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FUNCTIONAL CONTINUITY OF COMMUTATIVE m-CONVEX B_0-ALGEBRAS WITH COUNTABLE MAXIMAL IDEAL SPACES

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All algebras in this paper are commutative algebras over the field of complex numbers. A B_0-algebra is a locally convex completely metrizable topological algebra. The topology of such an algebra A can be given by means of an increasing sequence of seminorms

$$\|x\|_1 \leqslant \|x\|_2 \leqslant \ldots \tag{1}$$

for all elements x in A. The requirement of joint continuity of multiplication in a topological algebra means that the seminorms (1) can be chosen so that

$$\|xy\|_i \leqslant \|x\|_{i+1} \|y\|_{i+1} \tag{2}$$

for $i = 1, 2, \ldots$, and all elements x and y in A.

A B_0-algebra is said to be *locally multiplicatively-convex algebra* (shortly an *m-convex algebra*) if instead of (2) we have stronger relations

$$\|xy\|_i \leqslant \|x\|_i \|y\|_i \tag{3}$$

for all elements $x, y \in A$, $i = 1, 2, \ldots$ In case when the algebra in question possesses the unit element e, the seminorms satisfying (1) and (3) can be so chosen that they also satisfy

$$\|e\|_i = 1 \tag{4}$$

for all natural i. The m-convex B_0-algebras are also called *Fréchet algebras.* These algebras were introduced and studied in [1] and [5]. One of basic questions posed in [5] (Question 1, p. 50), considered also by S. Mazur, asks whether in a commutative m-convex B_0-algebra A all its multiplicative-linear functionals are automatically continuous. This problem, still unsolved, called attention of many authors. For an up to day bibliography the reader is referred to the book [3] devoted to this subject. The topological algebras in which all multiplicative-linear functionals must be automatically continuous are called by Michael *functionally continuous.* Since there is a one-to-one correspondence between multiplicative-linear functionals and maximal modular ideals, of codimension one, given by $f \leftrightarrow \ker f$, the problem of functional continuity of m-convex B_0-algebras is equivalent to the question whether in such algebras all such ideals are closed. We denote by $\mathfrak{M}(A)$ the maximal ideal space of A, i.e. the set of all its closed maximal modular ideals, or all its continuous multiplicative-linear functionals provided with the Gelfand topology. We also denote by $\mathfrak{M}^*(A)$ the space of all maximal modular ideals of codimension one (multiplicative-linear functionals) in A. If A is a Q-algebra, i.e. the algebra with open set of invertible elements in case when A possesses the unit element, or open set of quasi-invertible elements in general, then all its maximal modular ideals are closed and of codimension one, provided the algebra in question is a B_0-algebra. Otherwise, as we showed in [6] there are always maximal ideals which are of infinite codimension and dense in A. The strongest result on the problem of functional continuity of m-convex B_0-algebras seems to be the result of Arens [2] stating that if such an algebra is finitely generated (i.e. there is a finite number of elements in A (the system of generators) such that the algebra of all polynomials in these elements is dense) then it is functionally continuous. We shall need this result in the following form:

THEOREM A. *Let A be a commutative complex m-convex B_0-algebra and let $F \in \mathfrak{M}^*(A)$. Then for each finite number of elements $x_1, x_2, \ldots, x_n \in A$ there is a functional f in $\mathfrak{M}(A)$ such that*

$$F(x_i) = f(x_i) \tag{5}$$

for $i = 1, 2, \ldots, n$.

Using this result we prove in this paper that if the maximal ideal space $\mathfrak{M}(A)$ of a commutative m-convex B_0-algebra A is at most countable, then A is a functionally continuous algebra. As a corollary we obtain a result of Husain and Liang [4] stating that commutative m-convex B_0-algebras with orthogonal Schauder bases are functionally continuous. Our result reads as follows.

THEOREM. *Let A be a commutative m-convex B_0-algebra with at most countable maximal ideal space. Then all multiplicative linear functionals in A are continuous.*

Proof. Without loss of generality we can assume that the algebra A possesses the unit element e. Otherwise we could consider the algebra A_1, obtained from A by adjoining the unit e. It is the direct sum $A_1 = A \oplus Ce$ provided with seminorms given by the formula $\|x+\lambda e\|_i = \|x\|_i + |\lambda|$ for $x \in A$ and $\lambda \in C$. All elements in $\mathfrak{M}^*(A)$ extend to elements of $\mathfrak{M}^*(A_1)$ by setting $f(x+\lambda e) = f(x)+\lambda$. Also the cardinality of $\mathfrak{M}(A_1)$ is the same as that of $\mathfrak{M}(A)$. We can also assume that the space $\mathfrak{M}(A)$ is infinite, otherwise, by Theorem 13.6 in [5], A is a Q-algebra and the conclusion follows.

