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ON THE RELATIONSHIP BETWEEN MENGER SPACES AND WALD SPACES

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I. Introduction. Menger [2] defined a *statistical metric space* (briefly, an SM-space) to be an ordered pair (S, F), where S is a set and F is a transformation from $S\times S$ into the collection of all left-continuous distribution functions such that if p, q and r belong to S, then

I. $F_{pq}(0) = 0$ (F_{pq} denotes $F(p,q)$);

II. $F_{pq}(0+) = 1$ if and only if $p = q$;

III. $F_{pq} = F_{qp}$; and

IV. if $F_{pq}(x) = 1$ and $F_{qr}(y) = 1$, then $F_{pr}(x+y) = 1$.

A *Menger space* [3] is a triple (S, F, T), where T is a t-function (defined below) and (S, F) satisfies the above-mentioned definition with IV replaced by

IVm. $F_{pr}(x+y) \geqslant T[F_{pq}(x), F_{qr}(y)]$ for all $x \geqslant 0$ and $y \geqslant 0$.

A *t-function* is a function T from the square disc I^2 into I such that if a, b, c and d are elements of I, then

(1) $T(0,0) = 0$ and $T(a,1) = a$;

(2) $T(c,d) \geqslant T(a,b)$ if $c \geqslant a$ and $d \geqslant b$;

(3) $T(a,b) = T(b,a)$; and

(4) $T\big(a, T(b,c)\big) = T\big(T(a,b), c\big)$.

Also of interest in this regard are functions T which satisfy the above-mentioned definition with (4) omitted and "$T(a,1) = a$" weakened to "$T(1,1) = 1$ and $T(a,1) > 0$ if $a > 0$". Such functions will be called *t′-functions*. The t'-functions are partially ordered by the relation T is *weaker* than T' (T' is *stronger* than T) if and only if $T(a,b) \leqslant T'(a,b)$ for all a, b in I^2, and strict inequality holds for at least one pair a, b. Of interest have been the following t'-functions (listed in the increasing order of strength):

$T_1(a,b) = \max(a+b-1, 0)$;

$T_2(a,b) = ab$;

$T_3(a,b) = \min(a,b)$; and

$T_4(a,b) = \max(a,b)$.

T_1, T_2 and T_3 are actually t-functions, T_4 being the strongest of all t-functions.

In [4], Špaček introduced a notion which he called a *random metric.* Although quite different from the notion of an SM-space, random metrics have been related to SM-spaces. For example, Stevens in [5] uses the notion of a random metric to define a class of SM-spaces which he terms the *metrically generated spaces.* He gives a rather complete classification of the metrically generated spaces within the t-function framework of the Menger spaces. It is the purpose of this paper to give a similar classification scheme for another class of SM-spaces, the Wald spaces.

A *Wald space* [7] is an SM-space (S, F) such that if p, q and r belong to S, then

IVw. $F_{pr}(x) \geqslant (F_{pq} * F_{qr})(x)$ for all numbers x.

Schweizer and Sklar [3] prove the following two theorems, which give relationships between Wald spaces and Menger spaces:

THEOREM A. *If (S, F) is a Wald space, then (S, F, T_2) is a Menger space.*

THEOREM B. *If (S, F) is an SM-space such that* IVm *holds for $T = T_4$ and for all triples of distinct points of S, then (S, F) is a Wald space.*

(It was shown by example that this theorem cannot be strengthened by replacing T_4 by the weaker t'-function T_3.)

It will be shown here that Theorem A cannot be strengthened by replacing T_2 with a stronger function (or by a function which is not comparable to T_2) because of

THEOREM 1. *There exists a Wald space (S, F) such that T_2 is the strongest t-function T such that (S, F, T) is a Menger space.*

On the other hand, Theorem B will be strengthened. Let T_w be the t'-function such that

$$T_w(a, b) = \begin{cases} 0 & \text{if } a = 0 \text{ or } b = 0, \\ [\max(a, b)]^2 & \text{if } 0 < a \leqslant 1 \text{ and } 0 < b \leqslant 1. \end{cases}$$

T_w is weaker than T_4, but T_w and T_3 are not comparable.

