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THE PSEUDO-ARC AS AN INVERSE LIMIT WITH SIMPLE BONDING MAPS*

BY

C. WAYNE PROCTOR (NACOGDOCHES, TEXAS)

G. W. Henderson [4] has stated that in describing the pseudo-arc "inverse limits has seen little use as the description of the bonding functions were restatements of a chain construction, involved infinitely many different functions and thus at least as complicated as chains to use". In his paper, Professor Henderson proceeded to reduce the complexity by reducing the number of different bonding maps from infinitely many to one. The purpose of this paper is to reduce the complexity in a different manner by defining simple bonding maps without relying on a chain construction. These simple bonding maps have the property that each has a graph that looks like one "N" has been inserted into the graph of the identity function. It is hoped that the simple nature of these bonding maps will aid the reader in obtaining a better mental picture of the pseudo-arc.

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A simple function F is defined to be a function from the closed interval $[0, 1]$ onto $[0, 1]$ with the property that there exists a closed interval $[r, s]$ of $[0, 1]$ such that $F(x) = x$ if x is not in $[r, s]$ and the graph of F over $[r, s]$ is an "N" within $[r, s]\times[r, s]$, i.e.

$$F(x) = \begin{cases} 3x-2r & \text{if } r \leqslant x \leqslant (2r+s)/3, \\ -3x+2(r+s) & \text{if } (2r+s)/3 < x \leqslant (r+2s)/3, \\ 3x-2s & \text{if } (r+2s)/3 < x \leqslant s. \end{cases}$$

If f_i is a function from X_{i+1} onto X_i for $n \leqslant i < m$, then the notation f_n^m will denote the function from X_m onto X_n which is equal to the composite function $f_n \circ f_{n+1} \circ \ldots \circ f_{m-1}$.

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THEOREM. *The pseudo-arc is the inverse limit of an inverse limit system which has each coordinate space equal to* $[0, 1]$ *and has simple functions as bonding maps.*

Proof. Define X_i for each natural number i to be the ith coordinate space which is equal to $[0, 1]$. Let $\{[r_1, s_1], [r_2, s_2], [r_3, s_3], \dots\}$ denote the collection of all the closed intervals within $[0, 1]$ which have rational end points. Define f_1 from X_2 onto X_1 so that $f_1(x) = x$ if x is not in $[r_1, s_1]$ and the graph of f_1 over $[r_1, s_1]$ is an "N" within $[r_1, s_1] \times [r_1, s_1]$ as defined in the definition above. The bonding function f_2 from X_3 onto X_2 is defined so that it is the same as the identity function except over components C of $f_1^{-1}([r_2, s_2])$ with the property that $f_1(C)$ is $[r_2, s_2]$ in which case the function f_2 is defined so that its graph within the Cartesian product $C \times C$ is an "N". The function f_3 from X_4 onto X_3 is defined in a similar manner by requiring that $f_3(x) = x$ for all x which are not in components C of $f_2^{-1}([r_1, s_1])$ that have $f_2(C) = [r_1, s_1]$ and requiring that the graph of f_3 within $C \times C$ for such C be an "N". Through the use of mathematical induction it can be proven that there exists an infinite sequence of functions $f_1, f_2, f_3, f_4, \dots$ such that (1) f_1, f_2, f_3 are as defined above and (2) if i is a natural number such that $n(n+1)/2 < i \leqslant (n+1)(n+2)/2$ for some $n \geqslant 2$, the function f_i is the function from X_{i+1} onto X_i possessing the property that, if $k = i - n(n+1)/2$, f_i is the same as the identity function except over components C of $(f_k^i)^{-1}([r_{n+2-k}, s_{n+2-k}])$ which have $f_k^i(C) = [r_{n+2-k}, s_{n+2-k}]$ in which case the function f_i has an "N" as its graph within $C \times C$. The following table probably best describes how the functions are defined.

