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A SOLUTION TO THE PROBLEM OF B. V. RAO ON BOREL STRUCTURES

BY

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1. Introduction. Let X be a non-empty set and let $\mathscr{A}$ be a family of subsets of X. By $\sigma(\mathscr{A})$ we denote the least σ-algebra containing $\mathscr{A}$. A σ-algebra $\mathscr{C}$ is called *countably generated* (c.g.) if there is a countable family $\mathscr{A}$ such that $\mathscr{C} = \sigma(\mathscr{A})$. A countably generated σ-algebra of subsets of X is called *separable* if it contains all singletons $\{x\}$ for $x \in X$. If $\mathscr{C}$ is a σ-algebra on X and $Y \subset X$, then $\mathscr{C}(Y) = \{A \cap Y: A \in \mathscr{C}\}$.

B. V. Rao in [5] and K. P. S. Bhaskara Rao and B. V. Rao in [1] present several examples of separable σ-algebras whose intersection is not c.g. As they point out all of these examples are such that the intersection does not contain a separable σ-algebra. In [5] B. V. Rao raised the question (P 687) whether there are separable σ-algebras such that their intersection is not c.g. but contains a separable σ-algebra. In this paper we give a positive answer to this question.

It is worth mentioning that recently Grzegorek ([3], Theorem 0) proved under the Continuum Hypothesis the following strengthening of results of B. V. Rao [5]: If X is a set of cardinality 2^{ω}, then there are separable σ-algebras $\mathscr{C}$ and $\mathscr{D}$ on X such that for every uncountable $Y \subset X$ and every injection $f\colon X \xrightarrow{\text{onto}} X$ the σ-algebra $\mathscr{C}(Y) \cap (f(\mathscr{D}) \cap Y)$ does not contain a separable σ-algebra.

We precede our construction by two easy lemmas.

LEMMA 1. *If $\mathscr{C}$ is a σ-algebra of subsets of X and $Y \subset X$, then*

$$\sigma(\mathscr{C} \cup \{Y\}) = \{(C_1 \cap Y) \cup (C_2 \cap Y^c): C_1, C_2 \in \mathscr{C}\}.$$

For a cardinal number $\varkappa$ we denote by $\mathrm{cf}(\varkappa)$ the cofinality of $\varkappa$.

LEMMA 2. *Let $\mathscr{C} = \bigcup_{\alpha<\varkappa} \mathscr{C}_\alpha$, where $\mathscr{C}_\alpha$ are σ-algebras such that $\alpha < \beta < \varkappa$ implies $\mathscr{C}_\alpha \subset \mathscr{C}_\beta$. If $\mathrm{cf}(\varkappa) > \omega$ and for every $\alpha < \varkappa$ there exists $C \in \mathscr{C} \setminus \mathscr{C}_\alpha$, then $\mathscr{C}$ is a σ-algebra which is not c.g.*

Remark. If $\mathscr{C}$ is as in Lemma 2 and $\mathrm{cf}(\varkappa) = \omega$, then it follows by the Theorem of Broughton and Huff [2] that $\mathscr{C}$ is not a σ-algebra.

2. Example. By I we denote the interval $[0, 1]$. Let $\mathscr{B}$ be a σ-algebra of all Borel subsets of a square I^2. A standard transfinite induction establishes the existence of totally imperfect sets $E, F \subset I$ such that $I = E \cup F$ and $E \cap F = \emptyset$. We define sets

$$P = E \times E \cup F \times F \quad \text{and} \quad R = I \times E$$

and σ-algebras

$$\mathscr{A}_1 = \sigma(\mathscr{B} \cup \{P\}) \quad \text{and} \quad \mathscr{A}_2 = \sigma(\mathscr{B} \cup \{R\}).$$

We shall show that $\mathscr{A}_1$ and $\mathscr{A}_2$ have the desired properties.

PROPOSITION 1. *The σ-algebras $\mathscr{A}_1$, $\mathscr{A}_2$ and $\mathscr{B}$ are separable and*

$$\mathscr{B} \subset \mathscr{A}_1 \cap \mathscr{A}_2.$$

The proof is clear.

PROPOSITION 2. *The σ-algebra $\mathscr{A}_1 \cap \mathscr{A}_2$ is not countably generated.*

If $A \subset I^2$ and $x \in I$, then $A_x = \{y\colon (x, y) \in A\}$. Put

$$\mathscr{J} = \{A \subset I^2\colon \left|\{x \in I\colon |A_x| > \omega\}\right| \leqslant \omega\}.$$

Notice that $\mathscr{J}$ is a σ-ideal.

