COLLOQUIUM MATHEMATICUM

VOL. XLIV 1981 FASC. 1

DERIVED FUNCTORS OF $\varprojlim$ *AND ABELIAN* Ab3*- *AND* Ab4*-*CATEGORIES WITH ENOUGH INJECTIVES*

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In the paper we use the terminology of [3]. Throughout the paper $\mathscr{C}$ stands for an abelian Ab3*-category with enough injective objects, and I for a directed set considered as a category. We denote by $[I^0, \mathscr{C}]$ the category of all contravariant functors from I to $\mathscr{C}$. We know that $[I^0, \mathscr{C}]$ is an abelian Ab3*-category with enough injectives (cf. [3]). The inverse limit functor

$$\varprojlim : [I^0, \mathscr{C}] \to \mathscr{C}$$

is left exact, and by $\varprojlim^{(n)}$ we denote its n-th right derived functor (cf. [6] and [1]). The functors $\varprojlim^{(n)}$ were studied by many authors (cf. [2] and [4]-[7]).

In [6] Roos proved that if $\mathscr{C}$ is an abelian Ab3*- and Ab4*-category with enough injectives, then for every object A in $[I^0, \mathscr{C}]$ the object $\varprojlim^{(n)} A$ is isomorphic to the n-th cohomology object of the chain complex

$$\Pi^* A: 0 \to \Pi^0 A \xrightarrow{d^0} \Pi^1 A \xrightarrow{d^1} \Pi^2 A \to \ldots$$

with

$$\Pi^n A = \prod_{a_0 \leqslant \ldots \leqslant a_n} A(a_0, \ldots, a_n), \qquad A(a_0, \ldots, a_n) = A(a_0),$$

and the morphism d^n defined by the formula

$$\pi_{a_0, \ldots, a_{n+1}} \circ d^n = A(a_0 \to a_1) \circ \pi_{a_1, \ldots, a_{n+1}} + \sum_{i=1}^{n+1} (-1)^i \pi_{a_0, \ldots, \hat{a}_i, \ldots, a_{n+1}},$$

where $\pi_{a_0, \ldots, a_n}: \Pi^n A \to A(a_0, \ldots, a_n)$ is the structural projection, and $\hat{y}$ means y to be omitted.

The aim of this paper is to prove the theorem which is a generalization of the above-mentioned Roos result.

Before formulating the theorem we define a functor

$$\tilde{\ }: [I^0, \mathscr{C}] \to [I^0, \mathscr{C}]$$

as follows.

Given an object A in $[I^0, \mathscr{C}]$, we put

$$\tilde{A}(a) = \prod_{a_0 \leqslant a} A(a_0) \quad \text{for any } a \in I$$

and assume that $\tilde{A}(a \to \beta)\colon \tilde{A}(\beta) \to \tilde{A}(a)$ is the natural projection for $a \leqslant \beta$. Furthermore, if $f\colon A \to B$ is a morphism in $[I^0, \mathscr{C}]$, we put

$$\tilde{f}(a) = \prod_{a_0 \leqslant a} f(a_0) \quad \text{for any } a \in I.$$

THEOREM. *Let $\mathscr{C}$ be an abelian* Ab3*-*category with enough injectives. Then the following conditions are equivalent:*

(1) *$\mathscr{C}$ is an* Ab4*-*category;*

(2) $\varprojlim^{(n)} \tilde{A} = 0$, $n = 1, 2, 3, \ldots$, *for any directed set I and any object A in $[I^0, \mathscr{C}]$;*

(3) $\varprojlim^{(n)} A = H^n(\Pi^* A)$, $n = 0, 1, 2, \ldots$, *for any directed set I and any object A in $[I^0, \mathscr{C}]$;*

(4) $\varprojlim^{(1)} \tilde{A} = H^1(\Pi^* \tilde{A})$ *for any well-ordered set I and any object A in $[I^0, \mathscr{C}]$.*

Proof. (1) $\Rightarrow$ (2). Suppose that $\mathscr{C}$ is an abelian Ab3*- and Ab4*-category with enough injectives. Let I be a directed set and let A be an object of $[I^0, \mathscr{C}]$. There exists an exact sequence

$$0 \to A \xrightarrow{g^{-1}} Q^0 \xrightarrow{g^0} Q^1 \xrightarrow{g^1} Q^2 \to \ldots,$$

where Q^n $(n = 0, 1, 2, \ldots)$ are injective in $[I^0, \mathscr{C}]$. Since $\mathscr{C}$ is an abelian Ab4*-category, the functor $\tilde{\cdot}$ is exact (cf. [3]) and we have the exact sequence

$$0 \to \tilde{A} \xrightarrow{\tilde{g}^{-1}} \tilde{Q}^0 \xrightarrow{\tilde{g}^0} \tilde{Q}^1 \xrightarrow{\tilde{g}^1} \tilde{Q}^2 \to \ldots$$

