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STRICTLY BALANCED SUBMATROIDS IN RANDOM SUBSETS OF PROJECTIVE GEOMETRIES

BY

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1. Introduction. A random submatroid ω_r of the projective geometry $\mathrm{PG}(r-1, q)$ is obtained from $\mathrm{PG}(r-1, q)$ by deleting elements so that each element has, independently of all other elements, probability $1-p$ of being deleted and probability p of being retained. A recent paper of Oxley [4] gives a matroidal counterpart of several Bollobás' results [2]. In this paper we extend some Oxley's results using the method of Poisson convergence. Our proofs are very similar to those presented by Karoński [3].

In general, we shall follow Welsh [5] for all unexplained matroid terminology.

We shall consider projective geometries over a finite field $\mathrm{GF}(q)$ which we take as fixed. For $k = 1, 2, \ldots, r$ let

$$[r]_k = (q^r-1)(q^{r-1}-1)\ldots(q^{r-k-1}-1) \quad \text{and} \quad [r]_0 = 1.$$

Then $\begin{bmatrix} r \\ k \end{bmatrix}$, the number of rank-$k$-subspaces of $\mathrm{PG}(r-1, q)$, equals $[r]_k/[k]_k$.

The following estimation (see [4]) is useful:

$$q^{kr-\binom{k}{2}} \geqslant [r]_k \geqslant \beta q^{kr-\binom{k}{2}}, \tag{1}$$

where

$$0 < \beta = \prod_{n=1}^{\infty} (1-q^{-n}) < 1.$$

In the next parts we consider only simple matroids representable over $\mathrm{GF}(q)$, where q is fixed. If $\mathcal{M}$ is a family of matroids, then by an *$\mathcal{M}$-matroid* we mean a matroid isomorphic to a matroid from $\mathcal{M}$. By $b(\mathcal{M})$ we denote the number of all $\mathcal{M}$-matroids each of which has rank k and is a submatroid of $\mathrm{PG}(k-1, q)$.

Let X_n take values only from $Z^+ = \{0, 1, \ldots\}$. We write

$$X_n \rightsquigarrow \mathrm{Po}(\lambda) \quad \text{or} \quad \tilde{X}_n \rightsquigarrow N(0, 1)$$

when $\{X_n\}$ or $\{\tilde{X}_n\}$ converges in distribution to a Poisson distribution with expectation λ or to a standard normal distribution, respectively.

Let $\mathrm{Po}(\lambda, A)$ denote the probability that a Poisson random variable with expectation λ takes values in $A \subset Z^+$.

We say that $\{X_n\}$ is *Poisson convergent* if

$$\sup_{A \subset Z^+} |\Pr(X_n \in A) - \mathrm{Po}(\lambda_n, A)| \to 0. \tag{2}$$

Note that the Poisson convergence of $\{X_n\}$ implies that

$$X_n \rightsquigarrow \mathrm{Po}(\lambda) \quad \text{if } \mathrm{E}X_n \to \lambda,\ 0 < \lambda < \infty,$$

and

$$\tilde{X}_n \rightsquigarrow N(0, 1) \quad \text{if } \mathrm{E}X_n \to \infty,$$

where $\tilde{X}_n = (X_n - \lambda_n)/\lambda_n^{-1/2}$.

For details we refer the reader to [1].

2. Balanced matroids. Suppose that M is a matroid which has m elements and rank k and define the density of M as $d(M) = m/k$. Let

$$m_t(M) = \max\{d(F):\ F \subset M,\ \varrho F = t\}$$

for $t = 1, 2, \ldots, k$, where $F \subset M$ means that F is a proper submatroid of M.

Suppose that $\mathcal{M}$ is a family of matroids and $1 \leqslant t \leqslant \varrho\mathcal{M}$, where

$$\varrho\mathcal{M} = \max\{\varrho M:\ M \in \mathcal{M}\}.$$

Let

$$\varepsilon_t = \min_{\substack{M \in \mathcal{M} \\ \varrho M = t}} [d(M) - m_t(M)].$$

The balance index of $\mathcal{M}$ is defined as

$$\varepsilon(\mathcal{M}) = \min_{1 \leqslant t \leqslant \varrho\mathcal{M}} t\varepsilon_t. \tag{3}$$

If $\mathcal{M} = \{M\}$, we write simply $\varepsilon(M)$. A matroid M is *balanced* if $\varepsilon(M) \geqslant 0$, and is *strictly balanced* if $\varepsilon(M) > 0$.

