

ON THE EXISTENCE
OF CERTAIN CONFORMALLY RECURRENT METRICS

BY

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1. Introduction. Let M be a Riemannian manifold with a (possibly indefinite) metric g .

A tensor field $T^{i_1 \dots i_p}_{j_1 \dots j_q}$ of type (p, q) on M will be called *recurrent* if

$$(1) \quad T^{h_1 \dots h_p}_{i_1 \dots i_q} T^{i_1 \dots i_p}_{j_1 \dots j_q, k} = T^{h_1 \dots h_p}_{i_1 \dots i_q, k} T^{i_1 \dots i_p}_{j_1 \dots j_q},$$

where the comma denotes covariant differentiation with respect to g .

Relation (1) states that at any point $x \in M$ such that $T(x) \neq 0$ there exists a (unique) covariant vector a (called the *recurrence vector of T*) which satisfies the condition

$$(2) \quad T^{i_1 \dots i_p}_{j_1 \dots j_q, k}(x) = a_k T^{i_1 \dots i_p}_{j_1 \dots j_q}(x).$$

A Riemannian manifold M will be called *recurrent* [13] (*Ricci-recurrent* [7]) if its curvature tensor (Ricci tensor) is recurrent.

According to Adati and Miyazawa [1], an n -dimensional ($n \geq 4$) Riemannian manifold M will be called *conformally recurrent* if its Weyl conformal curvature tensor

$$(3) \quad C_{hijk} = R_{hijk} - \frac{1}{n-2}(g_{ij}R_{hk} - g_{ik}R_{hj} + g_{hk}R_{ij} - g_{hj}R_{ik}) \\ + \frac{R}{(n-1)(n-2)}(g_{hk}g_{ij} - g_{ik}g_{hj})$$

is recurrent.

If $C_{hijk,l} = 0$ everywhere on M and $\dim M \geq 4$, then M is said to be *conformally symmetric* [2].

Clearly, the class of conformally recurrent manifolds contains all conformally symmetric as well as all recurrent manifolds of dimension $n \geq 4$.

A conformally recurrent manifold (M, g) is said to be *simple* [8] (s.c.r. in short) if its metric is locally conformal to a non-conformally flat conformally symmetric one, i.e., if for each point $x \in M$ there exist a neighbourhood U of

x and a function p on U such that $\bar{g} = (\exp 2p)g$ is a non-conformally flat conformally symmetric metric.

Obviously, every non-conformally flat conformally symmetric manifold is necessarily s.c.r. The existence of essentially s.c.r. manifolds, i.e. of s.c.r. manifolds which are neither conformally symmetric nor recurrent, can be established [8] as follows:

THEOREM A. *Let M denote the Euclidean n -space ($n \geq 4$) endowed with the metric g_{ij} given by*

$$(4) \quad \begin{aligned} g_{ij} dx^i dx^j &= Q(dx^1)^2 + k_{\lambda\mu} dx^\lambda dx^\mu + 2dx^1 dx^n, \\ Q &= (Ak_{\lambda\mu} + Bc_{\lambda\mu}) x^\lambda x^\mu, \end{aligned}$$

where $i, j = 1, 2, \dots, n$, $\lambda, \mu = 2, 3, \dots, n-1$, $[k_{\lambda\mu}]$ is a symmetric and non-singular matrix, $[c_{\lambda\mu}]$ is a symmetric and non-zero matrix satisfying $k^{\lambda\mu} c_{\lambda\mu} = 0$ with $[k^{\lambda\mu}] = [k_{\lambda\mu}]^{-1}$, and A, B are functions of x^1 only such that $0 \neq B \neq \text{constant}$ and $A \neq cB$ ($c = \text{constant}$).

Then M is an essentially s.c.r. Ricci-recurrent manifold.

So, the class of s.c.r. manifolds is a natural extension of the class of non-conformally flat conformally symmetric ones.

Investigating s.c.r. manifolds the present author has proved the following results:

THEOREM B ([8], Theorem 1). *A Riemannian manifold M of dimension $n \geq 4$ is s.c.r. if and only if (i) $C_{hijk} \neq 0$ (everywhere on M), (ii) $C_{hijk,l} = a_l C_{hijk}$, (iii) the recurrence vector a_j is locally a gradient, and (iv) the Ricci tensor is a Codazzi one (i.e., $R_{ij,k} = R_{ik,j}$).*

THEOREM C ([8], Theorem 3). *The scalar curvature of a non-locally symmetric s.c.r. manifold vanishes.*

THEOREM D ([8], Theorem 4). *Let M be a non-locally symmetric s.c.r. manifold. Then M admits a unique function F such that*

$$(5) \quad FC_{hijk} = R_{ij} R_{hk} - R_{ik} R_{hj}.$$

F is said to be the *fundamental function* of M . Obviously, $F(x) = 0$ if and only if $\text{rank } R_{ij}(x) \leq 1$.