It is easy to check that if Q^n is injective, then $Q^n(a)$ is injective for each $a \in I$, and hence $\tilde{Q}^n$ is an injective object of $[I^0, \mathscr{C}]$. Moreover, a simple calculation shows that the sequence

$$\varprojlim \tilde{Q}^0 \xrightarrow{\varprojlim \tilde{g}^0} \varprojlim \tilde{Q}^1 \xrightarrow{\varprojlim \tilde{g}^1} \varprojlim \tilde{Q}^2 \to \ldots$$

is isomorphic to the exact sequence

$$\prod_{a \in I} Q^0(a) \xrightarrow{\prod_{a \in I} g^0(a)} \prod_{a \in I} Q^1(a) \xrightarrow{\prod_{a \in I} g^1(a)} \prod_{a \in I} Q^2(a) \to \ldots$$

Therefore, $\varprojlim^{(n)} \tilde{A} = 0$ for $n = 1, 2, 3, \ldots$

(2) $\Rightarrow$ (3). We apply arguments from [2], p. 31-33. Let A be an object of $[I^0, \mathscr{C}]$. Consider a sequence

$$(*) \qquad 0 \to A \xrightarrow{d^{-1}} F^0 \xrightarrow{d^0} F^1 \xrightarrow{d^1} F^2 \to \ldots$$

in $[I^0, \mathscr{C}]$ defined by (1)

(i) $F^n(a) = \prod_{a_0 \leqslant \ldots \leqslant a_n \leqslant a} F(a_0, \ldots, a_n)$, where $F(a_0, \ldots, a_n) = A(a_0)$;

(ii) $F^n(a \to \beta)$ is the projection for $a \leqslant \beta$;

(iii) $\pi^a_{a_0,\ldots,a_{n+1}} \circ d^n(a) = A(a_0 \to a_1) \circ \pi^a_{a_1,\ldots,a_{n+1}} + \sum_{i=1}^{n+1} (-1)^i \pi^a_{a_0,\ldots,\hat{a}_i,\ldots,a_{n+1}}$ for $n = 0, 1, 2, \ldots$ and $a_0 \leqslant \ldots \leqslant a_{n+1} \leqslant a$;

(iv) $\pi^a_{a_0} \circ d^{-1}(a) = A(a_0 \to a_1)$ for $a_0 \leqslant a$.

It is easy to check that $d^{n+1} \circ d^n = 0$ for $n \geqslant -1$. Moreover, if

$$\varepsilon^n(a): F^n(a) \to F^{n-1}(a) \qquad (n \geqslant 0,\ F^{-1} = A)$$

is a map defined by $\pi^a_{a_0,\ldots,a_{n-1}} \circ \varepsilon^n(a) = (-1)^n \pi^a_{a_0,\ldots,a_{n-1},a}$ for $n \geqslant 1$, $a_0 \leqslant \ldots \leqslant a_{n-1} \leqslant a$, and $\varepsilon^0(a) = \pi^a_a$, then

$$\varepsilon^{n+1}(a) \circ d^n(a) + d^{n-1}(a) \circ \varepsilon^n(a) = 1_{F^n(a)} \quad (n \geqslant 0) \quad \text{and} \quad \varepsilon^0(a) \circ d^{-1}(a) = 1_A.$$

Therefore, sequence $(*)$ is exact. It is clear that $F^m \approx \tilde{F}^{m-1}$ $(m \geqslant 0)$. Consequently, $\underleftarrow{\lim}^{(n)} F^m = 0$ for $n \geqslant 1$, $m \geqslant 0$, and from Proposition 7.18 in [1] it follows that $\underleftarrow{\lim}^{(n)} A$ is the n-th cohomology object of the sequence

$$0 \to \underleftarrow{\lim} F^0 \xrightarrow{\underleftarrow{\lim} d^0} \underleftarrow{\lim} F^1 \xrightarrow{\underleftarrow{\lim} d^1} \underleftarrow{\lim} F^2 \to \ldots$$

which is isomorphic to $\Pi^* A$.

(3) $\Rightarrow$ (4) is obvious.

