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ON THE LATTICE PROPERTIES OF INVARIANT FUNCTIONS FOR MARKOV OPERATORS ON $C(X)$

BY

RYSZARD RĘBOWSKI (WROCŁAW)

0. Introduction. Let $C(X)$ denote the Banach lattice of all real-valued continuous functions on a compact Hausdorff space X. By a *Markov operator* T we mean a linear and positive operator on $C(X)$, i.e., $Tf \geqslant 0$ for $f \geqslant 0$, with $T1 = 1$. A Markov operator T is called *strongly mean ergodic* (s.m.e.) if the Cesàro means

$$A_n f = n^{-1}(f + Tf + \ldots + T^{n-1}f)$$

converge in $C(X)$.

There are many examples of Markov operators for which C_T, the space of all T-invariant continuous functions, is not a sublattice of $C(X)$ (see, e.g., [3], [6]). Furthermore, as pointed out in [3], there exists a Markov operator for which C_T is not even a lattice with the canonical ordering inherited from $C(X)$. The lattice properties of the space C_T have been studied in [1], [3], and [4]. In [3] we have defined the *lattice boundary* $\partial_T(X)$ of the Markov operator T in the case where C_T forms a lattice as the set

$$\partial_T(X) = \bigcap \{x \in X: \ \operatorname{mod} f(x) = |f(x)|\},$$

where $\bigcap$ is the intersection over all f from C_T, and $\operatorname{mod} f$ denotes the lattice modulus of f in C_T. From the Proposition and Corollary 2 in [3] it follows that $\partial_T(X)$ is a closed T-invariant set and is equal to the conservative set provided T is s.m.e.

In the sequel let $\varkappa$ and Q be as in [1]. In [1] it is proved that

$$\partial T := \operatorname{cl}(\{x \in X: \ \varkappa(\delta_x) \in \operatorname{ex} Q\})$$

is a nonempty closed T-invariant subset of X and if, in addition, C_T is a lattice, we have the equality $\partial T = \partial_T(X)$, and therefore the lattice boundary is always nonempty (see Theorems 1 and 2 in [1]).

As in [4] let $\mathscr{D}$ denote the partition of X generated by the level sets of C_T. In $\mathscr{D}$ we distinguish those elements, called *ergodic sets*, which support at least one T-invariant measure. We denote the collection of all ergodic sets by $\mathscr{E}$.

Corollary 3 in [1] says that C_T is a lattice iff every f from $C(\partial T)$ which is constant on the cells of the restricted partition $\mathscr{D}|_{\partial T}$ extends uniquely to some $\tilde{f} \in C_T$. Note that this result does not say what the lattice modulus is.

In this note we continue investigations concerning the lattice properties of C_T. The main purpose is to establish conditions under which the lattice modulus of every $f \in C_T$ is equal to the pointwise limit $\lim A_n|f|(x)$ (Theorem 1). As we shall see, this property means that the lattice boundary has invariant measure one (see (2) in Theorem 1). In Section 2 we explain, in terms of the Sine ergodic decomposition of the Markov operator T, when C_T forms a sublattice of $C(X)$ (Theorem 2). This theorem generalizes a previous result on operators having topological ergodic decomposition, and contains the case of s.m.e. operators (see Theorem 2 and, respectively, Theorem 3 in [3]).

1. When does the lattice boundary have invariant measure one? Recall that by the *center* M of a Markov operator T we mean the closure of the union of the supports of all T-invariant probabilities. Equivalently, we can write

$$M = \bigcap \{x \in X: f(x) = 0\},$$

where the intersection is taken over all continuous functions f for which the Cesàro means $A_n|f|$ converge pointwise to zero (see [5]). It is well known that then M is always a subset of the *conservative set* $W := \bigcup \mathscr{E}$ (see [4]).

Further, we need the following proposition which supplements Theorem 1 in [1].

