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ON THE ADJOINTNESS BETWEEN OPERATIONS AND RELATIONS AND ITS IMPACT ON ATOMIC COMPACTNESS

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There is a naturally arising functor from the class of all structures of a given type (not necessarily finitary) to a suitable class of relational structures which preserves underlying sets and replaces each μ-ary operation by the $(\mu+1)$-ary relation which is its graph. (A functor which maps the class of all universal algebras of some type to a related class of relational structures is obtained as a special case.) This note * investigates the properties of this functor and their applications; it will be seen that this functor has a left adjoint (an explicit description of the adjoint is given) and that it preserves and reflects purity, pure-essentialness, atomic compactness and, in the finitary case, the existence of atomic compact hulls, although it does not reflect the existence of atomic compact extensions.

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1. Preliminaries. We assume that ordinals have been introduced in such a way that each ordinal is equal to the set of all ordinals less than it. For a set A and an ordinal λ, A^λ is the set of all functions from λ into A.

Recall that for a type $\sigma = \big((\lambda_i)_{i\in I}, (\mu_j)_{j\in J}\big)$ (where the λ_i and μ_j are arbitrary ordinals and we assume, for convenience, that $I\cap J = \emptyset$), a *structure of type* σ (*σ-structure*) is a triple $\big(A, (R_i)_{i\in I}, (f_j)_{j\in J}\big)$, where A is a set, $R_i \subseteq A^{\lambda_i}$ for each $i\in I$ and $f_j: A^{\mu_j} \to A$ for each $j\in J$. The usual custom of writing simply A for $\big(A, (R_i)_{i\in I}, (f_j)_{j\in J}\big)$ will be followed wherever it does not create confusion. If $I = \emptyset$, then a σ-structure is simply a universal algebra of type $(\mu_j)_{j\in J}$; if $J = \emptyset$, then structures of type σ are called *relational*.

The reader is referred to Grätzer [3] for basic results about structures.

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The *characteristic* a_σ of σ is the smallest infinite regular cardinal such that $a_\sigma > \lambda_i$ for all $i \in I$ and $a_\sigma > \mu_j$ for all $j \in J$. In the context of σ-structures, a set will be called *small* if it has fewer than a_σ elements.

For a structure $(A, (R_i)_{i\in I}, (f_j)_{j\in J})$, let

$$A^{\#} = A \times A \amalg \coprod_{i\in I} A^{\lambda_i},$$

where $\amalg$ denotes coproduct (i.e. disjoint union) in the category of sets.

A *homomorphism* from $(A, (R_i)_{i\in I}, (f_j)_{j\in J})$ to $(A', (R'_i)_{i\in I}, (f'_j)_{j\in J})$ (both σ-structures) is a mapping $h: A \to A'$ which is an algebra homomorphism $(A, (f_j)_{j\in J}) \to (A', (f'_j)_{j\in J})$ such that, for all $i \in I$, if $a \in R_i$, then $ha \in R'_i$, where ha is the functional composition of h and a.

The *kernel* of h is given by $\mathrm{Ker}(h) = \{(a, b) \in A \times A \mid h(a) = h(b)\}$. The *relational kernel* of h, $\mathrm{RKer}(h)$, is a subset of $A^{\#}$, and is given by

$$\mathrm{RKer}(h) = \mathrm{Ker}(h) \amalg \coprod_{i\in I} \{a \in A^{\lambda_i} \mid ha \in R'_i\}.$$

Recall that h is an *embedding* if it is one-one and, for all $a \in A^{\lambda_i}$, $a \in R_i$ if $ha \in R'_i$, i.e. h is an embedding iff

$$\mathrm{RKer}(h) \subseteq \Delta_A \amalg \coprod_{i\in I} R_i, \quad \text{where } \Delta_A = \{(a, a) \mid a \in A\}.$$

For algebras A, B, C, we have the familiar fact. Namely, let $f: A \to B$ be a homomorphism onto B and let $g: A \to C$ be a homomorphism. Then $\mathrm{Ker}(f) \subseteq \mathrm{Ker}(g)$ iff there is a homomorphism $h: B \to C$ with $hf = g$. The extension of this result to σ-structures is the following

LEMMA 1. *For σ-structures A, B, C, assume that $f: A \to B$ is a homomorphism onto B and $g: A \to C$ is any homomorphism. Then* $\mathrm{RKer}(f) \subseteq \mathrm{RKer}(g)$ *iff there exists $h: B \to C$ with $hf = g$.*