The following lemma is similar to Lemma 2.1 in [3].

LEMMA. *Let $\mathcal{B}$ be a family of balanced matroids each of which has m elements and rank k. Suppose that $M_1, \ldots, M_n$ are $\mathcal{B}$-matroids and for at least one pair M_i, M_j we have*

$$\sigma M_i \cap \sigma M_j \neq \emptyset \quad \textit{and} \quad F_n = M_1 \cup \ldots \cup M_n.$$

Then

$$|F_n| \geqslant \frac{m}{k}\varrho F_n + \varepsilon(\mathcal{B}). \tag{4}$$

Proof (by induction). The case $n = 2$. Let H have m elements and rank k and let M be a $\mathscr{B}$-matroid, $|\sigma H \cap \sigma M| = v > 0$, $|M \cap H| = u$. Since M is balanced, $u \leqslant mv/k - \varepsilon(\mathscr{B})$. Then

$$|M \cup H| \geqslant m + |H| - mv/k + \varepsilon(\mathscr{B}). \tag{5}$$

Substituting $H = M_1$ and $M = M_2$ we obtain

$$|F_2| \geqslant \frac{m}{k} \varrho F_2 + \varepsilon(\mathscr{B}).$$

For $n \geqslant 3$ we substitute $H = F_{n-1}$ and $M = M_n$ in (5) under the assumption that $v > 0$. Then

$$|F_n| \geqslant \frac{m}{k} \{|F_{n-1}| + k - \varrho(F_{n-1} \cap M_n) + \varepsilon(\mathscr{B})\} = \frac{m}{k} \varrho F_n + \varepsilon(\mathscr{B}),$$

which gives (4) and completes the proof.

The above Lemma will be used in this note only for the case $n = 2$.

3. Submatroid counts. Let $X_r = X_r(\mathscr{M})$ denote the number of $\mathscr{M}$-matroids in a random ω_r.

THEOREM 1. *If $\mathscr{B}$ is a family of balanced simple matroids each of which has m elements and rank k and is representable over* GF(q), *then*

$$\sup_{A \subset Z^+} |\Pr(X_r(\mathscr{B}) \in A) - \mathrm{Po}(\alpha^*(r), A)| \tag{6}$$
$$\leqslant 2p^m + \frac{4}{\alpha^*(r)} \sum_{k \leqslant l \leqslant 2k-1} \begin{bmatrix} r \\ l \end{bmatrix} \left\{ \begin{bmatrix} l \\ k \end{bmatrix} b(\mathscr{B}) \right\}^2 p^{lm/k + \varepsilon(\mathscr{B})},$$

where

$$\alpha^*(n) = \begin{bmatrix} r \\ k \end{bmatrix} b(\mathscr{B}) p^m.$$

Proof. By $\mathscr{B}^*$ we denote the set of all $\mathscr{B}$-matroids in PG$(r-1, q)$. For $M \in \mathscr{B}^*$ we define

$$X_M = \begin{cases} 1 & \text{if } M \in \omega_r, \\ 0 & \text{otherwise.} \end{cases}$$

Hence $X_r = \sum_{M \in \mathscr{B}^*} X_M$, $\Pr(X_M = 1) = p^m$ and $\mathrm{E}X_r = \alpha^*(r)$. Let

$$X_{r,M} = X_{r,M}(\mathscr{B}) = \sum_{\substack{F \in \mathscr{B}^* \\ F \cap M = \emptyset}} X_F$$

be a random variable counting $\mathscr{B}$-matroids in PG$(r-1, q)$ which are disjoint with a given $\mathscr{B}$-matroid M. Then as in [1], p. 352, and [3], p. 22, we obtain

$$\sup_{A \subset Z^+} |\Pr(X_r \in A) - \mathrm{Po}(\alpha^*(r), A)| \leqslant 2p^m + \frac{4}{\alpha^*(r)} \sum_{\substack{M, F \in \mathscr{B}^* \\ F \cap M = \emptyset, F \neq M}} \mathrm{E}(X_F X_M). \tag{7}$$

