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ON PROJECTIVENESS IN H-CLOSED SPACES

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This paper is a continuation of [5] and [6], where we have studied extremally disconnected resolutions for an arbitrary T_0-space. Our main object here is the projectiveness of extremally disconnected spaces.

Let $\mathscr{C}$ be a category and $\mathscr{A}$ a class of morphisms of $\mathscr{C}$. An object P of $\mathscr{C}$ is said to be *$\mathscr{A}$-projective* provided, for each $f\colon Y \to X$ from $\mathscr{A}$ and for each $g\colon P \to X$ from $\mathscr{C}$, there exists $\varphi\colon P \to Y$ in $\mathscr{C}$ such that the diagram

(1)
$$\begin{tikzcd} P \arrow[d, "g"'] \arrow[dr, "\varphi"] & \\ X & Y \arrow[l, "f"] \end{tikzcd}$$

commutes. Only the case where morphisms of $\mathscr{A}$ are epimorphisms is non-trivial. If $\mathscr{A}$ is the class of all epimorphisms of $\mathscr{C}$, then $\mathscr{A}$-projectiveness will be abbreviated to projectiveness.

Gleason [11] proved that, in the category of compact spaces and continuous maps, projective objects coincide with extremally disconnected spaces. This result was confirmed also by Rainwater [20] and Hager [12]. In the non-compact case for the class $\mathscr{A}$ there is usually taken the class of all perfect onto maps. There are many results in this direction, e.g., Flachsmeyer [10], Ponomarev [19], Strauss [21] in regular case, Mioduszewski and Rudolf [17] and [18] in completely regular case, and Banaschewski [1] and [2] in Hausdorff case. A categorial description of this situation was given by Banaschewski [2]. Recently, Dyckhoff [7] has given another categorial approach to projectiveness which extends the results of [2].

It will be shown in Theorem 1 of Section 1 that, in the category of all T_0-spaces and continuous maps, extremally disconnected spaces coincide with $\mathscr{A}$-projective spaces, where $\mathscr{A}$ is the class of all perfect separated maps. This is a simultaneous extension of the results of Dyckhoff [7] and of Banaschewski [2].

We shall also discuss the projectiveness in the category of H-closed spaces, which case does fall neither under the Banaschewski nor under the Dyckhoff schemes. It will be shown that here $\mathscr{A}$-projective spaces ($\mathscr{A}$ being the class of all perfect onto maps) coincide with extremally disconnected spaces. The greatest class $\mathscr{A}'$ such that all extremally disconnected spaces remain $\mathscr{A}'$-projective will be distinguished.

We shall show that $\mathscr{A}$-projective objects, $\mathscr{A}$ being the class of all onto maps, in the category of H-closed spaces must be the Katětov extensions of discrete spaces, and that exactly those spaces are $\mathscr{A}$-projective if we restrict the category to compact-like spaces.

A question of Mioduszewski and Rudolf [16] concerning the uniqueness of lifting maps by the Iliadis resolution is answered.

All maps are assumed to be continuous. Topological notions not defined here are as in Engelking [8].

1. Preliminary results. We recall some notions which can be found in [6]. A map $f\colon X \to Y$ is called *r.o.-minimal* (*r.o.* is the abbreviation of *regularly open*) if it is onto and the family

$$\{f^{-1}(U)\cap G\colon\ U \text{ is open in } Y \text{ and } G \text{ is regularly open in } X\}$$

is a base in X; it is called *irreducible* if it is onto and $\operatorname{cl} f(F) \neq Y$ for each regularly closed (shortly, r.c.) subset F, $F \neq X$. It is known [3] that if f is irreducible and r.o.-minimal, then $\operatorname{cl} f(F) \neq Y$ for an arbitrary closed F, $F \neq X$. An *extremally disconnected resolution* (shortly, *e.d. resolution*) is an irreducible r.o.-minimal map of an extremally disconnected (shortly, e.d.) space onto a given one. In [6] we have proved that, for each T_0-space X, there exists among its e.d. resolutions the greatest one, namely the Iliadis e.d. resolution $a\colon aX \to X$, and that a is a perfect separated map (a map is said to be *separated* if distinct points with the same image have disjoint neighbourhoods).

Let Top_0 be the category of all topological T_0-spaces and their continuous maps.

THEOREM 1. *Let $\mathscr{C}$ be a full subcategory of Top_0 such that $aX \in \mathscr{C}$ whenever $X \in \mathscr{C}$, and let $\mathscr{A}$ be a class of all perfect, separated, onto maps of $\mathscr{C}$. For each P from $\mathscr{C}$ the following are equivalent:*

(I) *P is e.d.*;

(II) *P is $\mathscr{A}$-projective in $\mathscr{C}$*;

(III) *each map $f\colon P \to X$, $X \in \mathscr{C}$, can be lifted over aX, i.e., there exists a map $\varphi\colon P \to aX$ such that $f = a\circ\varphi$.*

Proof. 1. The implication (I) $\Rightarrow$ (II) follows by a simple modification of the Gleason proof, where it is shown that, in the compact case, e.d. spaces are projective.