Intuitively, the common extension to infinitary structures of the definition of purity for finitary structures in Węglorz [8] and of purity for infinitary algebras in Nelson [5] would be the following:

A homomorphism $h: A \to B$ is pure if every small set of atomic formulae with constants in A, satisfiable in B in such a way that the constants from A are replaced by their images under h, is also satisfiable in A. More precisely, a homomorphism $h: A \to B$ is *pure* if every small subset of $A[X]^{\#}$ ($A[X]$ is the free extension of A, in the class of all σ-structures, by the set X) is contained in the relational kernel of a homomorphism $A[X] \to A$ over A whenever it is contained in the relational kernel of a homomorphism $A[X] \to B$ over h.

Note that if h is pure, then it is necessarily an embedding. An extension $B \supseteq A$ is called *pure* if the natural embedding $A \to B$ is pure.

A homomorphism $h: A \to B$ is called *pure-essential* if it is pure, and if gh pure implies that g is an embedding.

A σ-structure A is called *pure-injective* if, for every homomorphism $g: B \to A$ and every pure extension $C \supseteq B$, there is a homomorphism $C \to A$ extending g.

A structure is called a *pure-absolute retract* if it is a retract of each of its pure extensions.

A structure A is called *atomic compact* if, for every subset $F \subseteq A[X]^{\#}$, there is a homomorphism h: $A[X] \to A$ over A with $F \subseteq \mathrm{RKer}(h)$ whenever there is such a homomorphism for each small subset of F. For finitary structures, this is equivalent to the definition of atomic compactness in Węglorz [8] and for infinitary algebras this reduces to the definition of equational compactness in Nelson [5].

Let S_σ be the category of all σ-structures together with their homomorphisms. Recall (see Banaschewski [1]) that a diagram of structures and homomorphisms in S_σ is *over* a structure A iff all its structures are extensions of A and all its homomorphisms map A identically, and a *colimit over* A is the colimit of a diagram over A such that all the colimit maps are over A. An *a_σ-up-directed colimit* is the colimit of a diagram indexed by an a_σ-up-directed partially ordered set, i.e. a partially ordered set in which every subset with fewer than a_σ elements has an upper bound.

It is easy to see that, given homomorphisms $h_{a\beta}: E_a \to E_\beta$ over A (in S_σ) for all $a \leqslant \beta$ in the partially ordered a_σ-up-directed set Λ, E is the colimit over A of this diagram, with colimit maps $h_a: E_a \to E$, iff

$$h_\beta h_{a\beta} = h_a \quad \text{for all } a \leqslant \beta,$$
$$E = \bigcup_{a \in \Lambda} h_a(E_a)$$

and

$$\mathrm{RKer}(h_a) = \bigcup_{\beta \geqslant a} \mathrm{RKer}(h_{a\beta}) \quad \text{for each } a \in \Lambda.$$

The rather intimate connection between pure embeddings and a_σ-up-directed colimits can be seen in the following lemma:

LEMMA 2 (Banaschewski [1]). *A homomorphism h: $A \to B$ in S_σ is pure iff it is the a_σ-up-directed colimit over A of extensions retractable to A.*

The following characterization of atomic compact structures is, n the finitary case, essentially due to Węglorz [8]:

PROPOSITION 1. *For a σ-structure A, the following are equivalent:*

(1) *A is atomic compact,*

(2) *A is pure-injective (in S_σ),*

(3) *A is a pure-absolute retract (in S_σ).*

Proof. The proof of (1) $\Rightarrow$ (2) is the same as the proof of (1) $\Rightarrow$ (2) of Proposition 1 of Banaschewski and Nelson [2] if we replace "finite" by "small" and "Ker" by "RKer" wherever they appear, and use Lemma 1.

(2) $\Rightarrow$ (3) is, as usual, trivial.

