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A PARTICULAR CONFORMALLY SYMMETRIC SPACE

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1. Introduction. Conformally symmetric Riemannian spaces V_n ($n \geqslant 4$) defined by $C^h{}_{ijk,l} = 0$, where $C^h{}_{ijk}$ is the conformal curvature tensor and the comma denotes covariant differentiation with respect to the metric of the space, were for the first time studied by Chaki and Gupta [2] and, subsequently, by other authors (see [1] and [4]). In a recent paper [3] we have for the first time proved the existence of 4-dimensional conformally symmetric spaces which are neither symmetric nor conformally flat. In the present paper we prove the existence of n-dimensional ($n \geqslant 4$) spaces which are neither symmetric nor conformally flat.

2. Existence of a non-symmetric conformally symmetric-space. We consider the Riemannian n-space ($n \geqslant 4$) with coordinates x_i ($i = 1, 2, \ldots, n$) and metric

(1) $$ds^2 = g_{11}dx_1^2 + g_{22}dx_2^2 + \ldots + g_{nn}dx_n^2,$$

where

$$g_{11} = \varphi^2, \quad g_{22} = -x_1^2, \quad g_{hh} = \sin^2 x_{h-1} g_{h-1\,h-1} \ (2 < h < n), \quad g_{nn} = -\frac{1}{\varphi^2},$$

φ being given by

(2) $$\varphi^2 = -[1 + x_1 + x_1^2 + x_1^2 \log x_1]^{-1}.$$

For this space the non-zero components of the Riemann tensor are the following components together with those related to them:

(3)

$$R_{1221} = \frac{x_1}{\varphi}\frac{d\varphi}{dx_1}, \qquad R_{1kk1} = \sin^2 x_{k-1} R_{1\,k-1\,k-1\,1} \qquad (2 < k < n),$$

$$R_{1nn1} = -\frac{3}{\varphi^4}\left(\frac{d\varphi}{dx_1}\right)^2 + \frac{1}{\varphi^3}\frac{d^2\varphi}{dx_1^2},$$

$$R_{2332} = x_1^2 \sin^2 x_2 \left(1 + \frac{1}{\varphi^2}\right), \qquad R_{2kk2} = \sin^2 x_{k-1} R_{2\,k-1\,k-1\,2} \qquad (3 < k < n),$$

$$R_{2nn2} = -\frac{1}{\varphi^4} R_{1221}, \qquad R_{hkkh} = \sin^2 x_{h-1} R_{h-1\,kk\,h-1} \qquad (2 < h < k \leqslant n).$$

Taking covariant derivatives of the Riemann tensor, the Ricci tensor R_{ij} and the scalar curvature R, we obtain

(4)

$$R_{1221,1} = \frac{x_1}{\varphi}\frac{d^2\varphi}{dx_1^2} - \frac{3x_1}{\varphi^2}\left(\frac{d\varphi}{dx_1}\right)^2 - \frac{1}{\varphi}\frac{d\varphi}{dx_1}, \quad R_{1221,p} = 0, \quad p \neq 1,$$

$$R_{1kk1,1} = \sin^2 x_{k-1} R_{1\,k-1\,k-1\,1,1}, \quad R_{1kk1,p} = 0, \quad p \neq 1 \ (2 < k < n),$$

$$R_{1nn1,1} = \frac{1}{\varphi^3}\frac{d^3\varphi}{dx_1^3} - \frac{9}{\varphi^4}\frac{d^2\varphi}{dx_1^2} + \frac{d\varphi}{dx_1} + \frac{12}{\varphi^5}\left(\frac{d\varphi}{dx_1}\right)^3, \quad R_{1nn1,p} = 0, \quad p \neq 1,$$

$$R_{2332,1} = -2x_1 \sin^2 x_2 \left[\left(1 + \frac{1}{\varphi^2}\right) + \frac{x_1}{\varphi^3}\frac{d\varphi}{dx_1}\right], \quad R_{2332,p} = 0, \quad p \neq 1,$$

$$R_{2kk2,1} = \sin^2 x_{k-1} R_{2\,k-1\,k-1\,2,1}, \quad R_{2kk2,p} = 0, \quad p \neq 1 \ (3 < k < n),$$

$$R_{2nn2,1} = -\frac{1}{\varphi^4} R_{1221,1}, \quad R_{2nn2,p} = 0, \quad p \neq 1,$$

$$R_{hkkh,1} = \sin^2 x_{h-1} R_{h-1\,kk\,h-1,1}, \quad R_{hkkh,p} = 0, \quad p \neq 1 \ (2 < h < k \leqslant n);$$

