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ON ELEMENTARY EQUIVALENCE OF ABELIAN GROUPS

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In this paper necessary and sufficient conditions are given for two abelian groups of finite exponent to be elementarily equivalent, and for two divisible abelian groups to be elementarily equivalent. For each reduced abelian group $\mathfrak{A}$, a countable reduced abelian group is exhibited which is elementarily equivalent to $\mathfrak{A}$, and for two abelian groups $\mathfrak{A}$ and $\mathfrak{B}$, necessary and sufficient conditions are given in terms of the reduced and divisible factors of these groups such that $\mathfrak{A}$ and $\mathfrak{B}$ be elementarily equivalent.

Throughout, German type letters will denote abelian groups, and the corresponding Roman letters — the universes of these groups. For a set I and groups $\mathfrak{A}_i$ with $i \in I$, $\prod_{i \in I} \mathfrak{A}_i$ will denote the Cartesian product of the $\mathfrak{A}_i$'s and $\sum_{i \in I} \mathfrak{A}_i$ will denote the direct sum; $\mathfrak{A}^I$ will denote $\prod_{i \in I} \mathfrak{A}$, and $\mathfrak{A}^{(I)}$ will denote $\sum_{i \in I} \mathfrak{A}$. If m_i is a natural number for $i \in I$, $\sum_{i \in I} m_i$ will denote the usual sum; if this sum is not finite, we write $\sum_{i \in I} m_i = \omega$.

This dual usage of the symbol $\sum$ should cause the reader no difficulty as its meaning will always be clear from the context. If $\mathscr{F}$ is an ultrafilter on I, then $\mathfrak{A}^I/\mathscr{F}$ will denote the ultrapower of $\mathfrak{A}$ over $\mathscr{F}$. Let $\mathscr{P}$ denote the set of positive primes and put $\mathscr{P}^+ = \mathscr{P} \cup \{0\}$; for a group $\mathfrak{A}$, $p \in \mathscr{P}$, and $x \in A$, let $o(x)$ denote the order of x, set

$$A_p = \{x \in A\colon o(x) \text{ is a power of } p\},$$

and let $\exp(\mathfrak{A})$ be the exponent of $\mathfrak{A}$. Let ω be the set of non-negative integers and put $N = \omega - \{0\}$. For $n \in \omega$ and $p \in \mathscr{P}$, let $\mathfrak{Z}(p, n)$ be the cyclic group of order p^n and let $\mathfrak{Z}(p^\infty)$ be the quasicyclic group of type p^∞. Let $\mathfrak{D}$ be the rationals under addition and, for $p \in \mathscr{P}$, let

$$\mathfrak{D}(p) = \{m/n \in Q\colon p \text{ and } n \text{ are relatively prime}\}.$$

For groups $\mathfrak{A}$ and $\mathfrak{B}$ we write $\mathfrak{A} \equiv \mathfrak{B}$ iff $\mathfrak{A}$ is elementarily equivalent to $\mathfrak{B}$. If φ is a formula in the language of abelian groups, we write $\mathfrak{A} \models \varphi$ for "$\mathfrak{A}$ semantically yields φ".

We shall make extensive use of Szmielew's characterization of elementary equivalence between abelian groups [3], and, for the reader's convenience, we state her fundamental result in Theorem 1 in the sequel.

Definition 1. For an abelian group $\mathfrak{A}$ and $m, n \in \omega$,

(i) $x, y \in A$ are called *congruent modulo m*, denoted by $x \equiv y \pmod{m}$, if $x = y + mz$ for some $z \in A$;

(ii) $x_1, \ldots, x_n \in A$ are *independent modulo m* if, for every sequence of integers $m_1, \ldots, m_n$, it follows that

$$m_1x_1 + \ldots + m_nx_n = 0 \quad \text{implies} \quad m_i \equiv 0 \pmod{m} \text{ for } i = 1, \ldots, n;$$

(iii) $x_1, \ldots, x_n \in A$ are *strongly independent modulo m* if, for every sequence of integers $m_1, \ldots, m_n$, it follows that

$$m_1x_1 + \ldots + m_nx_n \equiv 0 \pmod{m} \quad \text{implies} \quad m_i \equiv 0 \pmod{m} \quad \text{for } i = 1, \ldots, n.$$

Definition 2. Given $\mathfrak{A}$, $p \in \mathscr{P}$, and $k \in N$, for $i = 1, 2, 3$, define the *i-th rank modulo* p^k of $\mathfrak{A}$, denoted by $\varrho^{(i)}[p, k]\mathfrak{A}$, as the maximum finite number, if it exists, of elements in A which are,

for $i = 1$, of order p^k and independent modulo p^k,

for $i = 2$, strongly independent modulo p^k,

for $i = 3$, of order p^k and strongly independent modulo p^k.