An analytic conformally recurrent manifold (M, g) will be called *special* if its metric is locally non-trivially conformal to a non-conformally flat conformally recurrent one, i.e., if for each point $x \in M$ there exist a (connected) neighbourhood U of x and a non-constant function p on U such that $\bar{g} = (\exp 2p)g$ is a non-conformally flat conformally recurrent metric.

From the above definitions it follows that every analytic non-conformally symmetric (and therefore each analytic essentially) s.c.r. manifold is necessarily special. The converse statement, as we shall show (Example 1), fails in general. Moreover, we shall construct (Example 2) a non-recurrent

conformally recurrent metric with non-vanishing scalar curvature (cf. Theorem C) which is not non-trivially conformal to any conformally recurrent metric.

The remainder of this paper deals with s.c.r. manifolds. In connection with Theorem D, there arises an interesting question whether there exist essentially s.c.r. manifolds with a constant as well as with a non-constant fundamental function. It will be shown (Example 3) that this existence problem has an affirmative answer. We shall also prove (Theorem 2) that the Ricci tensor of non-locally symmetric s.c.r. manifolds satisfies $\text{rank } R_{ij} \leq 2$, and that every s.c.r. manifold with a metric of index 1 (Theorem 1) is necessarily Ricci-recurrent. Theorem 5 deals with a class of conformally recurrent manifolds which admit on some neighbourhood of each point $x \in M$ satisfying $R_{ij}(x) \neq 0$ a non-zero parallel vector field, and Theorems 3 and 4 involve certain results on the Ricci tensor of non-locally symmetric s.c.r. manifolds. Finally, the last result (Theorem 6) deals with s.c.r. manifolds whose Ricci tensor is not parallel.

Unless stated otherwise, all manifolds under consideration are assumed to be connected and of class C^∞ . Their Riemannian metrics are not assumed to be definite.

2. Preliminaries. In the sequel we need the following results:

LEMMA 1 ([10], Lemma 2). *If a_j and P_{lhmjk} are numbers satisfying*

$$P_{lhmjk} = -P_{lhmkj}, \quad 2a_i P_{lhmjk} + a_j P_{lhmik} + a_k P_{lhmji} = 0,$$

then $a_j = 0$ or $P_{hijkl} = 0$.

LEMMA 2. *The Weyl conformal curvature tensor satisfies the following well-known relations:*

$$(6) \quad \begin{aligned} C_{hijk} &= -C_{ihjk} = -C_{hikj} = C_{jkhi}, \\ C_{hijk} + C_{hjki} + C_{hkij} &= 0, \quad C^r_{ijr} = C^r_{irk} = C^r_{rjk} = 0, \\ C^r_{ijk,r} &= \frac{n-3}{n-2} \left[(R_{ij,k} - R_{ik,j}) - \frac{1}{2(n-1)} (R_{,k} g_{ij} - R_{,j} g_{ik}) \right]. \end{aligned}$$

LEMMA 3 ([9], Theorem 1). *Suppose that M admits two conformally recurrent metrics g and \bar{g} conformally related by $\bar{g} = (\exp 2p)g$.*

Then

$$(a) \quad p_i C_{hijk} + p_j C_{hiki} + p_k C_{hkij} = 0$$

everywhere on M , $p_j = \partial_j p$.

(b) *At each point $x \in M$ such that $C_{hijk}(x) \neq 0$, we have $\bar{a}_j = a_j - 4p_j$ and $\bar{p}^r p_r = 0$, where a_j and \bar{a}_j denote recurrence vectors of C and \bar{C} , respectively.*

LEMMA 4 ([9], Corollary). *Let (M, g) be conformally recurrent and p a function on M . Then $\bar{g} = (\exp 2p)g$ is conformally recurrent if and only if p satisfies condition (a).*

LEMMA 5 ([8], Theorem 2). *Every s.c.r. manifold with a definite metric is locally symmetric.*

LEMMA 6 ([8], Lemma 5). *Let M be an s.c.r. manifold. Then the following two conditions are equivalent: (i) There exist $x \in M$ and exterior 2-forms A and B at x such that $C_{hijk}(x) = A_{hi} B_{jk}$. (ii) $C_{hijk} = e\omega_{hi}\omega_{jk}$, where $|e| = 1$ and ω is a (uniquely determined) recurrent absolute ([12], p. 204) 2-form of rank 2 on M .*

LEMMA 7 ([8], Theorem 5). *If an s.c.r. manifold is not Ricci-recurrent, then it admits a unique recurrent absolute exterior 2-form satisfying*

$$(7) \quad C_{hijk} = e\omega_{hi}\omega_{jk}$$

with $|e| = 1$, $\text{rank } \omega = 2$ and $\omega_{ri}\omega^r_j = 0$.