THEOREM 2. *If (S, F) is an SM-space such that* IVm *holds for $T = T_w$ and for all triples of distinct points of S, then (S, F) is a Wald space.*

Now, Theorem 2 cannot be strengthened by replacing T_w by a weaker t'-function because of

THEOREM 3. *If T' is a t'-function which is weaker than T_w, then there exists an SM-space (S, F) such that* IVm *holds for $T = T'$ and for all triples of distinct points, but such that (S, F) is not a Wald space.*

II. Proofs. Extensive use is made of the techniques and results of Thorp [6].

Proof of Theorem 1. The example constructed in the proof of Theorem 1 of [6] will be used. S is the set of all triples (a, b, n) such that $0 < a < 1$, $0 < b < 1$, and n is in $\{1, 2, 3\}$. The functions F_{pq} are defined as follows:

If p and q are in S and do not agree in the first two terms, then $F_{pq} = J$, where J is the distribution function such that $J(x) = 0$ if $x \leqslant 1$ and $J(x) = 1$ if $x > 1$. If $p = (a, b, 1)$, $q = (a, b, 2)$ and $r = (a, b, 3)$ belong to S, then

$$F_{pq}(x) = \begin{cases} 0 & \text{if } x \leqslant 1/2, \\ a & \text{if } 1/2 < x \leqslant 1, \\ 1 & \text{if } 1 < x, \end{cases} \qquad F_{qr}(x) = \begin{cases} 0 & \text{if } x \leqslant 1/2, \\ b & \text{if } 1/2 < x \leqslant 1, \\ 1 & \text{if } 1 < x, \end{cases}$$

$$F_{pr}(x) = \begin{cases} 0 & \text{if } x \leqslant 1, \\ ab & \text{if } 1 < x \leqslant 3/2, \\ 1 & \text{if } 3/2 < x. \end{cases}$$

Here, and in examples which follow, it will be implied without so stating that F_{pp}, F_{qq}, F_{qp}, etc. are constructed so that (S, F) is an SM-space. It is shown in [6] that T_2 is the strongest t-function T such that (S, F, T) is a Menger space (in fact, it is shown that T_2 is the strongest t'-function T such that IVm holds in this example).

Now, it will be shown that (S, F) is a Wald space. It follows from Lemma 3.1 of [3] that IVw need be verified only for triples of distinct points of S.

First, suppose $p = (a, b, 1)$, $q = (a, b, 2)$ and $r = (a, b, 3)$ belong to S. If G and H are distribution functions such that $G(0) = H(0) = 0$,

$$(G * H)(x) = \int_0^x G(x-y)\,dH(y) \leqslant G(x) \int_0^x 1\,dH(y) = G(x)H(x),$$

and $F_{pq} \geqslant F_{pr} F_{qr}$ and $F_{qr} \geqslant F_{qp} F_{pr}$, so it is only necessary to verify that $F_{pr}(x) \geqslant (F_{pq} * F_{qr})(x)$ for all x. We have

$$(F_{pq} * F_{qr})(x) = \left\{ \begin{matrix} 0 & \text{if } x \leqslant 1, \\ ab & \text{if } 1 < x \leqslant 3/2, \\ a+b-ab & \text{if } 3/2 < x \leqslant 2, \\ 1 & \text{if } 2 < x, \end{matrix} \right\} \leqslant F_{pr}(x) \quad \text{for all } x.$$

Now, suppose p, q and r are elements of S which do not all agree in the first two terms. Assume p and q fail to agree in the first two terms.

Then either p and r or else q and r fail to agree in the first two terms. Assume that q and r fail to agree in the first two terms. Then

$$F_{pr} * F_{rq} = F_{pr} * J \leqslant J \cdot F_{pr} \leqslant J = F_{pq},$$

similarly,

$$F_{qp} * F_{pr} \leqslant F_{qr}.$$

Also,

$$(F_{pq} * F_{qr})(x) = (J * J)(x) = J(x-1) \leqslant F_{pr}(x) \quad \text{for all } x.$$

Thus (S, F) is a Wald space.

Proof of Theorem 2. Suppose (S, F) is an SM-space such that IVm holds for $T = T_w$ and for all triples of distinct points. In showing that (S, F) is a Wald space, IVw need be checked only for triples of distinct points of S, so suppose p, q, r is such a triple. Let

$$x_1 = \max\{x \mid F_{pq}(x) = 0\} \quad \text{and} \quad x_2 = \max\{x \mid F_{qr}(x) = 0\}.$$

Now

$$\int_{-\infty}^{x_1} F_{qr}(x-y)\,dF_{pq}(y) = \int_{x-x_2}^{+\infty} F_{qr}(x-y)\,dF_{pq}(y) = 0,$$

so if $x_1 + x_2 < x$, then it follows that

$$(F_{pq} * F_{qr})(x) = \int_{x_1}^{x-x_2} F_{qr}(x-y)\,dF_{pq}(y) \leqslant F_{qr}(x-x_1)F_{pq}(x-x_2).$$