If such a maximum number does not exist, set $\varrho^{(i)}[p, k]\mathfrak{A} = \aleph_0$.

THEOREM 1 (Szmielew [3]). $\mathfrak{A} \equiv \mathfrak{B}$ *iff, for every* $p \in \mathscr{P}$ *and* $k \in N$, *we have*

(I) $\varrho^{(i)}[p, k]\mathfrak{A} = \varrho^{(i)}[p, k]\mathfrak{B}$ $(i = 1, 2, 3)$,

(II) $\exp(\mathfrak{A}) < \omega$ *iff* $\exp(\mathfrak{B}) < \omega$.

COROLLARY. *Any* $\mathfrak{A}$ *is elementarily equivalent to a group of the form*

$$\sum_{n \in N, p \in \mathscr{P}} \mathfrak{Z}(p, n)^{(\alpha(p,n))} + \sum_{p \in \mathscr{P}} \mathfrak{D}(p)^{(\beta(p))} + \sum_{p \in \mathscr{P}} \mathfrak{Z}(p^\infty)^{(\gamma(p))} + \mathfrak{D}^{(\delta)},$$

where $\alpha(p, n)$, $\beta(p)$ *and* $\gamma(p)$ *are cardinals not greater than* ω *and* $\delta \in \{0, 1\}$.

We shall also make use of the facts that

$$\varrho^{(i)}[p, k]\Big(\prod_{i \in I} \mathfrak{A}_i\Big) = \sum_{i \in I} \varrho^{(i)}[p, k]\mathfrak{A}_i \quad \text{and} \quad \prod_{i \in I} \mathfrak{A}_i \equiv \sum_{i \in I} \mathfrak{A}_i,$$

both of which are easy consequences of Theorem 1. The ranks of various commonly-used abelian groups are known and given in Theorem 1.9 of [3].

If $\exp(\mathfrak{A}) < \omega$, then it is well known that there are cardinals α_l such that $\mathfrak{A} \cong \mathfrak{C}_1^{(\alpha_1)} + \ldots + \mathfrak{C}_r^{(\alpha_r)}$, where $\mathfrak{C}_l = \mathfrak{Z}(p_l, n_l)$ with $p_l \in \mathscr{P}$, $n_l \in N$ for $1 \leqslant l \leqslant r$ and $p_k \neq p_l$ for $k \neq l$.

THEOREM 2. *Let* $\exp(\mathfrak{A}) < \omega$ *and* $\mathfrak{A} \cong \mathfrak{C}_1^{(\alpha_1)} + \ldots + \mathfrak{C}_r^{(\alpha_r)}$ *as above. Then* $\mathfrak{B} \equiv \mathfrak{A}$ *iff* $\mathfrak{B} \cong \mathfrak{C}_1^{(\beta_1)} + \ldots + \mathfrak{C}_r^{(\beta_r)}$, *where* $\min\{\alpha_l, \aleph_0\} = \min\{\beta_l, \aleph_0\}$ *for* $1 \leqslant l \leqslant r$.

Proof. Suppose $\exp(\mathfrak{A}) < \omega$, and $\mathfrak{A}$ is of the stated form. If $\mathfrak{B} \equiv \mathfrak{A}$, then

$$\exp(\mathfrak{B}) < \omega \quad \text{and} \quad \mathfrak{B} \cong \mathfrak{B}_1^{(\beta_1)} + \ldots + \mathfrak{B}_t^{(\beta_t)},$$

where $\mathfrak{B}_l = \mathfrak{Z}(q_l, m_l)$ for $1 \leqslant l \leqslant t$. Since $\mathfrak{A} \equiv \mathfrak{B}$, we have

$$\varrho^{(3)}[q_l, m_l]\mathfrak{A} = \varrho^{(3)}[q_l, m_l]\mathfrak{B} = \min\{\beta_l, \aleph_0\} > 0.$$

