, however, an attention is restricted to the "saturated" subsets of X (meaning, here, those which can be expressed as a union of components of X), some conclusions are possible.

It was shown by Borsuk [1] that if X and Y are compacta having the same shape, then every component of X has the shape of some component of Y; indeed, there is 1-1 correspondence Λ between the components of X and those of Y such that each component of X has the shape of the corresponding component of Y, and this correspondence may be chosen to be a homeomorphism between the space of components of X and the space of components of Y. This result is extended in this paper to show that every closed, saturated subset S of X has the shape of some subset of Y (namely, the union of all components of Y corresponding, under Λ, to components of X lying in S). The requirement that S be closed is essential if Borsuk's [3] definition of (strong) shape for metrizable spaces is used in the non-compact case, but with the less restrictive definition of Fox [6], it is sufficient that S be locally compact. It seems likely that weaker conditions on S will suffice in this case, and also for Borsuk's definition of "weak" shape [4], but this is not established here.

1. Definitions and notation. The concept of the *shape* of a compactum, introduced by Borsuk (see [1] and [2]), is well known, and the definitions will not be repeated here. Both Borsuk [3] and Fox [6] have given definitions of a shape applicable to arbitrary metrizable spaces; these definitions are equivalent for compacta [6], but differ for some non-compact spaces [7]. The class of all compacta having the same shape as a given compactum X is denoted, as usual, by $\mathrm{Sh}(X)$. For an arbitrary metrizable space X, the class of all metrizable spaces having the same shape as X in the (strong) sense of Borsuk is denoted by $\mathrm{Sh}_S(X)$, and the class of all those having the shape of X in the sense of Fox by $\mathrm{Sh}_F(X)$.

A subset K of a space Z is said to be *locally compact in* Z if each point of K has a neighborhood U in Z such that $K \cap \overline{U}$ is compact (where $\overline{U}$ denotes the closure of U in Z). A collection $\mathscr{A}$ of subsets of Z is said to be *discrete in* Z if each point of Z has a neighborhood which intersects at most one member of $\mathscr{A}$.

If X is a locally compact metrizable space and every component of X is compact, then the collection $\mathscr{C}_X$ of all components of X is easily seen to be upper semicontinuous; the decomposition space $X/\mathscr{C}_X$ is called the *space of components* of X, and will be denoted by $\square X$. The *projection map* $p\colon X \to \square X$ is defined, as usual, by $p(x) = C$ if $x \in C \in \mathscr{C}_X$.

2. Shapes of subsets of compacta. Suppose X and Y are compacta lying in the Hilbert cube Q and let

$$p\colon X \to \square X \quad \text{and} \quad q\colon Y \to \square Y$$

be the projection maps of X and Y onto their respective spaces of components $\square X$ and $\square Y$. It was shown by Borsuk (see [1], p. 237, and [5], p. 49) that if $\{f_k, X, Y\}$ and $\{g_k, Y, X\}$ are homotopically inverse fundamental sequences from X to Y and from Y to X, respectively, in (Q, Q), then there exists a unique homeomorphism $\Lambda\colon \square X \to \square Y$ such that, for each $a \in \square X$,

$$\{f_k, p^{-1}(a), q^{-1}(\Lambda(a))\} \quad \text{and} \quad \{g_k, q^{-1}(\Lambda(a)), p^{-1}(a)\}$$

are homotopically inverse fundamental sequences. In particular, the following result holds:

2.1. Theorem (Borsuk). *Suppose X and Y are compacta having the same shape, and let*

$$p\colon X \to \square X \quad \text{and} \quad q\colon Y \to \square Y$$

be the projection maps. Then there is a homeomorphism $\Lambda\colon \square X \to \square Y$ such that, for every point $a \in \square X$, $p^{-1}(a)$ and $q^{-1}(\Lambda(a))$ have the same shape.

2.2. Theorem. *Under the hypothesis of Theorem 2.1, there is a homeomorphism $\Lambda\colon \square X \to \square Y$ such that, for every compact set $A \subset \square X$, $p^{-1}(A)$ and $q^{-1}(\Lambda(A))$ have the same shape.*

Proof. It may be assumed that X and Y are subsets of the Hilbert cube Q. Let $\{f_k, X, Y\}$ and $\{g_k, Y, X\}$ be homotopically inverse fundamental sequences in (Q, Q); as indicated above, Borsuk proved that there is a homeomorphism $\Lambda\colon \square X \to \square Y$ such that, for each $a \in \square X$,

$$\{f_k, p^{-1}(a), q^{-1}(\Lambda(a))\} \quad \text{and} \quad \{g_k, q^{-1}(\Lambda(a)), p^{-1}(a)\}$$

are homotopically inverse fundamental sequences. Suppose A is a compact subset of $\square X$, and let $X_1 = p^{-1}(A)$ and $Y_1 = q^{-1}(\Lambda(A))$. It will be shown that $\{f_k, X_1, Y_1\}$ and $\{g_k, Y_1, X_1\}$ are homotopically inverse fundamental sequences.