	X_1	X_2	X_3	X_4	X_5	$\dots$
$[r_1, s_1]$	f_1	f_3	f_6	f_{10}	f_{15}	$\dots$
$[r_2, s_2]$	f_2	f_5	f_9	f_{14}	$\dots$	
$[r_3, s_3]$	f_4	f_8	f_{13}	$\dots$		
$[r_4, s_4]$	f_7	f_{12}	$\dots$			
$[r_5, s_5]$	f_{11}	$\dots$				
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The variable n refers to the nth diagonal of the table that runs from the lower left to the upper right, and k refers to the kth position of the function f_i in the $(n+1)$st diagonal. The closed interval at the left of the row containing f_i is the interval $[r_{n+2-k}, s_{n+2-k}]$ which is viewed as a subset of the coordinate space X_k which heads the column that contains f_i.

In order to see that the pseudo-arc is the inverse limit of the system that has $f_1, f_2, f_3, \dots$ as bonding maps, one need only observe that the inverse limit is hereditarily indecomposable since R. H. Bing [2] has established that all chainable hereditarily indecomposable nondegenerate continua are topologically equivalent to the pseudo-arc as defined by E. E. Moise [6], to the homogeneous plane continuum as defined by Bing [1] and to the hereditarily indecomposable continuum as defined by Knaster [5].

Suppose M is a nondegenerate subcontinuum of the inverse limit such that M is the union of two proper subcontinua H and K of M. There is a natural number k_0 having the property that neither of the k_0th projections $\pi_{k_0}(H)$ nor $\pi_{k_0}(K)$ is a subset of the other. There is a closed interval $[r_j, s_j]$ in the collection mentioned at the beginning of this proof and there is a closed interval $[a, b]$ with $a < r_j < s_j < b$ and with $\{a, r_j\}$ being a subset of one of the sets $\pi_{k_0}(H) - \pi_{k_0}(K)$ or $\pi_{k_0}(K) - \pi_{k_0}(H)$ and $\{s_j, b\}$ being a subset of the other. It may be assumed without loss of generality that $\{a, r_j\}$ is a subset of $\pi_{k_0}(H) - \pi_{k_0}(K)$ and $\{s_j, b\}$ is a subset of the difference $\pi_{k_0}(K) - \pi_{k_0}(H)$. Through solving the equation $j = n+2-k_0$ for n, one finds that in the $(k_0+j-2)+1$ diagonal of the table there is a function f_{i_0} which was defined in terms of $[r_j, s_j]$ being a subset of X_{k_0}, namely i_0 is equal to $k_0 + (k_0+j-2)(k_0+j-1)/2$. There must be a component C_1 of $(f_{k_0}^{i_0+1})^{-1}([a, b])$ within $\pi_{i_0+1}(M)$ such that $[a, b]$ is the image of C_1 under the function $f_{k_0}^{i_0+1}$. There must also be a component C_2 of $(f_{k_0}^{i_0})^{-1}([r_j, s_j])$ containing a point of $f_{i_0}(C_1)$ such that $[r_j, s_j]$ is the image of C_2 under the function $f_{k_0}^{i_0}$. Choose a_1 in $C_1 \cap (f_{k_0}^{i_0+1})^{-1}(a)$ and b_1 in $C_1 \cap (f_{k_0}^{i_0+1})^{-1}(b)$. Notice that C_2 cannot contain the images of a_1 and b_1 using the function f_{i_0}; otherwise, if for example $f_{i_0}(a_1)$ were in C_2, then $a = f_{k_0}^{i_0+1}(a_1) = f_{k_0}^{i_0}(f_{i_0}(a_1)) \in f_{k_0}^{i_0}(C_2) = [r_j, s_j]$ which would deny that $a < r_j$. Also notice that $f_{i_0}(a_1)$ is in $\pi_{i_0}(H) - \pi_{i_0}(K)$ because $a = f_{k_0}^{i_0+1}(a_1)$ which is in $\pi_{k_0}(H) - \pi_{k_0}(K)$ and that $f_{i_0}(b_1)$ is in $\pi_{i_0}(K) - \pi_{i_0}(H)$ for a similar reason. With these properties and the additional property that C_2 contains $\pi_{i_0}(H) \cap \pi_{i_0}(K)$ as a subset, it can be seen that $f_{i_0}(C_1)$ must be a closed interval containing C_2 within its interior. This shows that C_2 is a subset of the closed interval $[a_1, b_1]$ (or $[b_1, a_1]$ if $b_1 < a_1$) in X_{i_0+1} due to the manner in which f_{i_0} was defined in terms of C_2. Since $C_2 \subseteq [a_1, b_1]$ (or $[b_1, a_1]$) $\subseteq C_1 \subseteq \pi_{i_0+1}(M)$, the set C_2 is a subset of $\pi_{i_0+1}(M)$. With C_2 being a subset of the union of the two intervals $\pi_{i_0+1}(H)$ and $\pi_{i_0+1}(K)$, either $\pi_{i_0+1}(H)$ or $\pi_{i_0+1}(K)$ contains a subinterval of C_2 with length at least half the length of C_2. Suppose $\pi_{i_0+1}(H)$ has this property. Since the graph of f_{i_0} is an "N" within the square $C_2 \times C_2$, $f_{i_0}(\pi_{i_0+1}(H))$ must have C_2 as a subset. With $[r_j, s_j] = f_{k_0}^{i_0}(C_2) \subseteq f_{k_0}^{i_0}(f_{i_0}(\pi_{i_0+1}(H))) = \pi_{k_0}(H)$, the point s_j must be in $\pi_{k_0}(H)$ which is a contradiction. If $\pi_{i_0+1}(K)$ contained at least half of the interval C_2, another contradiction would be arrived at by a similar argument. This proves that the inverse limit is hereditarily indecomposable; thus, the inverse limit is the pseudo-arc.