LEMMA 8 ([8], Proposition 1). *Let M be an s.c.r. manifold. If M is not locally symmetric, then the relation*

$$(8) \quad R_{hl}C_{mijk} - R_{hm}C_{lij}k + R_{il}C_{hmjk} - R_{im}C_{hljk} \\ + R_{jl}C_{himk} - R_{jm}C_{hilk} + R_{kl}C_{hijm} - R_{km}C_{hijl} = 0$$

holds.

LEMMA 9 ([8], Lemma 8). *Let M be a non-locally symmetric s.c.r. manifold such that*

$$d_i C_{hijk} + d_j C_{hiki} + d_k C_{hlij} = 0$$

for some field d_j of non-zero vectors.

If C_{hijk} is not of the form (7), then $d_{i,j} = A_j d_i$ for a certain vector field A_j on M . Moreover, if $d_{i,j} = d_{j,i}$, then $\text{rank } R_{ij} \leq 1$.

LEMMA 10 ([8], Theorem 6). *Every s.c.r. manifold M with non-parallel Ricci tensor satisfies $\text{rank } R_{ij} \leq 2$. Moreover, if M is Ricci-recurrent, then $\text{rank } R_{ij} \leq 1$.*

LEMMA 11. *The Weyl conformal curvature tensor of every s.c.r. manifold satisfies the condition*

$$(9) \quad a_i C_{hijk} + a_j C_{hiki} + a_k C_{hlij} = 0,$$

where a_j is the recurrence vector of C .

The assertion is an immediate consequence of Lemma 3 and the definition of an s.c.r. manifold.

LEMMA 12. *If the Ricci tensor of a non-locally symmetric s.c.r. manifold is of the form*

$$(10) \quad R_{ij} = dd_i d_j,$$

where $|d| = 1$, then the equations

$$(11) \quad d_i C_{hljk} + d_j C_{hlki} + d_k C_{hlij} = 0,$$

$$(12) \quad d_i R_{hljk} + d_j R_{hlki} + d_k R_{hlij} = 0$$

hold.

Proof. Alternating (8) in h, l, m and making use of Lemma 2, we obtain

$$2(R_{il} C_{hmjk} + R_{im} C_{lhjk} + R_{ih} C_{mljk}) + R_{jm} C_{lhik} + R_{jl} C_{hmik} \\ + R_{hj} C_{mlik} + R_{kl} C_{hmji} + R_{km} C_{lhji} + R_{kh} C_{mlji} = 0,$$

which, in view of (10), yields

$$2d_i(d_l C_{hmjk} + d_m C_{lhjk} + d_h C_{mljk}) + d_j(d_l C_{hmik} + d_m C_{lhik} + d_h C_{mlik}) \\ + d_k(d_l C_{hmji} + d_m C_{lhji} + d_h C_{mlji}) = 0.$$

Putting $P_{lhmjk} = d_l C_{hmjk} + d_m C_{lhjk} + d_h C_{mlji}$ and applying Lemma 1, we easily obtain (11).

Relation (12) is an immediate consequence of (11), (10) and Theorem C. This completes the proof.

LEMMA 13 ([8], Proposition 2). *The curvature tensor of every s.c.r. manifold satisfies*

$$R_{hijk,lm} - R_{hijk,ml} = 0.$$

LEMMA 14 ([6], Theorem 1). *Suppose that the Weyl conformal curvature tensor of a Riemannian manifold M ($\dim M \geq 4$) satisfies the condition*

$$(13) \quad C_{hijk,l} = a_l C_{hijk},$$

where the recurrence vector a_j is assumed to be locally a gradient.

If M admits a symmetric parallel tensor h_{ij} (which is not a multiple of g_{ij}) and M is neither conformally flat nor recurrent, then

$$(14) \quad R_{ij} - \frac{1}{n} R g_{ij} = G \left(h_{ij} - \frac{1}{n} h g_{ij} \right)$$

for some function G on M , $h = g^{rs} h_{rs}$.