Let V be a neighborhood of Y_1 in Q. For each $a \in A$, V is a neighborhood of $q^{-1}(\Lambda(a))$ in Q and hence, since $\{f_k, p^{-1}(a), q^{-1}(\Lambda(a))\}$ is a fundamental sequence, there exists a neighborhood U'_a of $p^{-1}(a)$ in Q such that, for almost all k,

$$f_k | U'_a \simeq f_{k+1} | U'_a \quad \text{in } V.$$

Since $\mathscr{C}_X$ is upper semicontinuous, there is a neighborhood U_a of $p^{-1}(a)$ in Q such that $U_a \subset U'_a$ and $X \cap U_a$ is saturated with respect to $\mathscr{C}_X$. Since $U_a \subset U'_a$, $f_k | U_a \simeq f_{k+1} | U_a$ in V for almost all k, and since $X \cap U_a$ is open in X and is saturated with respect to $\mathscr{C}_X$, $p(X \cap U_a)$ is open in $\square X$. Then $\{p(X \cap U_a) | a \in A\}$ is an open covering of A in $\square X$, and since $\square X$ is compact and 0-dimensional, it follows easily that there exist disjoint open and closed subsets $W_1, W_2, \ldots, W_n$ of $\square X$ such that

$$A \subset \bigcup_{i=1}^{n} W_i$$

and $\{W_1, W_2, \ldots, W_n\}$ is a refinement of $\{p(X \cap U_a) | a \in A\}$. For each i, $1 \leqslant i \leqslant n$, let a_i be a point of A such that $W_i \subset p(X \cap U_{a_i})$ and let $K_i = p^{-1}(W_i)$. Then $K_1, K_2, \ldots, K_n$ are disjoint compact sets whose union contains X_1, and $K_i \subset U_{a_i}$ for $i = 1, 2, \ldots, n$. Hence there exist disjoint open sets $U_1, U_2, \ldots, U_n$ in Q such that $K_i \subset U_i \subset U_{a_i}$ for each i, $1 \leqslant i \leqslant n$. Since $U_i \subset U_{a_i}$,

$$f_k | U_i \simeq f_{k+1} | U_i \quad \text{in } V \text{ for almost all } k;$$

since $U_i \cap U_j = \emptyset$ for $i \neq j$, it follows that if

$$U = \bigcup_{i=1}^{n} U_i,$$

then, for almost all k, $f_k | U \simeq f_{k+1} | U$ in V. Hence $\{f_k, X_1, Y_1\}$ is a fundamental sequence from X_1 to Y_1 in (Q, Q). Similarly, $\{g_k, Y_1, X_1\}$ is a fundamental sequence from Y_1 to X_1.

Since, for each $a \in A$, the fundamental sequences

$$\{g_k f_k, p^{-1}(a), p^{-1}(a)\} \quad \text{and} \quad \{f_k g_k, q^{-1}(\Lambda(a)), q^{-1}(\Lambda(a))\}$$

are homotopic to the identity fundamental sequences on $p^{-1}(a)$ and $q^{-1}(\Lambda(a))$, respectively, an argument almost identical to the preceding one can be used to show that

$$\{g_k f_k, X_1, X_1\} \simeq \underline{i}_{X_1} \quad \text{and} \quad \{f_k g_k, Y_1, Y_1\} \simeq \underline{i}_{Y_1}.$$

Hence $\mathrm{Sh}(X_1) = \mathrm{Sh}(Y_1)$, as required.

Suppose X and Y are locally compact metrizable spaces with compact components. An example given in [5], p. 49, shows that the existence of a homeomorphism $\Lambda\colon \square X \to \square Y$ such that $p^{-1}(a)$ and $q^{-1}(\Lambda(a))$ have the same shape for every $a \in \square X$ does not imply (even for compacta) that $\mathrm{Sh}_S(X) = \mathrm{Sh}_S(Y)$. If, however, Λ is required to satisfy the stronger condition that $p^{-1}(A)$ and $q^{-1}(\Lambda(A))$ have the same shape for every compact set $A \subset \square X$, then it does follow, as shown below, that $\mathrm{Sh}_F(X) = \mathrm{Sh}_F(Y)$. (Even this stronger requirement on Λ does not insure that $\mathrm{Sh}_S(X) = \mathrm{Sh}_S(Y)$; see [4], or Section 5 of [7].)