From this it follows that $\mathfrak{B}_l \cong \mathfrak{C}_j$ for some j with $1 \leqslant j \leqslant r$. Similarly we can show that each $\mathfrak{C}_j$ is isomorphic to some $\mathfrak{B}_l$, so after a suitable rearrangement

$$\mathfrak{B} \cong \mathfrak{C}_1^{(\beta_1)} + \ldots + \mathfrak{C}_r^{(\beta_r)}$$

and

$$\min\{\alpha_l, \aleph_0\} = \varrho^{(3)}[p_l, n_l]\mathfrak{A} = \varrho^{(3)}[p_l, n_l]\mathfrak{B} = \min\{\beta_l, \aleph_0\} \quad \text{for } 1 \leqslant l \leqslant r.$$

For the converse, if $\mathfrak{B}$ has the stated representation, it is not difficult to verify the conditions of Theorem 1 to show that $\mathfrak{B} \equiv \mathfrak{A}$.

If $\mathfrak{A}$ is a divisible abelian group, then it is well known that

$$\mathfrak{A} \cong \mathfrak{D}^{(n_0)} + \sum_{p \in \mathscr{P}} \mathfrak{Z}(p^\infty)^{(n_p)}$$

for some choice of cardinals n_p with $p \in \mathscr{P}^+$.

THEOREM 3. *Let* $\mathfrak{A}$ *be a non-trivial divisible abelian group, say*

$$\mathfrak{A} \cong \mathfrak{D}^{(n_0)} + \sum_{p \in \mathscr{P}} \mathfrak{Z}(p^\infty)^{(n_p)}$$

as above. Then $\mathfrak{B} \equiv \mathfrak{A}$ *iff* $\mathfrak{B}$ *is non-trivial and*

$$\mathfrak{B} \cong \mathfrak{D}^{(l_0)} + \sum_{p \in \mathscr{P}} \mathfrak{Z}(p^\infty)^{(l_p)},$$

where l_0 *can be any cardinal consistent with* $\mathfrak{B} \neq \{0\}$ *and, for* $p \in \mathscr{P}$, l_p *is any cardinal such that* $\min\{l_p, \aleph_0\} = \min\{n_p, \aleph_0\}$.

Proof. Suppose that $\mathfrak{A}$ is non-trivial and has the stated form. If $\mathfrak{B} \equiv \mathfrak{A}$, then $\mathfrak{B}$ is divisible, so

$$\mathfrak{B} \cong \mathfrak{D}^{(l_0)} + \sum_{p \in \mathscr{P}} \mathfrak{Z}(p^\infty)^{(l_p)}$$

for cardinals l_p with $p \in \mathscr{P}^+$. Using Theorem 1 and basic facts on ranks, we infer that for $p \in \mathscr{P}$ and $k \in N$,

$$\min\{n_p, \aleph_0\} = \varrho^{(1)}[p, k]\mathfrak{A} = \varrho^{(1)}[p, k]\mathfrak{B} = \min\{l_p, \aleph_0\}.$$

For the converse, suppose $\{0\} \neq \mathfrak{B} \cong \mathfrak{D}^{(l_0)} + \sum_{p \in \mathscr{P}} \mathfrak{Z}(p^\infty)$, where l_0 is any cardinal consistent with $\mathfrak{B} \neq \{0\}$, and $\min\{l_p, \aleph_0\} = \min\{n_p, \aleph_0\}$ for $p \in \mathscr{P}$. Since $\mathfrak{B}$ is non-trivial, $\exp(\mathfrak{B}) = \aleph_0 = \exp(\mathfrak{A})$. For $p \in \mathscr{P}$ and $k \in N$,

$$\varrho^{(i)}[p, k]\mathfrak{A} = 0 = \varrho^{(i)}[p, k]\mathfrak{B} \quad \text{for } i = 2, 3.$$

Also

$$\varrho^{(1)}[p, k]\mathfrak{A} = \min\{n_p, \aleph_0\} = \min\{l_p, \aleph_0\} = \varrho^{(1)}[p, k]\mathfrak{B},$$

so, by Theorem 1, $\mathfrak{A} \equiv \mathfrak{B}$.