In order to see that the pseudo-arc can be obtained in terms of simple bonding maps, one needs only to observe that, for each i, the function f_i has the property there exist finitely many disjoint subintervals $[u_{i1}, v_{i1}]$, $[u_{i2}, v_{i2}]$, $\ldots$, $[u_{im_i}, v_{im_i}]$ such that $f_i(x) = x$ if x is not in $[u_{ij}, v_{ij}]$ for any $j \leqslant m_i$ and the graph of f_i over $[u_{ij}, v_{ij}]$ is an "N" for each $j \leqslant m_i$. Now for each i and each $j \leqslant m_i$, define f_{ij} to be the simple function from $[0, 1]$ onto $[0, 1]$ such that $f_{ij}(x) = x$ if x is not in $[u_{ij}, v_{ij}]$ and $f_{ij}(x) = f_i(x)$ if x is in $[u_{ij}, v_{ij}]$. Notice that $f_i = f_{i1} \circ f_{i2} \circ \ldots \circ f_{im_i}$. Define F_1 to be the simple function f_1. In general, for each natural number $n > 1$, F_n is defined to be f_{ij} if $n = m_1 + m_2 + \ldots + m_{i-1} + j$. It is easy to see that the inverse limit of the inverse limit system that has $F_1, F_2, F_3, \ldots$ as simple bonding maps is topologically equivalent to the inverse limit that we have shown to be the pseudo-arc. This completes the proof of the theorem.

REFERENCES

[1] R. H. Bing, *A homogeneous indecomposable plane continuum*, Duke Mathematical Journal 15 (1948), p. 729–742.

[2] – *Concerning hereditarily indecomposable continua*, Pacific Journal of Mathematics 1 (1951), p. 43–51.

[3] – *Snake-like continua*, Duke Mathematical Journal 18 (1951), p. 653–663.

[4] G. W. Henderson, *The pseudo-arc as an inverse limit with only one bonding map*, ibidem 31 (1964), p. 421–425.

[5] B. Knaster, *Un continu dont tout sous-continu est indécomposable*, Fundamenta Mathematicae 3 (1922), p. 247–286.

[6] E. E. Moise, *An indecomposable plane continuum which is homeomorphic to each of its nondegenerate subcontinua*, American Mathematical Society Translations 63 (1948), p. 581–594.

STEPHEN F. AUSTIN STATE UNIVERSITY
NACOGDOCHES, TEXAS 75962

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