Let then $\mathfrak{M}(A) = \{f_1, f_2, \ldots\}$. We shall construct an element z in A such that

$$f_i(z) \neq f_j(z) \tag{6}$$

for all natural $i \neq j$. To this end observe that for each natural k there is an element $y_k \in A$ such that

$$f_i(y_k) = 0 \tag{7}$$

for $i < k$, and

$$f_k(y_k) = 1 \tag{8}$$

for all indices k. In fact, for a given natural m and n, $m \neq n$, we can find an element x in A such that

$$\alpha = f_m(x) \neq f_n(x) = \beta.$$

Setting

$$y_{m,n} = \frac{x-\alpha e}{\beta - \alpha}$$

we have $f_m(y_{m,n}) = 0$ and $f_n(y_{m,n}) = 1$. Finally setting $y_k = \prod_{l<k} y_{l,k}$ we obtain an element satisfying relations (7) and (8).

We shall construct inductively a sequence $\{z_n\}$ of elements of A satisfying

$$f_k(z_m) = f_k(z_n) \tag{9}$$

for all indices k, m, n satisfying $k \leqslant m \leqslant n$;

$$f_k(z_n) \neq f_l(z_n) \tag{10}$$

for all indices k, l, n satisfying $k, l \leqslant n$;

$$\|z_{n+1} - z_n\|_n \leqslant 2^{-n} \tag{11}$$

for all natural n. To this end we put $z_1 = 0$ and assuming that we have already constructed elements $z_1, z_2, \ldots, z_n$ satisfying relations (9), (10), (11) we construct the element z_{n+1} in the following way. If $f_{n+1}(z_n) \neq f_k(z_n)$ for all $k \leqslant n$ we simply put $z_{n+1} = z_n$. If $f_{n+1}(z_n) = f_k(z_n)$ for some $k \leqslant n$ we put $z_{n+1} = z_n + \alpha y_{n+1}$, where the complex scalar α is chosen so that $f_{n+1}(z_{n+1}) = f_{n+1}(z_n) + \alpha \neq f_i(z_n)$ for all $i \leqslant n$, and $|\alpha| \cdot \|y_{n+1}\|_n \leqslant 2^{-n}$. The relations (4), (7) and (8) now show that we have (9), (10), and (11) for all involved indices not greater than $n+1$. The induction follows. The relation (11) and the completeness of A imply that the sequence $\{z_n\}$ converges in A. Define $z = \lim z_n$. By (9) we have $f_k(z_m) = f_k(z)$ for $k \leqslant m$ and thus the relation (10) implies the relation (6). Assume now in (5) $n = 2$, $x_1 = z$ and $x_2 = x$ – an arbitrary element in A. Relation (5) implies that there exists an index i_0 such that

$$F(z) = f_{i_0}(z) \tag{12}$$

and

$$F(x) = f_{i_0}(x). \tag{13}$$

But relation (12) determines the index i_0 uniquely, since the element z separates between the points of $\mathfrak{M}(A)$. By (13) we have $F = f_{i_0}$, since x was an arbitrary point in A. Conclusion follows.

An m-convex B_0-algebra is said to possess an *orthogonal basis* if there is a Schauder basis $\{e_i\}$ in A such that $e_i e_j = 0$ for $i \neq j$. Thus each element x in A can be written uniquely as $x = \sum_1^\infty \alpha_i(x) e_i$, the series being convergent in A and the coefficients $\alpha_i(x)$ being continuous linear functionals on A. Clearly any such algebra is commutative. Husain and Liang [4], cf. also [3], Theorem 3.47, have shown that every such algebra is functionally continuous. We shall obtain this result as a corollary to our theorem. In fact, let f be a non-zero continuous multiplicative-linear functional on A. There exists an index i_0 such that $f(e_{i_0}) \neq 0$. Otherwise for any element x in A we have

$$f(x) = f\Big(\sum_{i=1}^{\infty} \alpha_i(x) e_i\Big) = \sum_{i=1}^{\infty} \alpha_i(x) f(e_i) = 0$$

and $f = 0$. Also for $i \neq i_0$ we have $0 = f(e_{i_0} e_i) = f(e_{i_0}) f(e_i)$ and so $f(e_i) = 0$. This implies $f(x) = \sum_{i=1}^{\infty} \alpha_i(x) f(e_i) = \alpha_{i_0}(x) f(e_{i_0})$. Thus f is a scalar multiple of α_{i_0}, which implies that the space $\mathfrak{M}(A)$ is at most countable. We obtain

COROLLARY (Husain and Liang). *Let A be an m-convex B_0-algebra with an orthogonal basis. Then A is functionally continuous.*

Remark. The above corollary is formulated in a slightly more general way than the result in [4], since it is assumed there that the basis is unconditional, which is irrelevant in our proof.

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