LEMMA 1. *If $\mathfrak{A} \equiv \mathfrak{B}$ and $p \in \mathscr{P}$, then $\mathfrak{A}_p \equiv \mathfrak{B}_p$.*

Proof. Suppose $\mathfrak{A} \equiv \mathfrak{B}$ and $p \in \mathscr{P}$, but $\mathfrak{A}_p \not\equiv \mathfrak{B}_p$. In the case where one of $\exp(\mathfrak{A}_p)$, $\exp(\mathfrak{B}_p)$ is infinite and the other is finite a contradiction is easily obtained. Assume $\exp(\mathfrak{A}_p)$ and $\exp(\mathfrak{B}_p)$ are either both infinite or both finite. Then, by Theorem 1, $\varrho^{(i)}[q, k]\mathfrak{A}_p \neq \varrho^{(i)}[q, k]\mathfrak{B}_p$ for some $i \in \{1, 2, 3\}$, $q \in \mathscr{P}$, and $k \in N$. Assume without loss of generality that

$$\aleph_0 \geqslant \varrho^{(i)}[q, k]\mathfrak{A}_p = n > m = \varrho^{(i)}[q, k]\mathfrak{B}_p,$$

and choose distinct $x_1, \ldots, x_{m+1} \in A_p$ with the appropriate orders and independence property of Definition 2 (depending upon the value of i)- It can be shown that these elements satisfy the same independence prop. erty in $\mathfrak{A}$. Since $\mathfrak{A} \equiv \mathfrak{B}$, there are a set I and an ultrafilter $\mathscr{F}$ on I such that $\mathfrak{A}^I/\mathscr{F} \cong \mathfrak{B}^I/\mathscr{F}$ by an isomorphism Φ (see [1] and [2]). For $1 \leqslant j \leqslant m+1$, let $x_j' = \langle u_i \rangle_{i \in I}$, where $u_i = x_j$ for all $i \in I$. Set $\bar{x}_j' = x_j'/\mathscr{F}$, and let $y_j = \Phi(\bar{x}_j')$. From here it is easy to show that $y_j \in B_p^I/\mathscr{F}$ for $1 \leqslant j \leqslant m+1$ and that $y_1, \ldots, y_{m+1}$ have the appropriate independence property. Therefore,

$$\varrho^{(i)}[q, k](\mathfrak{B}_p^I/\mathscr{F}) \geqslant m+1,$$

whence $\varrho^{(i)}[q, k](\mathfrak{B}_p) \geqslant m+1$, since $\mathfrak{B}_p \equiv \mathfrak{B}_p^I/\mathscr{F}$. But this contradicts $\varrho^{(i)}[q, k]\mathfrak{B}_p = m$.

LEMMA 2. *If $\mathfrak{R}$ is reduced and*

$$\mathfrak{R} \equiv \sum_{n \in N} \mathfrak{Z}(p, n)^{(\alpha(n))} + \mathfrak{D}(p)^{(\beta)} + \mathfrak{Z}(p^\infty)^{(\gamma)}$$

for cardinals $\alpha(n)$, β and γ, then

$$\mathfrak{R} \equiv \sum_{n \in N} \mathfrak{Z}(p, n)^{(\alpha(n))} + \mathfrak{D}(p)^{(\beta)}.$$

Proof. If $\gamma = 0$, we are finished; so assume $\gamma \neq 0$. For convenience of notation, set

$$\mathfrak{A} = \sum_{n \in N} \mathfrak{Z}(p, n)^{(\alpha(n))}, \quad \mathfrak{B} = \mathfrak{D}(p)^{(\beta)} \quad \text{and} \quad \mathfrak{C} = \mathfrak{Z}(p^\infty)^{(\gamma)}.$$

Case 1. $\exp(\mathfrak{A}) = \aleph_0$.

In this case we have $\varrho^{(1)}[p, k]\mathfrak{A} = \aleph_0$ for $k \in N$, whence

$$\varrho^{(1)}[p, k](\mathfrak{A}+\mathfrak{B}+\mathfrak{C}) = \varrho^{(1)}[p, k](\mathfrak{A}+\mathfrak{B}).$$

Also,

$$\varrho^{(1)}[q, k](\mathfrak{A}+\mathfrak{B}+\mathfrak{C}) = 0 = \varrho^{(1)}[q, k](\mathfrak{A}+\mathfrak{B}) \quad \text{for } q \neq p \text{ and } k \in N.$$

Next, for $q \in \mathscr{P}$ and $k \in N$, we have

$$\varrho^{(2)}[q, k]\mathfrak{C} = \varrho^{(3)}[q, k]\mathfrak{C} = 0,$$

so

$$\varrho^{(i)}[q, k](\mathfrak{A}+\mathfrak{B}+\mathfrak{C}) = \varrho^{(i)}[q, k](\mathfrak{A}+\mathfrak{B}) \quad \text{for } i = 2, 3.$$

Since $\exp(\mathfrak{A}+\mathfrak{B}+\mathfrak{C}) = \aleph_0 = \exp(\mathfrak{A}+\mathfrak{B})$, it follows from Theorem 1 that $\mathfrak{A}+\mathfrak{B}+\mathfrak{C} \equiv \mathfrak{A}+\mathfrak{B}$.