Note that

$$\sum_{\substack{M,F\in\mathscr{B}^* \\ F\cap M\neq\emptyset, F\neq M}} \mathrm{E}(X_F X_M) \leqslant \sum_{k\leqslant l\leqslant 2k-1} \begin{bmatrix} r \\ l \end{bmatrix} \left\{ \begin{bmatrix} l \\ k \end{bmatrix} b(\mathscr{B}) \right\}^2 p^u,$$

where u stands for a lower bound for the number of elements in the union of matroids M and F, and l is the rank of $\sigma M \cap \sigma F$. By the Lemma we have $u \geqslant lm/k+\varepsilon(\mathscr{B})$ and we obtain the inequality

$$\sum_{\substack{M,F\in\mathscr{B}^* \\ F\cap M\neq\emptyset, F\neq M}} \mathrm{E}(X_F X_M) \leqslant \sum_{k\leqslant l\leqslant 2k-1} \begin{bmatrix} r \\ l \end{bmatrix} \left\{ \begin{bmatrix} l \\ k \end{bmatrix} b(\mathscr{B}) \right\}^2 p^{lm/k+\varepsilon(\mathscr{B})}.$$

Hence, we obtain the assertion.

To prove that the family of random variables $\{X_r\}$ is Poisson convergent one has to find when the right-hand side of inequality (7) tends to zero as $r\to\infty$.

We write

$$\alpha(r) = p^m q^{rk} b(\mathscr{S})([k]_k q^{\binom{k}{2}})^{-1},$$

$$\eta = \max\{1,\, b(\mathscr{S})([k]_k q^{\binom{k}{2}})^{-1}\},$$

where and in the sequel $\mathscr{S}$ denotes a family of strictly balanced simple matroids each of which has m elements and rank k and is representable over $\mathrm{GF}(q)$.

THEOREM 2. *If* $\alpha(r)\sim\lambda$, *where* λ *is some positive constant, and*

$$\eta q^{4k^2 - r\varepsilon(\mathscr{S})/m} = o(1),$$

then

$$X_r(\mathscr{S}) \rightsquigarrow \mathrm{Po}(\lambda).$$

If $\alpha(r)\to\infty$ *and*

$$\alpha(r) = o(\eta^{-1} q^{-4k^2 + r\varepsilon(\mathscr{S})/m}),$$

then

$$\tilde{X}_r(\mathscr{S}) \rightsquigarrow N(0,\, 1).$$

Proof. To prove this theorem we show that under suitable assumptions the right-hand side of (6) tends to zero. Note that using (1) we obtain

$$\begin{bmatrix} r \\ l \end{bmatrix} \left\{ \begin{bmatrix} l \\ k \end{bmatrix} b(\mathscr{S}) \right\}^2 \{\alpha^*(r)\}^{-1} p^{lm/k+\varepsilon(\mathscr{S})}$$

$$\leqslant q^{2lk-\binom{l}{2}} \{\beta^{[l]_l}\}^{-1} \eta \{\alpha(r)\}^{1-1/k-\varepsilon(\mathscr{S})/m} q^{-r\varepsilon(\mathscr{S})/m}$$

$$\leqslant \beta^{-1} q^{4k^2 - r\varepsilon(\mathscr{S})/m} \eta \{\alpha(r)\}^{1-1/k-\varepsilon(\mathscr{S})/m}.$$

Therefore

$$2p^m + \frac{4}{\alpha^*(r)} \sum_{k \leqslant l \leqslant 2k-1} \begin{bmatrix} r \\ l \end{bmatrix} \left\{ \begin{bmatrix} l \\ k \end{bmatrix} b(\mathscr{S}) \right\}^2 p^{lm/k + \varepsilon(\mathscr{S})} = 2p^m + \Theta,$$

where

$$\Theta = O(q^{4k^2 - r\varepsilon(\mathscr{S})/m} \eta)$$

for $\alpha(r) \to \lambda > 0$ and

$$\Theta = O\left(q^{4k^2 - r\varepsilon(\mathscr{S})/m} \eta \{\alpha(r)\}^{1-1/k-\varepsilon(\mathscr{S})/m}\right)$$

for $\alpha(r) \to \infty$. Hence both Θ and p^m tend to zero as $r \to \infty$, which completes the proof.

Theorem 1 is a matroid counterpart of Theorem 2.7 in [3]. The first statement of Theorem 2 is an extension of Theorems 3.1 and 3.12 in [4]. We obtain these theorems from Theorem 2 if $p \sim cq^{-rk/m}$, where c is an arbitrary positive constant and k and m are fixed or $\alpha(r) \sim \lambda$ for some positive constant λ, and $km^2 = o(r)$. This theorem is also a matroid counterpart of Theorems 2.8–2.10 in [3].

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