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EXCHANGE PROPERTIES AND BASIS PROPERTIES FOR CLOSURE OPERATORS

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Let $\mathscr{C}$ be a closure operator on a set A. A subset X of A is ($\mathscr{C}$-) *independent* if $x \notin \mathscr{C}(X - \{x\})$ for all $x \in X$; in this case X is a ($\mathscr{C}$-) *basis* for its closure $\mathscr{C}(X)$. It is a rare occasion that a closure operator has what we call the ($\mathscr{C}$-) *basis property*: that any two independent subsets with the same closure have the same cardinality. In the standard instances when this is true, for example for the generating operator $\mathscr{G}$ on a vector space (considered as a universal algebra — see [1]), it is usually a consequence of the well-known *exchange property*, which for a closure operator $\mathscr{C}$ may be stated as:

(1) *If* $y \in \mathscr{C}(X \cup \{x\})$ *and* $y \notin \mathscr{C}(X)$, *then* $x \in \mathscr{C}(X \cup \{y\})$.

However, there are other situations where the basis property holds, for instance when any closed set has a unique basis (this being true for $\mathscr{G}$ on a meet-semilattice, for example), but the exchange property fails. Motivated by the author's study of basis properties in various classes of semigroups we introduce here the following *weak exchange property*:

(2) *If* $\mathscr{C}(Y) = \mathscr{C}(X \cup \{x\})$, *then* $x \in \mathscr{C}(X \cup \{y\})$ *for some* $y \in Y$.

Clearly, the exchange property itself implies that (2) holds for all $y \in Y$, and therefore (1) implies (2).

The purpose of this note is to show that, for an *algebraic* closure operator $\mathscr{C}$ (i.e., such that if $a \in \mathscr{C}(X)$, then $a \in \mathscr{C}(X')$ for some finite subset X' of X), the weak exchange property is equivalent to the *strong basis property*, introduced by the author in [3], and which we now define.

Let $\mathscr{C}$ be any closure operator on the set A and let D be a $\mathscr{C}$-closed subset of A. A subset X of A is *D-independent* (with respect to $\mathscr{C}$) if $x \notin \mathscr{C}(D \cup X - \{x\})$ for all $x \in X$; in that case X is a *D-basis* for its *D-closure* $\mathscr{C}(D \cup X)$ (and is clearly minimal with respect to the property that its D-closure is that closed set).

Definition. The set A has the ($\mathscr{C}$-) *strong basis property* if any two D-independent subsets with the same D-closure have the same cardinality.

Denoting by M the minimum closed subset $\mathscr{C}(\emptyset)$ of A, it is clear that an independent subset of A is just an M-independent set, and the M-closure of any subset is just its ($\mathscr{C}$-) closure. Hence the basis property is a consequence of the strong basis property. As often occurs in mathematics, the strong basis property is in practice more amenable to proof, owing to the "inductive" property inherent in (iii) of the following theorem. Commentary on the theorem will follow. The reader is referred to the survey paper [2], Section 2, for the literature on $\mathscr{C}$-independence and other forms of "independence".

THEOREM. *Let $\mathscr{C}$ be an algebraic closure operator on the set A. The following are equivalent:*

(i) *A has the weak exchange property* (2) *with respect to $\mathscr{C}$.*

(ii) *A has the $\mathscr{C}$-strong basis property.*

(iii) *If $\mathscr{C}(D \cup \{a, b\}) = \mathscr{C}(D \cup \{c\})$ for some $\mathscr{C}$-closed subset D of A and elements a, b, c of A, then either $c \in \mathscr{C}(D \cup \{a\})$ or $c \in \mathscr{C}(D \cup \{b\})$.*

Each is implied by the exchange property (1) *and implies the $\mathscr{C}$-basis property.*

Proof. We prove (i) ⇒ (iii) ⇒ (ii) ⇒ (i). The last statement has already been demonstrated.

(i) ⇒ (iii). Putting $X = D$ and $Y = D \cup \{a, b\}$ in (2) it follows from (i) that either $c \in \mathscr{C}(D \cup \{a\})$ or $c \in \mathscr{C}(D \cup \{b\})$ or $c \in \mathscr{C}(D) \subseteq \mathscr{C}(D \cup \{a\})$.

(iii) ⇒ (ii). (The idea is implicit in the proof of [3], Theorem 2.3.) It is sufficient to show that for any $\mathscr{C}$-closed subset D of A and any subsets X and Y of D for which

$$|X| > |Y| \quad \text{and} \quad \mathscr{C}(D \cup X) = \mathscr{C}(D \cup Y)$$

we have $x \in \mathscr{C}(D \cup X - \{x\})$ for some $x \in X$. Moreover, since $\mathscr{C}$ is algebraic, it is further sufficient to suppose that X and Y are finite (and nonempty, the case $Y = \emptyset$ being trivial). Put

$$X = \{x_1, \ldots, x_m\} \quad \text{and} \quad Y = \{y_1, \ldots, y_k\}.$$

The proof is by induction on k.