LEMMA 3. *If $D \in \mathscr{A}_1 \cap \mathscr{A}_2$, then there exists $B \in \mathscr{B}$ such that the symmetric difference $D \,\Delta\, B$ is in $\mathscr{J}$.*

Proof. Let $D \in \mathscr{A}_1 \cap \mathscr{A}_2$. By Lemma 1 there are $B_1, B_2, C_1, C_2 \in \mathscr{B}$ such that

$$D = (B_1 \cap R) \cup (B_2 \cap R^c) = (C_1 \cap P) \cup (C_2 \cap P^c). \tag{1}$$

CLAIM. *$B_i \,\Delta\, C_j \in \mathscr{J}$ for $i, j = 1, 2$.*

Consider the case $i = j = 1$. For $x \in E$, $(B_1)_x \cap E = (C_1)_x \cap E$ since by (1) we have

$$B_1 \cap E \times E = C_1 \cap E \times E.$$

Hence

$$(B_1 \,\Delta\, C_1)_x \cap E = (B_1)_x \,\Delta (C_1)_x \cap E = \emptyset. \tag{2}$$

By the Mazurkiewicz–Sierpiński Theorem [4] the set

$$\{x \in I\colon |(B_1 \,\Delta\, C_1)_x| > \omega\}$$

is analytic and, by (2), disjoint with E, so it must be countable. The proof of the Claim in other cases is similar.

We shall check now that it suffices to put $B = C_1 \cup C_2$. We complete the proof by showing that, for each $i \in \{1, 2, 3, 4\}$, $A_i \cap (D \Delta B) \in \mathcal{J}$, where $A_1 = E \times E$, $A_2 = E \times F$, $A_3 = F \times E$, and $A_4 = F \times F$.

For $i = 1$ we have

$$\begin{aligned} A_1 \cap (D \Delta B) &= (A_1 \cap D) \Delta (A_1 \cap B) = (A_1 \cap D) \Delta [(A_1 \cap C_1) \cup (A_1 \cap C_2)] \\ &\subset [(A_1 \cap D) \Delta (A_1 \cap C_1)] \cup [(A_1 \cap D) \Delta (A_1 \cap C_2)] \\ &= [(A_1 \cap B_1) \Delta (A_1 \cap C_1)] \cup [(A_1 \cap B_1) \Delta (A_1 \cap C_2)] \\ &= [A_1 \cap (B_1 \Delta C_1)] \cup [A_1 \cap (B_1 \Delta C_2)]. \end{aligned}$$

Since $B_1 \Delta C_1 \in \mathcal{J}$ and $B_1 \Delta C_2 \in \mathcal{J}$, we have $A_1 \cap (D \Delta B) \in \mathcal{J}$.

For $i = 2, 3, 4$ the proof is similar.

Proof of Proposition 2. Let us well order the set

$$\mathcal{A}_1 \cap \mathcal{A}_2 \cap \mathcal{J} = \{N_\alpha : \alpha < 2^\omega\}.$$

Observe that

$$\mathcal{A}_1 \cap \mathcal{A}_2 = \bigcup_{\alpha < 2^\omega} \mathcal{C}_\alpha, \quad \text{where } \mathcal{C}_\alpha = \sigma(\mathcal{B} \cup \{N_\beta : \beta < \alpha\}).$$

Clearly, $\mathcal{C}_\alpha \subset \mathcal{A}_1 \cap \mathcal{A}_2$. Let now $D \in \mathcal{A}_1 \cap \mathcal{A}_2$. By Lemma 3 there are $B \in \mathcal{B}$ and $\alpha < 2^\omega$ such that

$$D \Delta B = N_\alpha, \quad D = (D \Delta B) \Delta B = N_\alpha \Delta B \in \mathcal{C}_{\alpha+1}.$$

We complete the proof by showing that $\bigcup_{\alpha < 2^\omega} \mathcal{C}_\alpha$ is not c.g. First notice that, for all $x \in I$, $\{x\} \times E \in \mathcal{A}_1 \cap \mathcal{A}_2$ since

$$\{x\} \times E = \{x\} \times I \cap R$$

and

$$\{x\} \times E = \begin{cases} \{x\} \times I \cap P & \text{for } x \in E, \\ \{x\} \times I \cap P^c & \text{for } x \in F. \end{cases}$$

On the other hand, for every $\alpha < 2^\omega$ there is $x \in I$ such that, for every $\beta < \alpha$, $(N_\beta)_x$ is countable, so $\mathcal{C}_\alpha(\{x\} \times I) = \mathcal{B}(\{x\} \times I)$. Since E is not a Borel set, $\{x\} \times E \notin \mathcal{C}_\alpha$. Thus Lemma 2 completes the proof.

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