If $x_1 + x_2 \geqslant x$, then $(F_{pq} * F_{qr})(x) = 0 \leqslant F_{pr}(x)$; so assume $x_1 + x_2 < x$. $F_{pq}(x-x_2) > 0$, so there is a $c > 0$ such that if $0 < t < c$, then

$$F_{pq}(x-x_2-t) > 0 \quad \text{and} \quad F_{qr}(x_2+t) > 0,$$

so that

$$F_{pr}(x) \geqslant \left[\max\left(F_{pq}(x-x_2-t),\, F_{qr}(x_2+t)\right)\right]^2.$$

Since F_{pq} is left continuous, it follows that

$$F_{pr}(x) \geqslant [F_{pq}(x-x_2)]^2.$$

It can be shown in a similar way that

$$F_{pr}(x) \geqslant [F_{qr}(x-x_1)]^2.$$

Thus it follows that

$$F_{pr}(x) \geqslant F_{pq}(x-x_2)F_{qr}(x-x_1) \geqslant (F_{pq} * F_{qr})(x).$$

Remark 1. Theorem 4.2 of [3] states that if (S, F) is *equilateral* (i.e. there is a distribution function G such that if p and q are distinct elements of S, then $F_{pq} = G$), then IVm holds for $T = T_4$ and for all triples of distinct points of S. The converse of this theorem is also true, for suppose (S, F) is such that IVm holds for $T = T_4$ and for all triples of distinct points of S. Let p and q be different points of S, and let $G = F_{pq}$. Then if p' is any other element of S,

$$F_{pq}(x) \geqslant T_4\big(F_{pp'}(x), F_{p'q}(0)\big) = F_{pp'}(x) \quad \text{for all } x.$$

Similarly,

$$F_{pp'}(x) \geqslant F_{pq}(x) \quad \text{for all } x,$$

so $F_{pp'} = F_{pq} = G$.

Furthermore, it can be shown that if q' is an element of S different from p and p', then $F_{p'q'} = F_{pp'} = G$. Therefore, Theorem B just states that every equilateral SM-space is a Wald space.

The following simple example satisfies the hypothesis of Theorem 2 but is not equilateral:

$S = (p, q, r)$, $F_{pq}(x) = 1/2$ if $0 < x \leqslant 1$, and $F_{pq}(x) = 1$ if $x > 1$, $F_{qr} = F_{pq}$ and $F_{pr} = F_{pq}^2$.

On the other hand, the example constructed in the proof of Theorem 1 is a Wald space which does not satisfy the hypothesis of Theorem 2.

Proof of Theorem 3. Suppose T is a t'-function which is weaker than T_w. Let a', b be such that $T(a', b) < T_w(a', b)$. It follows from the definition of T_w that a' and b are both positive. Assume $a' \leqslant b$; let $c = T(a', b)$, and let a be a positive number less than $\min(c^{1/2}, a')$, $T(a, b) \leqslant c < b^2$. Let S contain just three points 1, 2 and 3, and let F be defined as follows:

$$F_{12}(x) = F_{23}(x) = \begin{cases} 0 & \text{if } x \leqslant 0, \\ a & \text{if } 0 < x \leqslant 1, \\ c^{1/2} & \text{if } 1 < x \leqslant 2, \\ b & \text{if } 2 < x \leqslant 3, \\ 1 & \text{if } 3 < x, \end{cases}$$

$$F_{13}(x) = \begin{cases} 0 & \text{if } x \leqslant 0, \\ a^2 & \text{if } 0 < x \leqslant 1, \\ c & \text{if } 1 < x \leqslant 3, \\ 1 & \text{if } 3 < x. \end{cases}$$

It can be verified that IVm is satisfied for all triples of distinct points. In verifying that $F_{13}(x+y) \geqslant T\big(F_{12}(x), F_{23}(y)\big)$ in the ranges $0 < x+y < 1$

and $2 < x+y < 3$, the fact that T is actually weaker than T_w is necessary. However, (S, F) is not a Wald space, because

$$(F_{12}*F_{23})(3) = \int_0^3 F_{12}(3-y)\,dF_{23}(y) = c+2a(b-c^{1/2}) > c = F_{13}(3).$$