The following lemma seems to be well known:

LEMMA 15. *Let v^h denote a parallel vector field on M . Then the equations*

$$v_r R^r_{ijk} = 0, \quad v_r R^r_j = 0, \quad v_r R^r_{ijk,l} = 0, \quad v_r R^r_{j,l} = 0, \\ v^r R_{hijk,r} = 0, \quad v^r R_{ij,r} = 0, \quad v^r R_{,r} = 0$$

hold.

LEMMA 16 ([3], Theorem 1). Let M denote the Euclidean n -space ($n \geq 4$) endowed with the indefinite metric g_{ij} defined by

$$(15) \quad g_{ij} = \begin{cases} -2e & \text{if } i = j = 1, \\ \exp F_i & \text{if } i + j = n + 1, \\ 0 & \text{otherwise,} \end{cases}$$

where the functions $F_i = F_{n+1-i}$ are given by $F_1(x, y, \dots) = G(x, y) + A(x)$, $F_2(x, y, \dots) = G(x, y) + B(y)$, $F_\lambda(x, y, \dots) = G(x, y)$ for $\lambda = 3, \dots, n-2$, and $e = \text{constant} \neq 0$.

Then M is conformally recurrent.

LEMMA 17. Let M denote the Euclidean n -space ($n \geq 4$) endowed with metric (15). Then the Ricci tensor of M is a Codazzi tensor.

Proof. The reciprocal g^{ij} of g_{ij} is of the form

$$(16) \quad g^{ij} = \begin{cases} 2e \exp(-2F_1) & \text{if } i = j = n, \\ \exp(-F_i) & \text{if } i + j = n + 1, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, the only components of the Ricci tensor and Weyl conformal curvature tensor, which may not vanish, are those related to [3]:

$$(17) \quad \begin{aligned} R_{11} &= \frac{n-2}{4}(G_x^2 + 2G_x A_x - 2G_{xx}), & R_{12} &= \frac{n-2}{4}(G_x G_y - 2G_{xy}), \\ R_{22} &= \frac{n-2}{4}(G_y^2 + 2G_y B_y - 2G_{yy}), & C_{1212} &= e(G_{yy} - G_y B_y - G_y^2). \end{aligned}$$

As a consequence of (16), we have $g^{11} = g^{12} = g^{22} = 0$, which, in view of (17), implies $R = 0$ and

$$(18) \quad C^1{}_{ijk} = g^{1r} C_{rijk} = 0 = g^{2r} C_{rijk} = C^2{}_{ijk}.$$

Denote by U (if it exists) the open subset of M where C does not vanish. Since the recurrence vector a_j of C is on U the gradient of $f = \log|C_{1212}| - 3G - 2A - 2B$, we get $a_\alpha = f_{,\alpha} = 0$ for $\alpha = 3, \dots, n$, which, together with (18), yields $C^r{}_{ijk,r} = a_r C^r{}_{ijk} = 0$ on U . If now $C = 0$ on some neighbourhood of $x \in M$, then, obviously, we have $C^r{}_{ijk,r}(x) = 0$. Hence, by an elementary limit argument, $C^r{}_{ijk,r} = 0$ everywhere on M . But the last result, in view of (6) and $R = 0$, implies $R_{ij,k} = R_{ik,j}$, which completes the proof.

Let M denote the Euclidean n -space ($n \geq 4$) endowed with the metric g_{ij} given by

$$(19) \quad ds^2 = \sum_{a,b=1}^r g_{ab} dx^a dx^b + \sum_{A,B=r+1}^n g_{AB} dx^A dx^B,$$

where $[g_{ab}]$ and $[g_{AB}]$ are symmetric and non-singular matrices such that

$[g_{ab}]$ is independent of x^{r+1}, \dots, x^n and $[g_{AB}]$ is independent of x^1, \dots, x^r . The two parts of (19) are the metrics of M_r and M_{n-r} , called *decomposition spaces of M* .

Remark 1. It is well known that for the metric g_{ij} not only the Christoffel symbols but also the tensors R_{hijk} , $R_{hijk,l}$ etc. "decompose", i.e., every component with indices from both ranges $1, \dots, r$ and $r+1, \dots, n$ is zero, and every component with indices from only one range, say $1, \dots, r$, is equal to the component of the corresponding symbol or tensor for g_{ab} . Covariant differentiation in decomposition spaces is the same as in M with respect to corresponding coordinates.

3. Non-simple conformally recurrent metrics. We are now in a position to show the existence of certain non-simple conformally recurrent metrics.