Case 2. $\exp(\mathfrak{A}) = p^m < \aleph_0$.

For l, $n \in N$, set

$$\varphi_{l,n} = \forall x\big(o(x) = p^l \to \exists y \exists z(p^m z = y \wedge ny = p^m x)\big).$$

Then $\mathfrak{R} \models \varphi_{l,n}$; for $s \in \omega$, set $\psi_s = \exists x\big(o(x) = p^s\big)$. Then, since $\gamma > 0$, we have $\mathfrak{A}+\mathfrak{B}+\mathfrak{C} \models \psi_s$, whence $\mathfrak{R} \models \psi_s$. If we let

$$S = \{p^m x\colon o(x) < \omega \text{ and } x \in R\},$$

$\mathfrak{S}$ is a subgroup of $\mathfrak{R}$. It is easy to see that $x \in R$ and $o(x) < \omega$ imply $o(x) = p^l$ for some $l \in \omega$. Choose $y \in S$ with $y \neq 0$; then $y = p^m x$ for some $x \in R$ with $o(x) < \omega$, and $o(x) = p^l$ for some $l \in N$. Let $n \in N$. Then, since $\mathfrak{R} \models \varphi_{l,n}$, there exist w, $z \in R$ such that $p^m z = w$ and $nw = p^m x = y$, i.e. $w \in S$ and $nw = y$. Therefore, $\mathfrak{S} \neq \{0\}$ and $\mathfrak{S}$ is divisible, contrary to $\mathfrak{R}$'s being reduced.

LEMMA 3. *If* $\mathfrak{R} \equiv \mathfrak{A}+\mathfrak{D}^{(a)}$, $\mathfrak{R}$ *is reduced, and* $\exp(\mathfrak{A}) < \omega$, *then* $a = 0$.

Proof. Suppose $a > 0$ and $\exp(\mathfrak{A}) = m < \omega$. For $n \in N$, set

$$\varphi_n = \forall x \exists z \exists w(z = mw \wedge mx = nz).$$

Since $\mathfrak{A}+\mathfrak{D}^{(a)} \models \varphi_n$ for all $n \in N$, $\mathfrak{R} \models \varphi_n$. Set

$$B = \{mx\colon x \in R \text{ and } o(x) = \aleph_0\}.$$

Then, clearly, $B \neq 0$. Choose $y \in B$, $n \in N$; then $y = mx$, where $x \in R$ and $o(x) = \aleph_0$. Since $\mathfrak{R} \models \varphi_n$, there are z, $w \in R$ such that $z = mw$ and $y = mx = nz$. Now, $o(x) = \aleph_0$ implies $o(z) = o(w) = \aleph_0$. Therefore, $y = nz$ with $z \in B$, so the subgroup of $\mathfrak{R}$ generated by B is divisible and non-empty, contrary to $\mathfrak{R}$'s being reduced.

THEOREM 4. *If $\mathfrak{R}$ is reduced, then*

$$\mathfrak{R} \equiv \sum_{n\in N, p\in\mathscr{P}} \mathfrak{Z}(p, n)^{(\alpha(p,n))} + \sum_{p\in\mathscr{P}} \mathfrak{D}^{(\beta(p))}$$

for some choice of cardinals $\alpha(p, n)$, $\beta(p) \in \omega+1$.

Proof. If $\mathfrak{R}$ has a finite exponent, then $\mathfrak{R}$ is isomorphic to a direct sum of cyclic groups and we are finished. Thus assume $\mathfrak{R}$ has an infinite exponent. From the Corollary to Theorem 1 we know that

$$\mathfrak{R} \equiv \mathfrak{A}+\mathfrak{B}+\mathfrak{C}+\mathfrak{D}^m,$$

where $\mathfrak{A} = \sum_{n\in N, p\in\mathscr{P}} \mathfrak{Z}(p, n)^{(\alpha(p,n))}$, $\mathfrak{B} = \sum_{p\in\mathscr{P}} \mathfrak{D}(p)^{(\beta(p))}$, $\mathfrak{C} = \sum_{p\in\mathscr{P}} \mathfrak{Z}(p^\infty)^{(\gamma(p))}$

for some choice of $\alpha(p, n)$, $\beta(p)$, $\gamma(p) \in \omega+1$ and $m \in \{0, 1\}$.