2. The implication (II) $\Rightarrow$ (III) follows by the remark that, by [6], the Iliadis resolution is perfect and separated.

3. (III) $\Rightarrow$ (I). In view of (III), also the identity $i\colon P \to P$ can be lifted over aP, i.e., there exists a map $\varphi\colon P \to aX$ such that $i = a\circ\varphi$. Clearly, φ is an embedding. Since $a(\mathrm{cl}\varphi(P)) = P$ and a is irreducible and r.o.-minimal, $\varphi(P)$ is dense in aP. Thus P is e.d.

Let **HCl** be a full subcategory of $\mathbf{Top}_0$ consisting of all H-closed spaces.

THEOREM 2. *An H-closed space is e.d. iff it is $\mathscr{A}$-projective in* **HCl**, *where $\mathscr{A}$ is the class of all perfect onto maps from* **HCl**.

Proof. Clearly, maps from Hausdorff spaces are separated. It is known that irreducible maps are skeletal (see [16]). In [4] it was shown that preimages of H-closed spaces under perfect skeletal maps are H-closed. The conclusion follows by Theorem 1.

Note. The last assertion cannot be obtained by using the categorial scheme given by Banaschewski [2], since **HCl** is not closed with respect to the "pullback diagrams". Since preimages of locally H-closed spaces under skeletal perfect maps are locally H-closed (see [4]), H-closed spaces in Theorem 2 can be replaced by locally H-closed ones.

It was shown by Henriksen and Jerison [13] that, in the compact case, the lifting map from Theorem 1 is uniquely determined iff

$$\mathrm{Int} f^{-1}(\mathrm{cl}\, U) = \mathrm{Int}\,\mathrm{cl} f^{-1}(U) \tag{2}$$

for each r.o. subset U of X. Maps satisfying (2) are called in [16] the *Henriksen-Jerison maps* (shortly, *HJ-maps*). In [16] Mioduszewski and Rudolf have shown that each HJ-map $f\colon Y \to X$, where X is Hausdorff and Y is e.d. Hausdorff, lifts over aX and such a lifting is unique. They asked if the assumption that f is an HJ-map is essential. The answer is positive:

THEOREM 3. *A map $f\colon Y \to X$, where X and Y are from* $\mathbf{Top}_0$ *and Y is e.d., admits a unique lifting over aX iff it is an HJ-map.*

Proof. 1. The proof of the necessity does not differ, in virtue of Theorem 1, from that for the compact case.

2. To show the converse let us suppose that there exist two different maps $g, h\colon Y \to X$ such that $a\circ g = a\circ h$. There exists a point $x\in Y$ such that $h(x) \neq g(x)$. Since a is separated (cf. [6], Lemma 10), there exists a closed-open set G, $G\subset aX$, which contains exactly one from the points $h(x)$ and $g(x)$. Since a is irreducible, $G = \mathrm{cl}\, a^{-1}(U)$ for a certain r.o. set U, $U\subset X$ (see [16], p. 27). If $g(x)\in \mathrm{cl}\, a^{-1}(U)$, then

$$x\in g^{-1}\big(\mathrm{cl}\, a^{-1}(U)\big) = \mathrm{cl}\,\mathrm{Int}\, g^{-1}\big(\mathrm{cl}\, a^{-1}(U)\big) \subset \mathrm{cl}\,\mathrm{Int} f^{-1}(\mathrm{cl}\, U).$$

Thus, f being an HJ-map, $x\in \mathrm{cl} f^{-1}(\mathrm{Int}\,\mathrm{cl}\, U) = \mathrm{cl} f^{-1}(U)$.

On the other hand, $h(X) \notin \mathrm{cl}\, a^{-1}(U)$. Then $x \notin h^{-1}\big(\mathrm{cl}\, a^{-1}(U)\big)$ and, finally,

$$x \notin \mathrm{cl}\, h^{-1}\big(a^{-1}(U)\big) = \mathrm{cl}\, f^{-1}(U);$$

a contradiction.

2. Projectiveness for H-closed spaces. In the present section we shall be concerned with $\mathscr{A}$-projective objects in the category HCl for $\mathscr{A}$ being a class of epimorphisms which are not necessarily perfect.