Example 1. Let M denote the Euclidean n -space ($n \geq 4$) endowed with the metric g_{ij} , whose only non-zero components are

$$g_{11}(x^1, \dots, x^n) = \frac{1}{2} \sum_{i=2}^{n-1} e_i (x^i)^2 + (n-2) e_{n-1} \exp x^{n-1},$$

$$g_{1n} = g_{n1} = 1, \quad g_{ii} = e_i \quad (i = 2, \dots, n-1), \quad |e_i| = 1.$$

Then M is a non-simple essentially (i.e. neither conformally symmetric nor recurrent) special conformally recurrent Ricci-recurrent manifold whose scalar curvature vanishes. The Weyl conformal curvature tensor does not vanish at any point and the recurrence vectors a_j of C and b_j of R_{ij} are both non-null everywhere.

Proof. It is easy to verify (cf. [11], equ. (44)) that in the above metric the only non-zero components of R_{hijk} , R_{ij} and C_{hijk} are those related to

$$R_{1\lambda\lambda 1} = \frac{1}{2} e_\lambda, \quad R_{1n-1n-11} = \frac{1}{2} e_{n-1} (1 + (n-2) \exp x^{n-1}),$$

$$R_{11} = \frac{n-2}{2} (1 + \exp x^{n-1}), \quad C_{1\lambda\lambda 1} = -\frac{1}{2} e_\lambda \exp x^{n-1},$$

$$C_{1n-1n-11} = \frac{n-3}{2} e_{n-1} \exp x^{n-1}, \quad \lambda = 2, 3, \dots, n-2.$$

It can be also found that

$$R_{1n-1n-11, n-1} = \frac{n-2}{2} e_{n-1} \exp x^{n-1}, \quad R_{11, n-1} = \frac{n-2}{2} \exp x^{n-1},$$

$$C_{1\lambda\lambda 1, n-1} = -\frac{1}{2} e_\lambda \exp x^{n-1}, \quad C_{1n-1n-11, n-1} = \frac{n-3}{2} e_{n-1} \exp x^{n-1},$$

and that all other components are zero.

From the above equations it follows that M is a non-recurrent conformally recurrent manifold whose recurrence vector $a_j = \delta_j^{n-1}$. One can also easily verify that M is Ricci-recurrent and that the recurrence vector b_j is given by $b_j = \partial_j \log(1 + \exp x^{n-1})$. Moreover, both recurrence vectors are non-null vectors everywhere and $R = 0$.

Since $R_{1n-1,1} = 0$ and $R_{11,n-1} \neq 0$, the Ricci tensor is not a Codazzi one. So, by Theorem B, M cannot be s.c.r. On the other hand, it is easy to see that every function $p(x^1)$ of x^1 only satisfies condition (a). Thus, by Lemma 4, M is special. This completes the proof.

Example 1 yields

COROLLARY 1. *For each $n \geq 4$, there exist n -dimensional essentially special conformally recurrent Ricci-recurrent manifolds which are not s.c.r., and whose Weyl conformal curvature tensor does not vanish everywhere.*

Example 2. Let M denote the Euclidean n -space ($n \geq 4$) endowed with a metric g of the form (19), where g_{ab} is 2-dimensional with non-constant scalar curvature Q , and g_{AB} is an $(n-2)$ -dimensional metric of constant curvature such that its scalar curvature S is a non-zero constant.

If $E = Q/2 + S/((n-2)(n-3))$ does not vanish at any point of M , then g is a non-recurrent conformally recurrent metric whose Weyl conformal curvature tensor does not vanish everywhere. Moreover, g has a non-identically vanishing scalar curvature and is not non-trivially conformal to any conformally recurrent metric on M .

Proof. One can easily verify that in the above described metric the only non-zero components of C_{hijk} and $C_{hijk,l}$ are

$$(20) \quad \begin{aligned} C_{abcd} &= \frac{n-3}{n-1} EG_{bcad}, & C_{ABCD} &= \frac{2}{(n-1)(n-2)} EG_{BCAD}, \\ C_{aABb} &= \frac{3-n}{(n-1)(n-2)} Eg_{AB}g_{ab}, & C_{aABb,d} &= \frac{3-n}{2(n-1)(n-2)} Q_{,d} g_{AB}g_{ab}, \\ C_{abcd,e} &= \frac{n-3}{2(n-1)} Q_{,e} G_{bcad}, & C_{ABCD,a} &= \frac{1}{(n-1)(n-2)} Q_{,a} G_{BCAD}, \end{aligned}$$

where $a, b, c, d, e = 1, 2$, $A, B, C, D = 3, \dots, n$, $G_{bcad} = g_{bc}g_{ad} - g_{ac}g_{bd}$, and $G_{BCAD} = g_{BC}g_{AD} - g_{AC}g_{BD}$.