Case 1. $\mathfrak{A}+\mathfrak{B}+\mathfrak{C}$ has a finite exponent.

In this case $\gamma(p) = \beta(p) = 0$ for all $p \in \mathscr{P}$, and $\mathfrak{A}$ has a finite exponent. Thus $\mathfrak{R} \equiv \mathfrak{A}+\mathfrak{D}^m$, where $\mathfrak{R}$ is reduced and $\mathfrak{A}$ has a finite exponent; so, by Lemma 3, $m = 0$.

Case 2. $\mathfrak{A}+\mathfrak{B}+\mathfrak{C}$ has an infinite exponent.

In this case $\mathfrak{R} \equiv \mathfrak{A}+\mathfrak{B}+\mathfrak{C}+\mathfrak{D}^m \equiv \mathfrak{A}+\mathfrak{B}+\mathfrak{C}$, where the second equivalence follows from Theorem 1. Now,

$$\mathfrak{A}+\mathfrak{B}+\mathfrak{C} \cong \sum_{p\in\mathscr{P}} \mathfrak{B}(p),$$

$$\text{where } \mathfrak{B}(p) = \sum_{n\in N} \mathfrak{Z}(p, n)^{(\alpha(p,n))} + \mathfrak{D}(p)^{(\beta(p))} + \mathfrak{Z}(p^\infty)^{(\gamma(p))}.$$

By Lemma 1,

$$\mathfrak{R}_p \equiv \Big(\sum_{p\in\mathscr{P}} \mathfrak{B}(p)\Big)_p \cong \sum_{n\in N} \mathfrak{Z}(p, n)^{(\alpha(p,n))} + \mathfrak{Z}(p^\infty)^{(\gamma(p))}.$$

But, since $\mathfrak{R}$ is reduced, so is $\mathfrak{R}_p$. Hence, by Lemma 2,

$$\sum_{n\in N} \mathfrak{Z}(p, n)^{(\alpha(p,n))} + \mathfrak{Z}(p^\infty)^{(\gamma(p))} \equiv \sum_{n\in N} \mathfrak{Z}(p, n)^{(\alpha(p,n))}.$$

Therefore

$$\mathfrak{B}(p) \equiv \sum_{n\in N} \mathfrak{Z}(p, n)^{(\alpha(p,n))} + \mathfrak{D}(p)^{(\beta(p))},$$

so

$$\mathfrak{R} \equiv \sum_{p\in\mathscr{P}} \mathfrak{B}(p) \equiv \sum_{p\in\mathscr{P}} \Big(\sum_{n\in N} \mathfrak{Z}(p, n)^{(\alpha(p,n))} + \mathfrak{D}(p)^{(\beta(p))}\Big) \cong \mathfrak{A}+\mathfrak{B},$$

where we have made use of the fact that $\mathfrak{A}_i \equiv \mathfrak{B}_i$ for $i \in I$ implies

$$\sum_{i\in I} \mathfrak{A}_i \equiv \sum_{i\in I} \mathfrak{B}_i.$$

THEOREM 5. *Let* $\mathfrak{D}_1$, $\mathfrak{D}_2$ *be divisible, and* $\mathfrak{R}_1$, $\mathfrak{R}_2$ *reduced. Then* $\mathfrak{D}_1+\mathfrak{R}_1 \equiv \mathfrak{D}_2+\mathfrak{R}_2$ *iff*

(i) $$\exp(\mathfrak{D}_1+\mathfrak{R}_1)<\aleph_0 \quad \textit{iff} \quad \exp(\mathfrak{D}_2+\mathfrak{R}_2)<\aleph_0,$$

(ii) $$\mathfrak{R}_1 \equiv \mathfrak{R}_2 \equiv \sum_{n\in N, p\in\mathscr{P}} \mathfrak{Z}(p,n)^{(\alpha(p,n))}+\sum_{p\in\mathscr{P}}\mathfrak{D}(p)^{(\beta(p))}$$

for some $\alpha(p,n)$, $\beta(p)\in\omega+1$,

(iii) $$\mathfrak{D}_1 \cong \sum_{p\in\mathscr{P}}\mathfrak{Z}(p^\infty)^{(\gamma(p))}+\mathfrak{D}^{(\alpha)} \quad \textit{and} \quad \mathfrak{D}_2 \cong \sum_{p\in\mathscr{P}}\mathfrak{Z}(p^\infty)^{(\gamma(p)^*)}+\mathfrak{D}^{(\alpha^*)},$$

where α, α^* *can be any cardinals consistent with condition* (i) *and, for each* $p\in\mathscr{P}$, $\gamma(p)$, $\gamma(p)^*$ *can be any cardinals consistent with condition* (i) *unless* $\sum_{n\in N}\alpha(p,n)<\omega$ *in which case* $\min\{\aleph_0,\gamma(p)\}=\min\{\aleph_0,\gamma(p)^*\}$.