(4) $\Rightarrow$ (1). Suppose to the contrary that (1) is not satisfied. Let $\{g(\xi): B(\xi) \to C(\xi)\}_{\xi<\gamma}$ be a family of epimorphisms indexed by ordinal numbers such that $\prod_{\xi<\gamma} g(\xi)$ is not an epimorphism. Without loss of generality we can assume that

$$(**) \qquad \prod_{\xi<\beta} g(\xi) \text{ is an epimorphism for every } \beta < \gamma.$$

It is clear that γ is a limit number. We put $\{\xi : \xi < \gamma\} = I$. For any $\xi < \gamma$ we have an exact sequence

$$0 \to A(\xi) \xrightarrow{f(\xi)} B(\xi) \xrightarrow{g(\xi)} C(\xi) \to 0.$$

(1) $\pi^a_{a_0,\ldots,a_n}: F^n(a) \to F(a_0, \ldots, a_n)$ denotes the structural projection.

Putting $A(\xi \to \xi') = 0$, $B(\xi \to \xi') = 0$ and $C(\xi \to \xi') = 0$ for all $\xi \leqslant \xi' < \gamma$ we define objects A, B and C in $[I^0, \mathscr{C}]$. By (**) we have an exact sequence

$$0 \to \tilde{A} \xrightarrow{\tilde{f}} \tilde{B} \xrightarrow{\tilde{g}} \tilde{C} \to 0$$

and the induced long exact sequence

$$0 \to \varprojlim \tilde{A} \xrightarrow{\varprojlim \tilde{f}} \varprojlim \tilde{B} \xrightarrow{\varprojlim \tilde{g}} \varprojlim \tilde{C} \to \varprojlim{}^{(1)} \tilde{A} \to \ldots$$

Since, obviously,

$$\varprojlim \tilde{g} = \prod_{\xi < \gamma} g(\xi)$$

and it is not an epimorphism, we infer that $\varprojlim{}^{(1)} \tilde{A} \neq 0$.

On the other hand, we show that $H^1(\Pi^* \tilde{A}) = 0$, i.e. the sequence

$$\prod_{a_0 < \gamma} \tilde{A}(a_0) \xrightarrow{d^0} \prod_{a_0 \leqslant a_1 < \gamma} \tilde{A}(a_0, a_1) \xrightarrow{d^1} \prod_{a_0 \leqslant a_1 \leqslant a_2 < \gamma} \tilde{A}(a_0, a_1, a_2)$$

is exact. But this sequence is isomorphic to the sequence

$$\prod_{a \leqslant a_0 < \gamma} A(a, a_0) \xrightarrow{d'^0} \prod_{a \leqslant a_0 \leqslant a_1 < \gamma} A(a, a_0, a_1) \xrightarrow{d'^1} \prod_{a \leqslant a_0 \leqslant a_1 \leqslant a_2 < \gamma} A(a, a_0, a_1, a_2),$$

where

$$\pi'_{a, a_0, \ldots, a_n} \circ d'^n = \sum_{i=0}^{n+1} (-1)^i \pi'_{a, a_0, \ldots, \hat{a}_i, \ldots, a_{n+1}}.$$

We define morphisms

$$s^n \colon \prod_{a \leqslant a_0 \leqslant \ldots \leqslant a_n < \gamma} A(a, a_0, \ldots, a_n) \to \prod_{a \leqslant a_0 \leqslant \ldots \leqslant a_{n-1} < \gamma} A(a, a_0, \ldots, a_{n-1}) \qquad (n = 1, 2)$$

by $\pi'_{a, a_0, \ldots, a_{n-1}} \circ s^n = \pi'_{a, a, a_0, \ldots, a_{n-1}}$. It is easy to check that

$$d'^0 \circ s^1 + s^2 \circ d'^1 = 1_{\Pi A(a, a_0, a_1)}, \quad a \leqslant a_0 \leqslant a_1 < \gamma.$$

Therefore, our sequence is exact and we get a contradiction. The proof of the theorem is complete.

REFERENCES

[1] I. Bucur and A. Deleanu, *Introduction to the theory of categories and functors*, London 1968.

[2] C. U. Jensen, *Les foncteurs dérivés de* $\varprojlim$ *et leurs applications en théorie des modules*, Lecture Notes in Mathematics 254 (1972).

[3] N. Popescu, *Abelian categories with applications to rings and modules*, London - - New York 1973.
[4] T. Porter, *Stability of algebraic inverse systems, I. Stability, weak stability and weakly-stable socle*, Fundamenta Mathematicae (to appear).
[5] — *Stability of algebraic inverse systems, II. Purity and localization*, Journal of Pure and Applied Algebra 7 (1976), p. 133-143.
[6] J. E. Roos, *Sur les foncteurs dérivés de* $\varprojlim$. *Applications*, Comptes Rendus Hebdomadaires des Séances de l'Académie des Sciences, Paris, 252 (1961), p. 3702-3704.
[7] D. Simson, *Purity and homological dimensions in locally finitely presentable Grothendieck categories* (in Polish), Thesis, Toruń 1974.

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Reçu par la Rédaction le 31. 12. 1977