2.3. LEMMA. *Suppose X and Y are locally compact metrizable spaces with compact components, and let*

$$p\colon X \to \square X \quad \textit{and} \quad q\colon Y \to \square Y$$

be the projection maps. If there is a homeomorphism $\Lambda\colon \square X \to \square Y$ such that $p^{-1}(A)$ and $q^{-1}(\Lambda(A))$ have the same shape for every compact set A in $\square X$, then $\mathrm{Sh}_F(X) = \mathrm{Sh}_F(Y)$.

Proof. It is easy to see that $\square X$ is locally compact and 0-dimensional, and hence can be covered by a collection $\mathscr{A} = \{A_\mu \mid \mu \in \mathfrak{m}\}$ of disjoint open sets with compact closures. Clearly, each A_μ is itself compact, and $\mathscr{A}$ is discrete in $\square X$.

For each $\mu \in \mathfrak{m}$, let $X_\mu = p^{-1}(A_\mu)$ and $Y_\mu = q^{-1}(\Lambda(A_\mu))$. Then

$$X = \bigcup_{\mu \in \mathfrak{m}} X_\mu \quad \text{and} \quad Y = \bigcup_{\mu \in \mathfrak{m}} Y_\mu,$$

and $\{X_\mu \mid \mu \in \mathfrak{m}\}$ and $\{Y_\mu \mid \mu \in \mathfrak{m}\}$ are discrete collections of compact subsets of X and Y, respectively. For each $\mu \in \mathfrak{m}$, A_μ is a compact subset of $\square X$ and hence, by hypothesis, $p^{-1}(A_\mu)$ and $q^{-1}(\Lambda(A_\mu))$ have the same shape. Hence $\mathrm{Sh}(X_\mu) = \mathrm{Sh}(Y_\mu)$ for every $\mu \in \mathfrak{m}$, and it follows from Theorem 4.2 of [7] that $\mathrm{Sh}_F(X) = \mathrm{Sh}_F(Y)$.

2.4. THEOREM. *Suppose X and Y are compacta having the same shape, and let $p\colon X \to \square X$ and $q\colon Y \to \square Y$ be the projection maps. Then there is a homeomorphism $\Lambda\colon \square X \to \square Y$ such that, for every locally compact set K in $\square X$, $p^{-1}(K)$ and $q^{-1}(\Lambda(K))$ have the same shape in the sense of Fox.*

Proof. By Theorem 2.2, there is a homeomorphism $\Lambda\colon \square X \to \square Y$ such that, for every compact set A in $\square X$, $p^{-1}(A)$ and $q^{-1}(\Lambda(A))$ have the same shape.

Suppose K is a locally compact subset of $\square X$ and let $X_1 = p^{-1}(K)$ and $Y_1 = q^{-1}(\Lambda(K))$. Let $p_1 = p|X_1$, $q_1 = q|Y_1$ and $\Lambda_1 = \Lambda|K$. Then $\square X_1 = K$, $\square Y_1 = \Lambda(K)$, $p_1\colon X_1 \to \square X_1$ and $q_1\colon Y_1 \to \square Y_1$ are the projection maps, and $\Lambda_1\colon \square X_1 \to \square Y_1$ is a homeomorphism. Since X_1 and Y_1 are locally compact spaces with compact components and, for every compact subset A of $\square X_1$, the sets $p_1^{-1}(A) = p^{-1}(A)$ and $q_1^{-1}(\Lambda_1(A)) = q^{-1}(\Lambda(A))$ have the same shape, it follows from Lemma 2.3 that $\mathrm{Sh}_F(X_1) = \mathrm{Sh}_F(Y_1)$, as required.