Remark 2. Let T^+ be the collection of all t'-functions T' such that if (S, F) is an SM-space for which IVm holds for $T = T'$ and for all triples of distinct points of S, then (S, F) is a Wald space. Theorems 2 and 3 show that T_w is a minimal element of T^+, in that T_w is an element of T^+, but no element of T^+ is weaker than T_w. However, it has not been shown that T_w is the absolute minimum of T^+, in the sense that every other element of T^+ is stronger than T_w. In fact, T^+ has no absolute minimum, because the following is an example of an element of T^+ which is not comparable to T_w:

$$T'(a, b) = \begin{cases} [\max(a, b)]^2 & \text{if } \max(a, b) \leqslant 1/2 \text{ and } \min(a, b) < 1/8, \\ 1/4 & \text{if } \max(a, b) > 1/2 \text{ and } \min(a, b) < 1/8, \\ 1 & \text{if } \min(a, b) \geqslant 1/8. \end{cases}$$

T' is a t'-function, but T' is not comparable with T_w because

$$T'(1, 1/9) < T_w(1, 1/9) \quad \text{and} \quad T'(1/2, 1/2) > T_w(1/2, 1/2).$$

Now, suppose T' does not belong to T^+. Then there exists an SM-space (S, F) such that IVm holds for $T = T'$ and for every triple of distinct points of S, but (S, F) is not a Wald space. Then there exists a triple p, q, r of distinct points of S and a number x such that

$$F_{pr}(x) < (F_{pq}*F_{qr})(x).$$

Assume that $F_{pq}(x) \geqslant F_{qr}(x)$. Then $F_{pq}(x) > 1/2$, otherwise

$$\begin{aligned} F_{pr}(x) &\geqslant T'\big(F_{pq}(x), 0\big) \\ &= [F_{pq}(x)]^2 \geqslant F_{pq}(x)F_{qr}(x) \geqslant \int_0^x F_{pq}(x-y)\,dF_{qr}(y). \end{aligned}$$

Therefore, $F_{pr}(x) \geqslant T'\big(F_{pq}(x), 0\big) = 1/4$. Let

$$x_1 = \max\{y \mid F_{qr}(y) \leqslant 1/8\}.$$

Then $F_{pq}(x-x_1) \leqslant 1/8$, otherwise there is some $y < x-x_1$ such that $F_{pq}(y) > 1/8$, $F_{qr}(x-y) > 1/8$ and $F_{pr}(x) \geqslant T'\big(F_{pq}(y), F_{qr}(x-y)\big) = 1$. So

$$\begin{aligned} &(F_{pq}*F_{qr})(x) \\ &\quad = \int_0^{x-x_1} F_{qr}(x-y)\,dF_{pq}(y) + \int_{x-x_1}^{x} F_{qr}(x-y)\,dF_{pq}(y) \leqslant 1/8+1/8. \end{aligned}$$

Thus

$$1/4 \leqslant F_{pr}(x) < (F_{pq} * F_{qr})(x) \leqslant 1/4,$$

and this is a contradiction.

A simple example of an SM-space (S, F) such that IVm holds for $T = T'$ (but not for $T = T_w$) and for all triples of distinct points of S is the following:

$S = (p, q, r)$, $F_{qr}(x) = 1/4$ if $0 < x \leqslant 1$, $F_{qr}(x) = 1$ if $x > 1$, and $F_{pq}(x) = F_{qr}(x) = F_{pr}^2(2x)$ for all x.

Remark 3. In contrast to Remark 2, T_2 is the absolute maximum of the collection T^- of all t-functions T such that if (S, F) is a Wald space, then (S, F, T) is a Menger space.

Remark 4. In [1], the results of Stevens [5] are extended to a class of SM-spaces which properly contains both the metrically generated spaces and the Wald spaces. Following the motivation behind the definition of a Wald space, a *W-space* is defined to be an SM-space for which there exists a pair (d, P) (called a *stochastic metric* in [1]) such that P is a probability measure and d is a transformation from $S \times S$ into the collection of all P-measurable random variables such that if p, q, r is a triple of points of S,

(i) $d_{pq} \geqslant 0$ almost surely (a.s.) (d_{pq} denotes $d(p, q)$);

(ii) $d_{pq} = 0$ a.s. if and only if $p = q$;

(iii) $d_{pq} = d_{qp}$ a.s.;

(iv) $P(d_{pq} + d_{qr} < x) \leqslant P(d_{pr} < x)$ for all x;

and F_{pq} is the distribution function for d_{pq}.

Triangular inequalities other than (iv) are also investigated in [1]. Among the theorems proved are the following:

(1) T_1 is the absolute maximum of the collection U^- of all t'-functions T such that if (S, F) is a W-space, then (S, F, T) is a Menger space; and

(2) T_3 is a minimal element of the collection U^+ of all t'-functions T' such that if (S, F) is an SM-space such that IVm holds for $T = T'$ and for all triples of distinct points of S, then (S, F) is a W-space.

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