Let $P(\mathscr{A}, \mathscr{C})$ be the class of all $\mathscr{A}$-projective objects in the category $\mathscr{C}$. Theorem 2 shows that if $\mathscr{A}$ is the class of all perfect onto maps of HCl, then $P(\mathscr{A}, \mathsf{HCl})$ is equal to the class of all e.d. H-closed spaces. Clearly, if $\mathscr{B} \subset \mathscr{A}$, then $P(\mathscr{A}, \mathscr{C}) \subset P(\mathscr{B}, \mathscr{C})$. In [16] it was shown that no e.d. dense in itself H-closed space belongs to $P(\mathsf{Epi}, \mathsf{HCl})$, where Epi denotes the class of all onto maps of HCl. Our purpose is to find the greatest class $\mathscr{A}'$ of epimorphisms for which the class Ed of all e.d. H-closed spaces equals $P(\mathscr{A}', \mathsf{HCl})$.

THEOREM 4. *The class $\mathscr{A}'$ of epimorphisms of HCl, consisting of maps which admit a restriction to an H-closed subspace being perfect and onto, is the greatest class for which $P(\mathscr{A}', \mathsf{HCl}) = \mathsf{Ed}$.*

This assertion can be deduced from the more general

THEOREM 4′. *Let $\mathscr{C}$ be a full subcategory of Top_0 such that $aX \in \mathscr{C}$ whenever $X \in \mathscr{C}$, a being the Iliadis resolution, and, for each $f\colon X \to Y$ from $\mathscr{C}$, the map $f\colon X \to f(X)$ belongs to $\mathscr{C}$. Let $\mathscr{A}'$ be the class of all separated onto maps from $\mathscr{C}$ which admit a restriction being a perfect onto map from $\mathscr{C}$. Then $\mathscr{A}'$ is the greatest class of separated onto maps of $\mathscr{C}$ such that $P(\mathscr{A}', \mathscr{C})$ equals the class of all e.d. spaces from $\mathscr{C}$.*

Proof. It is easy to see that $P(\mathscr{A}', \mathscr{C}) = P(\mathscr{A}, \mathscr{C})$, $\mathscr{A}$ being the class of all perfect separated onto maps of $\mathscr{C}$. Thus, by Theorem 1, $P(\mathscr{A}', \mathscr{C})$ equals the class of all e.d. spaces of $\mathscr{C}$. Let $\mathscr{B}$ be a class of separated onto maps of $\mathscr{C}$ such that $P(\mathscr{B}, \mathscr{C}) = P(\mathscr{A}, \mathscr{C})$. We show that $\mathscr{B} \subset \mathscr{A}$. Indeed, if $f\colon Y \to X$ belongs to $\mathscr{B}$, then there exists a map $\varphi\colon aX \to Y$ from $\mathscr{C}$ such that $a = f \circ \varphi$, aX being $\mathscr{B}$-projective. Since a is perfect and onto, the map

$$f|\varphi(aX)\colon \varphi(aX) \to Y$$

is perfect and onto (see Engelking [9]). But $f|\varphi(aX) \in \mathscr{C}$, in view of $\varphi(aX) \in \mathscr{C}$. Thus $f \in \mathscr{A}'$, which completes the proof.

Let τZ be the Katětov H-closed extension of Z. It is known that if Z is e.d., then so is τZ.

THEOREM 5. *Projective objects in HCl are either finite or equal to τD, D being a discrete space.*

Proof. Let P be projective in HCl. Since aP is H-closed, there exists a map $s\colon P \to aP$ such that $a \circ s = i$, i being an identity on P. Clearly,

s is an embedding. But $s(P)$ is closed in aP and $a(s(P)) = P$. Thus $aP = P$ topologically, since a: $aP \to P$ is irreducible and r.o.-minimal. Therefore P is e.d.

The space P, being e.d. and Hausdorff, is Urysohn. Hence the topology on P generated by the family of all r.o. sets is compact, i. e., there exists a contraction c: $P \to Y$, where Y is compact. Denote by dY the set Y with the discrete topology, and by d: $dY \to Y$ the natural contraction. Since Y is regular, there exists an extension $\tilde{d}$ of d onto the Katětov extension τdY of dY. But P is projective in HCl, and so there exists a map φ: $P \to \tau dY$ such that the diagram

$$\begin{tikzcd} \tau dY \arrow[d, "\tilde{d}"'] & \\ Y & P \arrow[l, "c"] \arrow[ul, "\varphi"'] \end{tikzcd}$$

commutes. Let X be the set of all accumulation points of P. Since φ is an embedding, we have $\varphi(X) \subset \tau dY \setminus dY$, and since the topology in $\tau dY \setminus dY$ is discrete, X is a discrete subspace of P. Thus X is nowhere dense in P. This means that P is an H-closed extension of a discrete space $D = P \setminus X$ whenever P is not finite.

Since τD is the greatest H-closed extension of D, there exists a map k: $\tau D \to P$ being an identity on D. But P is projective; hence there exists a map l: $P \to \tau D$ such that $k \circ l$ is an identity on P. Clearly, $l|D$ is an identity on D. Thus $P = \tau D$; τD being the greatest H-closed extension of D. This completes the proof.