(5)

$$R_{11,1} = -\frac{1}{\varphi}\frac{d^3\varphi}{dx_1^3} + \frac{d^2\varphi}{dx_1^2}\left[\frac{9}{\varphi^2}\frac{d\varphi}{dx_1} - \frac{n-2}{x_1\varphi}\right] - \frac{12}{\varphi^3}\left(\frac{d\varphi}{dx_1}\right)^3 + $$
$$+ \frac{3(n-2)}{x_1\varphi^2}\left(\frac{d\varphi}{dx_1}\right)^2 + \frac{n-2}{x_1^2\varphi}\frac{d\varphi}{dx_1}, \quad R_{11,p} = 0, \quad p \neq 1,$$

$$R_{22,1} = \frac{2}{x_1\varphi^4}\left[\varphi x_1^2 \frac{d^2\varphi}{dx_1^2} - 3x_1^2\left(\frac{d\varphi}{dx_1}\right)^2 + (n-4)\varphi x_1 \frac{d\varphi}{dx_1} + (n-3)\varphi^2(1+\varphi^2)\right],$$
$$R_{22,p} = 0, \quad p \neq 1,$$

$$R_{hh,1} = \sin^2 x_{h-1} R_{h-1\,h-1,1}, \quad R_{hh,p} = 0, \quad p \neq 1 \ (2 < h < n),$$

$$R_{nn,1} = -\frac{1}{\varphi^4} R_{11,1}, \quad R_{nn,p} = 0, \quad p \neq 1;$$

(6) $$R_{,1} = \frac{2}{\varphi^2}\left[-\frac{1}{\varphi}\frac{d^3\varphi}{dx_1^3} + \frac{d^2\varphi}{dx_1^2}\left\{\frac{9}{\varphi^2}\frac{d\varphi}{dx_1} - \frac{2(n-2)}{x_1\varphi}\right\} - \frac{12}{\varphi^3}\left(\frac{d\varphi}{dx_1}\right)^3 + \right.$$
$$\left. + \frac{6(n-2)}{x_1\varphi^2}\left(\frac{d\varphi}{dx_1}\right)^2 - \frac{n^2-7n+10}{x_1^2\varphi}\frac{d\varphi}{dx_1} - \frac{n^2-5n+6}{x_1^3}(1+\varphi^2)\right],$$
$$R_{,p} = 0, \quad p \neq 1.$$

The conformal curvature tensor C_{hijk} is given by

(7) $$C_{hijk} = R_{hijk} + g_{ij}L_{hk} - g_{ik}L_{hj} - g_{hj}L_{ik} + g_{hk}L_{ij},$$

where

(8) $$L_{ij} = \frac{1}{2-n}\left[R_{ij} - \frac{R}{2(n-1)}\, g_{ij}\right].$$

Taking covariant derivatives of L_{ij} and using (5) and (6), we get

(9)
$$L_{11,1} = \frac{1}{n-1}\left[\frac{1}{\varphi}\frac{d^3\varphi}{dx_1^3} - \frac{d^2\varphi}{dx_1^2}\left\{\frac{g^3}{\varphi^2}\frac{d\varphi}{dx_1} + \frac{3-n}{x_1\varphi}\right\} + \frac{12}{\varphi^3}\left(\frac{d\varphi}{dx_1}\right)^3 + \frac{3(3-n)}{x_1\varphi^2}\left(\frac{d\varphi}{dx_1}\right)^2 - \right.$$
$$\left. - \frac{2(n-3)}{x_1^2\varphi}\frac{d\varphi}{dx_1} - \frac{n-3}{x_1^3}(1+\varphi^2)\right], \quad L_{11,p} = 0, \quad p \neq 1,$$

$$L_{22,1} = \frac{x_1^2}{(2-n)(n-1)\varphi^2}\left[-\frac{1}{\varphi}\frac{d^3\varphi}{dx_1^3} + \frac{d^2\varphi}{dx_1^2}\left\{\frac{9}{\varphi^2}\frac{d\varphi}{dx_1} + \frac{2}{x_1\varphi}\right\} - \frac{12}{\varphi^3}\left(\frac{d\varphi}{dx_1}\right)^3 - \right.$$
$$\left. - \frac{6}{x_1\varphi^2}\left(\frac{d\varphi}{dx_1}\right)^2 + \frac{n^2-3n-2}{x_1^2\varphi}\frac{d\varphi}{dx_1} + \frac{n(n-3)}{x_1^3}(1+\varphi^2)\right], \quad L_{22,p} = 0, \quad p \neq 1,$$

$$L_{hh,1} = \sin^2 x_{h-1} L_{h-1\,h-1,1}, \quad L_{hh,p} = 0, \quad p \neq 1 \ (2 < h < n),$$

$$L_{nn,1} = \frac{1}{\varphi^4} L_{11,1}, \quad L_{nn,p} = 0, \quad p \neq 1.$$

Lastly, taking covariant derivatives of C_{hijk} and using (5), (6) and (9), we obtain