Proof. Suppose the conditions hold. Then, by condition (i) and Theorem 1, it suffices to show

$$\varrho^{(i)}[p,k](\mathfrak{D}_1+\mathfrak{R}_1)=\varrho^{(i)}[p,k](\mathfrak{D}_2+\mathfrak{R}_2) \quad \text{for } i=1,2,3,\ p\in\mathscr{P}, \text{ and } k\in N.$$

For $i=2$, 3, $p\in\mathscr{P}$ and $k\in N$, we have

$$\begin{aligned}\varrho^{(i)}[p,k](\mathfrak{D}_1+\mathfrak{R}_1) &= \varrho^{(i)}[p,k]\mathfrak{D}_1+\varrho^{(i)}[p,k]\mathfrak{R}_1=\varrho^{(i)}[p,k]\mathfrak{R}_1\\ &= \varrho^{(i)}[p,k]\mathfrak{R}_2=\varrho^{(i)}[p,k]\mathfrak{D}_2+\varrho^{(i)}[p,k]\mathfrak{R}_2\\ &= \varrho^{(i)}[p,k](\mathfrak{D}_2+\mathfrak{R}_2).\end{aligned}$$

For $p\in\mathscr{P}$, $k\in N$,

$$\varrho^{(1)}[p,k](\mathfrak{D}_1+\mathfrak{R}_1)=\gamma(p)+\sum_{n\geqslant k}\alpha(p,n)$$

and

$$\varrho^{(1)}[p,k](\mathfrak{D}_2+\mathfrak{R}_2)=\gamma(p)^*+\sum_{n\geqslant k}\alpha(p,n)^*.$$

Now, $\mathfrak{R}_1\equiv\mathfrak{R}_2$ implies

$$\sum_{n\geqslant k}\alpha(p,n)=\sum_{n\geqslant k}\alpha(p,n)^*.$$

If $\sum_{n\in N}\alpha(p,n)<\omega$, then either $\gamma(p)<\omega$ in which case

$$\gamma(p)=\gamma(p)^* \quad \text{and} \quad \varrho^{(1)}[p,k](\mathfrak{D}_1+\mathfrak{R}_1)=\varrho^{(1)}[p,k](\mathfrak{D}_2+\mathfrak{R}_2),$$

or $\gamma(p)\geqslant\aleph_0$ in which case

$$\gamma(p)^*\geqslant\aleph_0 \quad \text{and} \quad \varrho^{(1)}[p,k](\mathfrak{D}_1+\mathfrak{R}_1)=\aleph_0=\varrho^{(1)}[p,k](\mathfrak{D}_2+\mathfrak{R}_2).$$

Conversely, suppose $\mathfrak{D}_1+\mathfrak{R}_1 \equiv \mathfrak{D}_2+\mathfrak{R}_2$. Condition (i) follows from Theorem 1. By Theorem 4,

$$\mathfrak{R}_1 \equiv \mathfrak{R}_1^*, \quad \text{where } \mathfrak{R}_1^* = \mathfrak{A}+\mathfrak{B}$$

$$\text{with } \mathfrak{A} = \sum_{n\in N, p\in\mathscr{P}} \mathfrak{Z}(p,n)^{(\alpha(p,n))} \text{ and } \mathfrak{B} = \sum_{p\in\mathscr{P}} \mathfrak{D}(p)^{(\beta(p))},$$

and

$$\mathfrak{R}_2 \equiv \mathfrak{R}_2^*, \quad \text{where } \mathfrak{R}_2^* = \mathfrak{A}^*+\mathfrak{B}^*$$

$$\text{with } \mathfrak{A}^* = \sum_{n\in N, p\in\mathscr{P}} \mathfrak{Z}(p,n)^{(\alpha(p,n)^*)} \text{ and } \mathfrak{B}^* = \sum_{p\in\mathscr{P}} \mathfrak{D}(p)^{(\beta(p)^*)}.$$