Suppose $k = 1$. Set $E = \mathscr{C}(D \cup X - \{x_1, x_2\})$. Then

$$D \subseteq E \subseteq \mathscr{C}(D \cup X),$$

so

$$\mathscr{C}(D \cup X) = \mathscr{C}(D \cup \{y_1\}) \subseteq \mathscr{C}(E \cup \{y_1\}) \subseteq \mathscr{C}(D \cup X)$$

and

$$\mathscr{C}(E \cup \{y_1\}) = \mathscr{C}(D \cup X) = \mathscr{C}(E \cup \{x_1, x_2\}).$$

By (iii), either $y_1 \in \mathscr{C}(E \cup \{x_1\})$, whence

$$x_2 \in \mathscr{C}(E \cup \{x_1\}) = \mathscr{C}(D \cup X - \{x_2\})$$

or

$$y_1 \in \mathscr{C}(E \cup \{x_2\}),$$

whence

$$x_1 \in \mathscr{C}(E \cup \{x_2\}) = \mathscr{C}(D \cup X - \{x_1\}).$$

Next suppose that $k = n \geqslant 2$ and that the result is true for $k < n$. Set

$$F = \mathscr{C}(D \cup Y - \{y_1\}).$$

Then

$$\mathscr{C}(D \cup X) \subseteq \mathscr{C}(F \cup X) \subseteq \mathscr{C}(F \cup Y) = \mathscr{C}(D \cup Y) = \mathscr{C}(F \cup \{y_1\})$$

and

$$\mathscr{C}(F \cup X) = \mathscr{C}(F \cup \{y_1\}).$$

Applying the conclusion of the case $k = 1$ sufficiently many times leads to the conclusion that $y_1 \in \mathscr{C}(F \cup \{x_i\})$ for some i, $1 \leqslant i \leqslant m$. Thus

$$\mathscr{C}(F \cup \{x_i\}) = \mathscr{C}(F \cup \{y_1\}) = \mathscr{C}(D \cup X),$$

which may be rewritten in the form

$$\mathscr{C}(\mathscr{C}(D \cup \{x_i\}) \cup (Y - \{y_1\})) = \mathscr{C}(\mathscr{C}(D \cup \{x_i\}) \cup (X - \{x_i\})).$$

Now $|X - \{x_i\}| = m - 1 > k - 1 = |Y - \{y_1\}|$, so by hypothesis there exists $x \in X - \{x_i\}$ such that

$$x \in \mathscr{C}(\mathscr{C}(D \cup \{x_i\}) \cup (X - \{x_i\})) = \mathscr{C}(D \cup X - \{x\}),$$

as required.

(ii) $\Rightarrow$ (i). Suppose $\mathscr{C}(Y) = \mathscr{C}(X \cup \{x\}) = G$, say, and put $D = \mathscr{C}(X)$. Then

$$\mathscr{C}(D \cup Y) = \mathscr{C}(X \cup \{x\}).$$

Since $\mathscr{C}$ is algebraic, $x \in \mathscr{C}(D \cup Y')$ for some finite subset Y' of Y. By discarding elements from Y', if necessary, it may be assumed that Y' is a D-*basis* for G. If Y' is empty, then $x \in \mathscr{C}(D) = D = \mathscr{C}(X)$. Otherwise, by the strong basis property, $|Y'| = |\{x\}|$, so $Y' = \{y\}$ for some $y \in Y$. Then

$$x \in \mathscr{C}(D \cup \{y\}) = \mathscr{C}(X \cup \{y\}).$$

The example of *meet-semilattices*, mentioned earlier, offers a rather trivial instance of a variety of universal algebras in which the generating operator $\mathscr{G}$ has the strong basis property (D-independent subsets uniquely generate their D-closures for any closed subset, that is, subsemilattice, D), but the exchange property fails, in general. For example, let Y be the meet-semilattice

$$\langle x, z, 0 \mid x \wedge z = 0 \rangle$$

with three elements. Then, putting

$$X = \{z\}, \quad \mathscr{G}(X) = X \quad \text{and} \quad \mathscr{G}(X \cup \{x\}) = Y$$

gives

$$0 \in \mathscr{G}(X \cup \{x\}), \quad 0 \notin \mathscr{G}(X) \quad \text{and} \quad x \notin \mathscr{G}(X \cup \{0\}) = \{z, 0\}.$$

However, the failure of the exchange property in semilattices is mimicked in much larger classes of semigroups and associated types of algebras. Free semigroups and free monoids provide cases in point, in the varieties of semigroups and of monoids, respectively. In these cases "D-bases" are again unique.

Nontrivial instances were studied by the author for inverse semigroups and groups in [3] and [4], and more recently for semigroups and monoids in [5].

An inverse semigroup is a semigroup with an additional unary operation $^{-1}$ satisfying the identities

$$xx^{-1}x = x, \quad (x^{-1})^{-1} = x \quad \text{and} \quad x^{-1}xy^{-1}y = y^{-1}yx^{-1}x.$$

(Alternatively (see [6]), they are von Neumann regular semigroups in which the idempotent elements commute.) Subvarieties include the varieties of groups and of semilattices (where $x^{-1} = x$). In [3] it was shown that every *free* inverse semigroup has the strong basis property (with respect to $\mathscr{G}$). In [4] those inverse semigroups with the strong basis property were found, at least modulo the analogous problem for groups.

In [4] it was also shown that every finite p-group has the strong basis property (generalizing the Burnside Basis Theorem) and that any finite group with the basis property is solvable. There exists a group with 20 elements which has the basis property but not the strong basis property. The reader is referred to [4] for further results.

Various classes of semigroups with the strong basis property (with respect to $\mathscr{G}$) are found in [5].

We conclude with an interesting demonstration of the power of the *strong* basis property (with respect to the generating operator). If an algebra $A = \langle A, F \rangle$ has the strong basis property, then so does $A' = \langle A, F' \rangle$, where F' is obtained from F by the adjunction of a set of nullary operations. For if $D \subseteq A$ is a subalgebra of A', then it is also a subalgebra of A and, conversely, any subalgebra of A containing D is then a subalgebra of A'; thus the D-independent sets are the same in A and A'.

For example, if a monoid (semigroup with identity element) has the strong basis property as a *semigroup*, then it has the strong basis property as a *monoid*.

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