(10)
$$C_{1221,1} = \frac{(n-3)x_1^2}{(2-n)(n-1)}\left[\frac{1}{\varphi}\frac{d^3\varphi}{dx_1^3} - \frac{d^2\varphi}{dx_1^2}\left(\frac{9}{\varphi^2}\frac{d\varphi}{dx_1} + \frac{2}{x_1\varphi}\right) + \frac{12}{\varphi^3}\left(\frac{d\varphi}{dx_1}\right)^3 + \right.$$
$$\left. + \frac{6}{x_1\varphi^2}\left(\frac{d\varphi}{dx_1}\right)^2 + \frac{4}{x_1^2\varphi}\frac{d\varphi}{dx_1} + \frac{2}{x_1^3}(1+\varphi^2)\right], \quad C_{1221,p} = 0, \quad p \neq 1,$$

$$C_{1hh1,1} = \sin^2 x_{h-1} C_{1\,h-1\,h-1\,1,1}, \quad C_{1hh1,p} = 0, \quad p \neq 1 \ (2 < h < n),$$

$$C_{1nn1,1} = \frac{2-n}{x_1^2\varphi^2} C_{1221,1}, \quad C_{1nn1,p} = 0, \quad p \neq 1,$$

$$C_{2332,1} = \frac{2x_1^2\sin^2 x_2}{(n-3)\varphi^2} C_{1221,1}, \quad C_{2332,p} = 0, \quad p \neq 1,$$

$$C_{2kk2,1} = \sin^2 x_{k-1} C_{2\,k-1\,k-1\,2,1}, \quad C_{2kk2,p} = 0, \quad p \neq 1 \ (3 < k < n),$$

$$C_{2nn2,1} = -\frac{1}{\varphi^4} C_{1221,1}, \quad C_{2nn2,p} = 0, \quad p \neq 1,$$

$$C_{hkkh,1} = \sin^2 x_{h-1} C_{h-1\,kk\,h-1,1}, \quad C^{hkkh,p} = 0, \quad p \neq 1 \ (2 < h < k \leqq n).$$

It can be verified that, for the value of φ given by (2), the expression

$$\frac{1}{\varphi}\frac{d^3\varphi}{dx_1^3}-\frac{d^2\varphi}{dx_1^2}\left(\frac{9}{\varphi^2}\frac{d\varphi}{dx_1}+\frac{2}{x_1\varphi}\right)+\frac{12}{\varphi^3}\left(\frac{d\varphi}{dx_1}\right)^3+\frac{6}{x_1\varphi^2}\left(\frac{d\varphi}{dx_1}\right)^2+ \\ +\frac{4}{x_1^2\varphi}\frac{d\varphi}{dx_1}+\frac{2}{x_1^3}(1+\varphi^2)$$

is equal to zero. Hence it follows from (10) that $C_{hijk,l}=0$. Further, it can be verified from (4) that, for the value of φ given by (2), the condition for a symmetric space is not satisfied. Again from the expression for C_{1221}, i.e.

$$\frac{3-n}{(2-n)(n-1)}\left[-\frac{x_1^2}{\varphi}\frac{d^2\varphi}{dx_1^2}+\frac{3x_1^2}{\varphi^2}\left(\frac{d\varphi}{dx_1}\right)^2+\frac{2x_1}{\varphi}\frac{d\varphi}{dx_1}+1+\varphi^2\right],$$

it can be verified that $C_{1221}\neq 0$ for the value of φ given by (2).

Thus the space with metric (1), where φ is given by (2), is conformally symmetric but is neither symmetric nor conformally flat.

Finally, it may be mentioned that this statement remains valid if instead of (2) the value of φ is given by

(11) $$\varphi^2 = -[1+ax_1+x_1^2(b+c\log x_1)]^{-1},$$

where a, b and c are arbitrary constants.

REFERENCES

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