PROPOSITION. *$\partial T \subset W$. If, in addition, C_T is a lattice, then ∂T is a union of certain invariant cells in $\mathscr{D}$.*

Proof. To prove the first part it suffices to show that

$$\{x \in X: \varkappa(\delta_x) \in \operatorname{ex} Q\} \subset W.$$

First recall that $\varkappa(\delta_x) = \varkappa(\delta_y)$ iff $f(x) = f(y)$ for every $f \in C_T$, so $\mathscr{D}$ is the partition into the level sets of the mapping $x \to \varkappa(\delta_x)$. Now notice that if for certain $D \in \mathscr{D}$

$$D \cap \{x \in X: \varkappa(\delta_x) \in \operatorname{ex} Q\} \neq \emptyset,$$

then, by the definition of $\mathscr{D}$, D must be a subset of $\{x \in X: \varkappa(\delta_x) \in \operatorname{ex} Q\}$. Therefore, if we show that those D are invariant, we will get the inclusion $\partial T \subset W$, since every closed invariant set D carries an invariant measure and W is closed. Let $x \in D$ and $\varkappa(\delta_x) \in \operatorname{ex} Q$. By the argument used in the proof of Theorem 1 in [1] there exists a closed subset $X_x \subset \partial T$ such that $\varkappa^{-1}(\varkappa(\delta_x))$ is the set of all probability measures on X_x. For any y in X we have $y \in X_x$ iff $\varkappa(\delta_y) = \varkappa(\delta_x)$, so in fact $X_x = D$. Since

$$\varkappa(T^* \delta_x) = \varkappa(\delta_x),$$

we have $\operatorname{supp} T^* \delta_x \subset D$, whence D is invariant. If, in addition, C_T is a lattice, then by Remark 1 in [1] the closure in the definition of ∂T can be dropped.

Therefore, the last statement of our Proposition follows immediately from the above proof.

Using the positivity of T it is easy to check that the existence of the continuous pointwise limit $\lim A_n|f|(x)$ for every $f \in C_T$ implies that C_T forms a lattice with the modulus

$$(*) \qquad \operatorname{mod} f(x) = \lim A_n|f|(x), \qquad x \in X.$$

Clearly, on $\partial_T(X)$ we have $\operatorname{mod} f = |f|$. On the other hand, if C_T is a lattice, then by Theorem 2 in [1] we have

$$\partial_T(X) \neq \emptyset \quad \text{and} \quad \operatorname{mod} f = |f| \text{ on } \partial_T(X).$$

The following theorem explains when equation $(*)$ for the lattice modulus holds.

THEOREM 1. *For the Markov operator T the following conditions are equivalent*:

(1) *For every $f \in C_T$ there exists $\tilde{f} \in C_T$ such that $|f| = \tilde{f}$ a.e. for every invariant probability measure.*

(2) *C_T is a lattice and $\partial_T(X)$ has invariant measure one.*

(3) *C_T is a lattice and $\partial_T(X) = W$.*

(4) *For every continuous function f which is constant on each ergodic set there exists $\tilde{f} \in C_T$ such that $f = \tilde{f}$ a.e. for every invariant probability measure.*

(5) *For every $f \in C_T$ there exists a continuous pointwise limit* $\lim A_n|f|(x)$, *which defines the lattice modulus in C_T.*

Proof. To prove $(1) \Rightarrow (2)$ it suffices to show that for $f \in C_T$ the lattice modulus is equal to $\tilde{f}$. First, notice that $\tilde{f} \geqslant |f|$. Indeed, for a fixed point $x \in X$ let (n') denote a subnet along which $A_{n'}^*\delta_x$ converges weak* to some T-invariant measure μ_x. Then, by the positivity of T, we have by assumption

$$\tilde{f}(x) = \int \tilde{f} d\mu_x = \int |f| d\mu_x \geqslant |f(x)|.$$

Now, let $g \in C_T$ with $g \geqslant |f|$. If we take

$$h(x) = \min\{g(x), \tilde{f}(x)\},$$

then $Th \leqslant h$ and

$$h(x) \geqslant \int h d\mu_x = \int \tilde{f} d\mu_x = \tilde{f}(x).$$

This proves that $\tilde{f}$ is the least continuous invariant function majorizing $|f|$, which means that $\operatorname{mod} f = \tilde{f}$.