Thus

$$\mathfrak{D}_1+\mathfrak{R}_1^* \equiv \mathfrak{D}_1+\mathfrak{R}_1 \equiv \mathfrak{D}_2+\mathfrak{R}_2 \equiv \mathfrak{D}_2+\mathfrak{R}_2^*. \tag{1}$$

From basic group theory we know that

$$\mathfrak{D}_1 \cong \sum_{p\in\mathscr{P}} \mathfrak{Z}(p^\infty)^{(\gamma(p))}+\mathfrak{D}^{(\delta)} \quad \text{and} \quad \mathfrak{D}_2 \cong \sum_{p\in\mathscr{P}} \mathfrak{Z}(p^\infty)^{(\gamma(p)^*)}+\mathfrak{D}^{(\delta^*)}$$

for cardinals δ, δ^*, so, for $p\in\mathscr{P}$, $k\in N$ and $i = 2, 3$, we have

$$\varrho^{(i)}[p,k]\mathfrak{R}_1^* = \varrho^{(i)}[p,k](\mathfrak{R}_1^*+\mathfrak{D}_1) = \varrho^{(i)}[p,k](\mathfrak{R}_2^*+\mathfrak{D}_2) = \varrho^{(i)}[p,k]\mathfrak{R}_2^*,$$

where the middle equality follows from (1) and Theorem 1. In particular, note that, for each $l\in N$,

$$\min\{\alpha(p,l),\aleph_0\} = \sum_{n\in N} \varrho^{(3)}[p,l]\big(\mathfrak{Z}(p,n)^{(\alpha(p,n))}\big) = \varrho^{(3)}[p,l]\mathfrak{R}_1^* \tag{2}$$
$$= \varrho^{(3)}[p,l]\mathfrak{R}_2^* = \min\{\alpha(p,l)^*,\aleph_0\}.$$

If $\varrho^{(1)}[p,k]\mathfrak{R}_1^* \neq \varrho^{(1)}[p,k]\mathfrak{R}_2^*$, say without loss of generality

$$\aleph_0 \geqslant \varrho^{(1)}[p,k]\mathfrak{R}_1^* = m > n = \varrho^{(1)}[p,k]\mathfrak{R}_2^*,$$

then

$$\sum_{l\geqslant k} \alpha(p,l) = m > n = \sum_{l\geqslant k} \alpha(p,l)^*$$

which contradicts (2). To complete the verification of condition (ii) it suffices to show that $\exp(\mathfrak{R}_1^*)$ is finite iff $\exp(\mathfrak{R}_2^*)$ is finite. Suppose $\exp(\mathfrak{R}_1^*) = \aleph_0 > n = \exp(\mathfrak{R}_2^*)$. Then $\beta(p)^* = 0$ for all $p\in\mathscr{P}$, and there are only finitely many non-zero $\alpha(p,n)^*$'s for $n\in N$. Set

$$k = \max\{n\colon \alpha(p,n)^* > 0\}.$$

From (2) we know that $\alpha(p,l) = 0$ for all $l > k$; so if $\beta(p) = 0$ for all $p\in\mathscr{P}$, then $\exp(\mathfrak{R}_1^*) < \aleph_0$, contrary to our assumption. On the other hand, if there is a $p\in\mathscr{P}$ such that $\beta(p) > 0$, then for $l > k$ we would have $\varrho^{(2)}[p,l]\mathfrak{R}_1^* \geqslant \beta(p) > 0$, whereas $\varrho^{(2)}[p,l]\mathfrak{R}_2^* = 0$, contrary to

$$\varrho^{(2)}[p,l]\mathfrak{R}_1^* = \varrho^{(2)}[p,l]\mathfrak{R}_2^*.$$

Finally, to prove the remaining part of condition (iii) suppose

$$\sum_{n\in N} a(p, n) < \omega.$$

From (2) we know that $a(p, n) = a(p, n)^*$ for all $n \geqslant k$, so condition (iii) then follows from

$$\gamma(p) + \sum_{n\geqslant k} a(p, n) = \varrho^{(1)}[p, k](\mathfrak{D}_1 + \mathfrak{R}_1) = \varrho^{(1)}[p, k](\mathfrak{D}_2 + \mathfrak{R}_2)$$
$$= \gamma(p)^* + \sum_{n\geqslant k} a(p, n)^*.$$

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