Thus, by (20), g satisfies (13) with a_j given by

$$a_j = \partial_j \log |Q + 2S/((n-2)(n-3))|.$$

Since $S = \text{constant}$ and $Q \neq \text{constant}$ by assumption, the scalar curvature $R (= Q + S)$ does not identically vanish. Moreover, in view of $S \neq 0$, g cannot be recurrent ([13], Theorem 2.1).

Suppose now that \bar{g} is another conformally recurrent metric on M such that $\bar{g} = (\exp 2p)g$ for some function p on M . Then, by Lemma 3, condition

(a) holds. But (a), in view of (20) and $E \neq 0$, yields $\partial_A p = \partial_a p = 0$, which, evidently, completes the proof.

Since $C \neq 0$ everywhere, we have

COROLLARY 2. *For each $n \geq 4$, there exist n -dimensional essentially conformally recurrent metrics with $C \neq 0$ which are not non-trivially conformal to any conformally recurrent metric. Such metrics are never special.*

Clearly, the decomposition metrics in (19) can be chosen to be as in Example 2 and so that g is positive definite.

Hence, by Corollary 2, we have

COROLLARY 3. *For each $n \geq 4$, there exist n -dimensional essentially conformally recurrent manifolds with a positive definite metric.*

4. Simple conformally recurrent manifolds. In the first place we shall show the existence of non-Ricci-recurrent essentially s.c.r. metrics.

Example 3. Let M denote the Euclidean n -space ($n \geq 4$) endowed with metric (15). Define functions A, B and G by

$$(21) \quad \begin{aligned} G(x, y) &= x + \int \exp H dy, & B &= H - 2 \int \exp H dy, \\ A &= -\frac{2}{3} \left(x - \frac{4}{(n-2)^2} \int F dx \right), \end{aligned}$$

where $H = H(y)$ is an arbitrary function of y only, and F is a given constant or a non-constant function of x only. Moreover, let $e = 1$. Then M is essentially s.c.r. and its fundamental function is F .

Proof. In view of (17), we get

$$(22) \quad \begin{aligned} C_{1212} &= \exp 2H, & R_{12} &= \frac{n-2}{4} \exp H, & R_{22} &= \frac{3(2-n)}{4} \exp 2H, \\ R_{11} &= \frac{2-n}{12} \left(1 - \frac{16F}{(n-2)^2} \right), & f &= \int \exp H dy - \frac{1}{3} \left(5x + \frac{16}{(n-2)^2} \int F dx \right). \end{aligned}$$

One can easily verify that $R_{12}^2 - R_{11} R_{22} = F \exp 2H = FC_{1212}$ and that the recurrence vector of C is the gradient of f . Thus, by Lemmas 16 and 17 and Theorems B and D, M is s.c.r. and its fundamental function is F .

Assume $F = 0$ everywhere on M . Then, in view of (21) and (22), we have

$$\begin{aligned} R_{11,1} &= \frac{n-2}{18}, & R_{11,2} &= \frac{2-n}{6} \exp H, & R_{22,1} &= \frac{n-2}{2} \exp 2H & \text{and} & & R_{22,2} \\ &= \frac{3(2-n)}{2} \exp 3H. \end{aligned}$$

Hence, M is Ricci-recurrent whose recurrence vector is the gradient of $h = -\frac{2}{3}x + 2 \int \exp H dy$. Since $h_y = 2 \exp H \neq \exp H = f_y$ and $C_{1212} \neq 0$, M is not recurrent.

Suppose now that F does not identically vanish. Then, by Theorem D, there exists an open subset $U \subset M$ such that $\text{rank } R_{ij} > 1$ on U . On the

other hand, relations $R_{ij,k} = R_{ik,j}$ and $R_{ij,k} \neq 0$ show that the condition $R_{ij,k} = b_k R_{ij}$ implies $\text{rank } R_{ij} = 1$. Thus, M cannot be Ricci-recurrent and, therefore, is not recurrent. Since $f \neq \text{constant}$, M is essentially s.c.r. This completes the proof.

Example 3 shows that there exist essentially s.c.r. manifolds with non-parallel Ricci tensor whose fundamental function does not vanish at any point. Since such manifolds, which follows easily from Lemma 10 and Theorem D, cannot be Ricci-recurrent, we have

COROLLARY 4. *For each $n \geq 4$, there exist n -dimensional essentially s.c.r. manifolds which are not Ricci-recurrent.*

Remark 2. It is easy to prove that for the metric (4) we have

$$\text{index of } [g_{ij}] = \text{index of } [k_{\lambda\mu}] + 1,$$

the index of a symmetric matrix being understood as the number of negative entries in its diagonal form (for the details see Remark 1 of [4]).