$(2) \Rightarrow (3)$. Since C_T is a lattice, we have $\partial T = \partial_T(X)$ (see Theorem 2 in [1]) and, by the Proposition, $\partial_T(X) \subset W$. If E is now an ergodic set, then by assumption we have

$$E \cap \partial_T(X) \neq \emptyset,$$

and therefore $E \subset \partial_T(X)$ (see the Proposition).

Implication $(3) \Rightarrow (4)$ is a special case of Corollary 3 in [1].

(4) $\Rightarrow$ (1) follows from the inclusion $M \subset W$ (see [4]).

(5) $\Rightarrow$ (2). Since $\operatorname{mod} f \geqslant |f|$ and $\lim A_n(\operatorname{mod} f - |f|)(x) = 0$ for every $x \in X$, we have $\operatorname{mod} f = |f|$ a.e., i.e., $\mu(\partial_T(X)) = 1$ for every invariant probability measure μ.

(2) $\Rightarrow$ (5). For a fixed x let (n') denote a subnet along which $A_{n'}^{*}\delta_x$ converges to some invariant measure μ_x in the weak* topology. Since, for every $f \in C_T$ and $x \in X$, $\lim T^n|f|(x)$ exists and is equal to $\lim A_n|f|(x)$, we have

$$\lim A_n|f|(x) = \lim A_{n'}|f|(x) = \int |f| d\mu_x = \int \operatorname{mod} f \, d\mu_x = \operatorname{mod} f(x).$$

This concludes the proof of Theorem 1.

We observe that, for T s.m.e., condition (5) is fulfilled trivilly, and therefore in this case $\partial_T(X) = W$ (for another proof of this fact see Corollary 2 in [3]).

Remark. The Example in [3] shows that ∂T need not have invariant measure one even if C_T is a lattice. Consequently, the lattice modulus of $f \in C_T$ need not be the limit of the Cesàro means $A_n|f|$.

2. C_T as a sublattice of $C(X)$. We have noticed that for the Markov operator T the boundary ∂T is the closure of the union of the collection $\mathscr{E}_1$, consisting of all ergodic sets contained in $\{x \in X\colon \varkappa(\delta_x) \in \operatorname{ex} Q\}$. In [3] we have considered an example of a Markov operator for which the class of invariant elements of $\mathscr{E}$ is larger than $\mathscr{E}_1$. Indeed, in that example C_T consists of affine functions on $X \subset [-1, 1]$, $\mathscr{E}$ consists of the invariant sets $\{0\}$, $\{-1\}$, $\{1\}$, but $\partial T = \partial_T(X) = \{-1, 1\}$. On the other hand, if T s.m.e., then every ergodic set is invariant (see [6]), and as we have already observed in [3] we have $\mathscr{E}_1 = \mathscr{E}$.

Now we prove that if, in addition, every level set $D \in \mathscr{D}$ is invariant, then C_T is a sublattice of $C(X)$ and vice versa. In particular, we see in virtue of Theorem 1 in [3] that, for T s.m.e., C_T is a sublattice of $C(X)$ iff the conservative set is the whole space X (for another proof see Theorem 1 and Corollary 2 in [3] and Theorem 9 in [2]). Furthermore, it is worth noting that also Theorem 2 in [3] is a simple consequence of the following

THEOREM 2. *If T is a Markov operator, then C_T is a sublattice of $C(X)$ iff each cell of $\mathscr{D}$ is T-invariant.*

Proof. If C_T is a sublattice, then $\partial_T(X) = X$ and by the Proposition each $D \in \mathscr{D}$ is T-invariant. To see the converse, for every $D \in \mathscr{D}$, $f \in C_T$ and $x \in D$ we have

$$T|f|(x) = \int |f| dT^*\delta_x = \int_D |f| dT^*\delta_x = |f(x)|,$$

since $f = \text{const}$ on D.

Remark. Example 6 in [6] shows that the equality $W = X$ does not imply that each D from $\mathscr{D}$ is invariant.

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INSTITUTE OF MATHEMATICS
TECHNICAL UNIVERSITY OF WROCŁAW
WYBRZEŻE WYSPIAŃSKIEGO 27
50-370 WROCŁAW, POLAND

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