A subset F of a space X is said to be *regularly embedded* (shortly, *r.ebd.*) if, for each $x \in F$, there exists an open neighbourhood U of x such that $\mathrm{cl}\, U \cap F = \emptyset$. A space is said to be *compact-like* if it is H-closed and Urysohn.

Let us note the following properties of r.ebd. subsets of a given space:

LEMMA 1. *An intersection of a family of r.ebd. subsets is r.ebd.*

LEMMA 2. *A subset of a compact-like space is compact-like iff it is r.ebd.*

LEMMA 3. *If a map f: $X \to Y$ is onto, X is compact-like and Y is Hausdorff, then there exists a subset F of X which is compact-like and such that $f|F$ is irreducible.*

Proof. Let us consider a chain L of r.ebd. subsets of X which are carried onto whole Y. By Lemma 1, $\bigcap L$ is r.ebd. in X. Since Y is Hausdorff, counterimages of points under f are r.ebd. in X. Hence, by Lemmas 1 and 3, the family $\{A \cap f^{-1}(x): A \in L\}$, x being a distinguished point of Y, is a chain of H-closed subsets and, by a theorem of Katětov [15], it has a non-empty intersection. Thus $f(\bigcap L) = Y$. Therefore, by the Kuratowski-Zorn Lemma, in X there exists a minimal (in the sense of inclusion) r.ebd. subset F of X for which $f(F) = Y$.

By Lemma 2, it suffices to show that $f|F$ is irreducible. Suppose that there exists an r.c. subset E of F such that $f(F) = Y$. Since F is H-closed, so is E. Thus, by Lemma 2, E is r.ebd. in X. Hence $E = F$, which completes the proof.

LEMMA 4. *Each irreducible map of an Urysohn space onto an e.d. space is one-to-one.*

The proof of this lemma can be obtained by a simple modification of the proof of Theorem 2 from [3].

LEMMA 5. *If in the pullback diagram*

$$\begin{tikzcd} Z \arrow[d, "g"'] & T \arrow[l] \arrow[d] \\ X & Y \arrow[l, "f"] \end{tikzcd}$$

spaces Y and Z are compact-like and X is Urysohn, then T is compact-like

Proof. To prove this it suffices to show, by Lemma 2, that

$$T = \{(y, z) \in Y \times Z \colon f(y) = g(z)\}$$

is r.ebd. in $Y \times Z$, $Y \times Z$ being compact-like. If $(y, z) \notin T$, then $f(y) \neq g(z)$. Hence, X being Urysohn, there exist open sets U and V containing $f(x)$ and $g(z)$, respectively, such that $\mathrm{cl}\, U \cap \mathrm{cl}\, V = \emptyset$. It is easy to check that $W = f^{-1}(U) \times g^{-1}(V)$ is an open neighbourhood of (y, z), and $T \cap \mathrm{cl}\, W = \emptyset$.

Let CL be a category of all compact-like spaces and continuous maps. We get

THEOREM 6. *The Katětov extensions of discrete spaces are projective in* CL.

Proof. Let $g\colon \tau D \to X$, and let $f\colon Y \to X$ be a map onto, X and Y being compact-like. Clearly, τD is compact-like as an H-closed e.d. space. Let us consider the pullback diagram

$$\begin{tikzcd} \tau D \arrow[d, "g"'] & T \arrow[l, "\varphi"'] \arrow[d, "\varphi"'] \\ X & Y \arrow[l, "f"] \end{tikzcd}$$

By Lemma 5, T is compact-like. Since f is onto, so is φ. Then, by Lemma 3, there exists a compact-like space Z, $Z \subset T$, such that $\varphi|Z$ is irreducible. By Lemma 4, $\varphi|Z$ is one-to-one. Since $\varphi|Z$ is irreducible, isolated points of Z are carried onto D. Let us note that $(\varphi|Z)^{-1}(D)$ is dense in Z. Indeed, $\mathrm{cl}(\varphi|Z)^{-1}(D)$, being r.c. in Z, is carried onto τD. Since $\varphi|Z$ is irreducible, $\mathrm{cl}(\varphi|Z)^{-1}(D) = Z$. Thus Z is an H-closed extension of D and is not less than τD. But since τD is the greatest one, $\varphi|Z$ is a homeomorphism.

It is easy to see that $g = f \circ \psi \circ (\varphi|Z)^{-1}$. Therefore, τD is projective in **CL**.

THEOREM 7. *Projective objects in* **CL** *are exactly those which are either finite spaces or Katětov extensions of discrete spaces.*

Proof. Clearly, finite spaces are projective in **CL**. By Theorem 6, τD for D discrete are also projective in **CL**. The inverse implication was shown, in fact, in the proof of Theorem 5.

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