Thus, in view of Theorem A and Remark 2, we have

COROLLARY 5. *For each $n \geq 4$, there exist n -dimensional essentially s.c.r. manifolds with metrics of indices from the range $\{1, 2, \dots, n-1\}$. Such manifolds have never definite metrics (Lemma 5).*

Remark 3. If M is s.c.r., then $T = C_{rijk} C^r_{hlm}$ satisfies condition (2) at each point of M . So, if T vanishes at some point of M , then it vanishes everywhere on M . The existence of essentially s.c.r. manifolds satisfying $T \neq 0$ ($T = 0$) can be proved by a similar argument as in [4] (see [4], Lemma 6 and Theorems 2 and 4).

Remark 4. Every s.c.r. manifold with a metric of index 1 (or $(n-1)$) satisfies $T \neq 0$. The proof is similar to that of Theorem 4 of [4] (for the details see Theorems 3 and 4 of [4]).

THEOREM 1. *Every s.c.r. manifold M with a metric of index 1 (or $(n-1)$) is Ricci-recurrent.*

Proof. Suppose that M is not Ricci-recurrent. Then, by Lemma 7, we have $\omega_r C^r_{ijk} = 0$. Transvecting now (7) with C^h_{lmq} and using the last result, we obtain $T = 0$, a contradiction (Remark 4). This completes the proof.

THEOREM 2. *The Ricci tensor of every non-locally symmetric s.c.r. manifold M satisfies $\text{rank } R_{ij} \leq 2$. Moreover, if the Weyl conformal curvature tensor is not of the form (7), then $\text{rank } R_{ij} \leq 1$.*

Proof. Assume (7) and let $x \in M$. If $R_{ij}(x) \neq 0$, we may choose a vector u^i at x such that $u^i u^j R_{ij} = d$, $|d| = 1$. Then, by (7), we have $FC_{hijk} u^h u^k = -eFw_i w_j$, where $w_j = u^r \omega_{rj}$ and $(R_{ij} R_{hk} - R_{ik} R_{hj}) u^h u^k = dR_{ij} - d_i d_j$, where $d_j = u^r R_{rj}$. Hence, by (5), $R_{ij} = dd_i d_j - edFw_i w_j$, which shows that $\text{rank } R_{ij} \leq 2$.

Suppose now that (7) does not hold. Then, by Lemma 7, M is Ricci-

recurrent. If the Ricci tensor is not parallel, then the assertion is an immediate consequence of Lemma 10. Assume therefore $R_{ij,k} = 0$. Since M is not locally symmetric by assumption, M cannot be conformally symmetric. Hence, there exists a point $y \in M$ such that the recurrence vector a_j of C does not vanish on some neighbourhood U of y . By Lemmas 6 and 11, the assumptions of Lemma 9 are satisfied. Thus, because of $a_{i,j} = a_{j,i}$, $\text{rank } R_{ij} \leq 1$ on U . Since R_{ij} is parallel, $\text{rank } R_{ij} \leq 1$ extends to the whole of M . This completes the proof.

THEOREM 3. *The Ricci tensor of a non-locally symmetric s.c.r. manifold satisfies*

$$R_{im} C_{hljk} + R_{jm} C_{hlki} + R_{km} C_{hlij} = 0.$$

The above result has been proved for essentially conformally symmetric manifolds, i.e., for conformally symmetric manifolds which are neither conformally flat nor locally symmetric. But its proof, as one can easily verify ([5], Theorem 7) requires only Theorems D and 2 and Lemma 12. So, Theorem 3 remains also true for non-locally symmetric s.c.r. manifolds.

THEOREM 4. *Let M be a non-locally symmetric s.c.r. manifold. Then at each point $x \in M$ such that $R_{ij}(x) \neq 0$ we have a relation of the form*

$$(23) \quad R_{hijk} = R_{ij} B_{hk} + R_{hk} B_{ij} - R_{ik} B_{hj} - R_{hj} B_{ik}$$

for some symmetric tensor B_{ij} at x .

Proof. By Theorems 2 and D, we have two cases. If $\text{rank } R_{ij}(x) = 1$, say $R_{ij} = dd_i d_j$, where $|d| = 1$, then, by Lemma 12, equation (12) holds.

Now, with help of (12), we can follow step by step a proof of Walker (see [13], p. 45) to obtain

$$R_{hijk} = d_i d_j D_{hk} + d_h d_k D_{ij} - d_i d_k D_{hj} - d_h d_j D_{ik},$$

where $D_{ij} = D_{ji} = v^r v^s R_{rijs}$ and v^i is chosen so that $v^r d_r = 1$.