2.5. COROLLARY. *Suppose X and Y are compacta having the same shape and f is a map of X onto a 0-dimensional compactum Z. Then there is a map g of Y onto Z such that, for every locally compact set L in Z, $f^{-1}(L)$ and $g^{-1}(L)$ have the same shape in the sense of Fox.*

Proof. Let $p\colon X\to\square X$ and $q\colon Y\to\square Y$ be the projection maps and let $\Lambda\colon \square X\to\square Y$ be the homeomorphism given by Theorem 2.4. It follows from the monotone-light factorization theorem (see [8], p. 141) that there is a map $r\colon \square X\to Z$ such that $f = r\circ p$. Let $s = r\circ\Lambda^{-1}$ and $g = s\circ q$. Then the diagram

$$\begin{tikzcd}
X \arrow[dr, "f"] \arrow[dd, "p"'] & & Y \arrow[dl, "g"'] \arrow[dd, "q"] \\
& Z & \\
\square X \arrow[ur, "r"] & & \square Y \arrow[ul, "s"'] \arrow[ll, "\Lambda^{-1}"']
\end{tikzcd}$$

is commutative, and all maps shown are surjections.

Suppose K is a locally compact subset of Z and let $K = r^{-1}(L)$, $H = s^{-1}(L)$. Then $f^{-1}(L) = p^{-1}(K)$ and $g^{-1}(L) = q^{-1}(H)$, and, moreover,

$$H = s^{-1}(L) = \Lambda\big(r^{-1}(L)\big) = \Lambda(K).$$

Hence $f^{-1}(L) = p^{-1}(K)$ and $g^{-1}(L) = q^{-1}\big(\Lambda(K)\big)$, and since K is a locally compact subset of $\square X$, it follows from the choice of Λ that

$$\mathrm{Sh_F}\big(f^{-1}(L)\big) = \mathrm{Sh_F}\big(g^{-1}(L)\big).$$

Remark. Let X' be the union of a countable, discrete collection of circles, Y' the union of a countable, discrete collection of Warsaw circles, and X and Y the one-point compactifications of X' and Y', respectively. It is not difficult to show that $\mathrm{Sh}(X) = \mathrm{Sh}(Y)$. However, by the example of Section 5 of [7] (given earlier by S. Nowak; see [5]), $\mathrm{Sh_S}(X') \neq \mathrm{Sh_S}(Y')$ and it easily follows that no non-closed saturated subset of X has the same strong shape, in the sense of Borsuk, as *any* subset of Y. In particular, it cannot be concluded in Theorem 2.4 that

$$\mathrm{Sh_S}\big(p^{-1}(K)\big) = \mathrm{Sh_S}\Big(q^{-1}\big(\Lambda(K)\big)\Big)$$

unless K is closed in $\square X$.

QUESTION 1. Can the requirement in Theorem 2.4 that K be locally compact be deleted, or replaced by a weaker condition (e.g., that K be an F_σ-set in $\square X$)? (**P 888**)

QUESTION 2. If X and Y are metrizable spaces with $\mathrm{Sh_F}(X) = \mathrm{Sh_F}(Y)$, must there be a 1-1 correspondence Φ between the components of X

and those of Y such that, for each component X_0 of X, $\mathrm{Sh}_F(X_0) = \mathrm{Sh}_F(\Phi(X_0))$? If $\mathscr{C}_X$ and $\mathscr{C}_Y$ are upper semicontinuous, can Φ be chosen to be a homeomorphism from $\Box X$ to $\Box Y$? (**P 889**)

REFERENCES

[1] K. Borsuk, *Concerning homotopy properties of compacta*, Fundamenta Mathematicae 62 (1968), p. 223-254.

[2] — *Concerning the notion of the shape of compacta*, Proceedings of the International Symposium on Topology and its Applications, Herceg-Novi 1968 and Beograd 1969, p. 98-104.

[3] — *On the concept of shape for metrizable spaces*, Bulletin de l'Académie Polonaise des Sciences, Série des sciences mathématiques, astronomiques et physiques, 18 (1970), p. 127-132.

[4] — *Some remarks concerning the theory of shape in arbitrary metrizable spaces*, Proceedings of the Third Prague Symposium on General Topology, Prague 1972, p. 77-81.

[5] — *Theory of shape*, Lecture Notes, Series 28, Matematisk Instituut, Aarhus Universitet, 1972.

[6] R. H. Fox, *On shape*, Fundamenta Mathematicae 74 (1972), p. 47-71.

[7] S. Godlewski and S. Nowak, *On two notions of shape*, Bulletin de l'Académie Polonaise des Sciences, Série des sciences mathématiques, astronomiques et physiques, 20 (1972), p. 387-393.

[8] G. T. Whyburn, *Analytic topology*, American Mathematical Society Colloquium Publications 28, New York 1942.

Reçu par la Rédaction le 16. 9. 1972