The last equation leads immediately to (23). Assume now $\text{rank } R_{ij}(x) = 2$. In this case relation (23) with $B_{ij} = \frac{1}{2F} R_{ij} + \frac{1}{n-2} g_{ij}$ is a consequence of (3), (5) and Theorem C. This completes the proof.

THEOREM 5. *Let M ($\dim M \geq 4$) be a non-conformally flat Riemannian manifold whose Weyl conformal curvature tensor satisfies (13) with a locally gradient recurrence vector. If for each point $x \in M$ satisfying $R_{ij}(x) \neq 0$ there exists a non-trivial parallel vector field on some neighbourhood of x , then M is Ricci-recurrent.*

Proof. Let $x \in M$ be such that $R_{ij}(x) \neq 0$. Denote by v_i a non-trivial parallel vector field on some (connected) neighbourhood U of x . Setting $h_{ij} = v_i v_j$ we obtain a symmetric parallel tensor on U which is not a multiple of g_{ij} . Since C cannot vanish on U , either the manifold U is recurrent (and

therefore Ricci-recurrent) or it satisfies the assumptions of Lemma 14. Hence, by Lemmas 14 and 15, equation (14) yields

$$(24) \quad R = (1 - n)hG.$$

On the other hand, differentiating (14) covariantly, we get

$$(25) \quad R_{ij,k} - \frac{1}{n}R_{,k}g_{ij} = G_{,k} \left(h_{ij} - \frac{1}{n}hg_{ij} \right),$$

which, together with Lemma 15, implies $v^k G_{,k} = 0$. Contracting now (25) with g^{ik} and taking into account $v^r G_{,r} = 0$, we easily obtain

$$(26) \quad (n-2)R_{,j} = -2hG_{,j}.$$

Hence, by (24) and (26), we have $hG_{,j} = 0$. Suppose that the constant $h = 0$. Then, in view of (24), we find $R = 0$. But the last result reduces (14) to the form $R_{ij} = Gh_{ij}$, which shows that U is Ricci-recurrent. If now $G = \text{constant}$ on U , then (26) implies $R = \text{constant}$. Equation (25) can therefore be written as $R_{ij,k} = 0$. Thus, U is Ricci-recurrent, which, evidently, completes the proof.

THEOREM 6. *Let M be an s.c.r. manifold with non-parallel Ricci tensor. Then the following two conditions are equivalent: (i) M is Ricci-recurrent. (ii) For each point x of M which satisfies $R_{ij}(x) \neq 0$, there exists a non-trivial parallel vector field on some neighbourhood of x .*

Proof. Let M be Ricci-recurrent and $x \in M$ be such that the condition $R_{ij}(x) \neq 0$ holds. Then, by Lemma 10, $R_{ij} = dd_i d_j \neq 0$ ($|d| = 1$) on some neighbourhood of x . Using (2) and (10), we get $d_i(d_{j,k} - \frac{1}{2}b_k d_j) + d_j(d_{i,k} - \frac{1}{2}b_k d_i) = 0$, b_j being the recurrence vector of R_{ij} . But the last equation implies

$$(27) \quad d_{i,j} = \frac{1}{2}b_j d_i.$$

On the other hand, Lemma 13 yields $0 = R_{ij,kl} - R_{ij,lk} = (b_{k,l} - b_{l,k})R_{ij}$. Hence, $b_j = b_{,j}$ for some function b . Now, let $D_j = (\exp(-\frac{1}{2}b))d_j$. Taking into account (27) we can easily verify that D_j is parallel. Since $d_j \neq 0$, D_j is not trivial.

The implication (ii) \Rightarrow (i) follows immediately from Theorem 5. This completes the proof.

Remark 5. Let M denote the Euclidean n -space ($n \geq 4$) endowed with metric (4), where B , $[k_{\lambda\mu}]$ and $[c_{\lambda\mu}]$ are such as in Theorem A, and $A = 0$. Then M is a non-locally symmetric s.c.r. manifold whose Ricci tensor vanishes. Hence, there exist non-locally symmetric s.c.r. manifolds with vanishing Ricci tensor. Such manifolds are necessarily recurrent. The considered in Theorem 4 set of points x satisfying $R_{ij}(x) \neq 0$ can therefore be

empty. On the other hand, there exist non-locally symmetric s.c.r. manifolds which are recurrent and whose Ricci tensor does not vanish. Examples can be easily obtained from metric (4).

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