(3) $\Rightarrow$ (1). Suppose that A is a pure-absolute retract, and that $S \subseteq A[X]^{\#}$ is a set each of whose small subsets F is contained in the relational kernel of a homomorphism $h_F : A[X] \to A$ over A. Then, by Lemma 1, there exists a homomorphism $g_F : A[X]_F/\theta_F \to A$ such that $g_F \nu_F = h_F$, where, for an arbitrary subset $T \subseteq A[X]^{\#}$, θ_T is the congruence on $A[X]$ generated by $T \cap (A[X] \times A[X])$, and $A[X]_F$ is the σ-structure whose underlying algebra is the same as that of $A[X]$, and whose i-th relation (for $i \in I$) is the i-th relation of $A[X]$ together with $T \cap A[X]^{\lambda_i}$, and $\nu_T : A[X] \to A[X]_T/\theta_T$ is the natural homomorphism.

It follows that the natural map $A \to A[X]_F/\theta_F$ has a retraction for each small $F \subseteq S$. Now the set K of all small subsets of S, ordered by set--inclusion, forms a partially ordered a_σ-up-directed set. For $F, G \in K$ with $F \subseteq G$, let

$$h_{FG} : A[X]_F/\theta_F \to A[X]_G/\theta_G \quad \text{and} \quad h_F : A[X]_F/\theta_F \to A[X]_S/\theta_S$$

be the natural homomorphisms. Since each h_F maps onto $A[X]_S/\theta_S$,

$$A[X]_S/\theta_S = \bigcup_{F \in K} h_F(A[X]_F/\theta_F).$$

It is also clear that $h_F = h_G h_{FG}$ for $F \subseteq G$, and since K is a_σ-up-directed, we have

$$\mathrm{RKer}(h_F) = \bigcup_{G \supseteq F} \mathrm{RKer}(h_{FG}) \quad \text{for all } F \in K.$$

Since all the above-given structures are extensions of A, and all maps are over A, it follows that $A[X]_S/\theta_S$ is the a_σ-up-directed colimit over A of the $A[X]_F/\theta_F$ ($F \in K$), and hence it is a pure extension of A.

Since A is a pure-absolute retract, there is a retraction $g : A[X]_S/\theta_S \to A$, and then $S \subseteq \mathrm{RKer}(g\nu_S)$. This shows that A is atomic compact.

2. The functor. For a particular type σ, let, as before, $\boldsymbol{S}_\sigma$ be the category of all σ-structures, together with their homomorphisms, and let $\boldsymbol{R}_\sigma$ be the category of all relational structures of type $(\emptyset, (\lambda_i)_{i \in I \cup J})$, where $\lambda_i = \mu_i + 1$ for $i \in J$. (Note that $\boldsymbol{S}_\sigma$ and $\boldsymbol{R}_\sigma$ have the same characteristic, namely a_σ, and so "small" will have the same meaning with respect to both categories.)

For $(A, (R_i)_{i \in I}, (f_j)_{j \in J}) \in \boldsymbol{S}_\sigma$, let $\Psi(A) = (A, (R_i)_{i \in I \cup J})$, where $R_j = \{a \in A^{\lambda_j} \mid f_j(a|\mu_j) = a(\mu_j)\}$ for $j \in J$. Then it is clear that, for $A, B \in \boldsymbol{S}_\sigma$, a set mapping h from A to B is an $\boldsymbol{S}_\sigma$-homomorphism iff it is an $\boldsymbol{R}_\sigma$-homomorphism. Consequently, by defining Ψ to be the "identity map" on morphisms we obtain a functor from $\boldsymbol{S}_\sigma$ to $\boldsymbol{R}_\sigma$.

Now Ψ is clearly full and faithful and product-preserving, and it is straightforward to check that it also preserves equalizers. In addition,

S_σ is complete and locally small, and it is easy to see that Ψ admits a "solution set" for every relational structure in R_σ; consequently, the Adjoint Functor Theorem (Mitchell [4], p. 124) guarantees the existence of a functor $\Phi : R_\sigma \to S_\sigma$ which is left adjoint to Ψ.

Moreover, an explicit description of Φ can be given as follows.

For $(A, (R_i)_{i\in I\cup J}) \in R_\sigma$, let θ be the equivalence relation on A generated by

$$\{(x, y) \in A \times A \mid \text{there exist } \alpha, \beta \in R_j \text{ with } \alpha|\mu_j = \beta|\mu_j \text{ and } \alpha(\mu_j) = x,\ \beta(\mu_j) = y\}.$$

Then A/θ is, in the obvious way, a partial S_σ-structure; $\Phi(A)$ is the S_σ-structure freely generated by this partial structure.

Consequently, for $A \in S_\sigma$, if θ is an equivalence relation on A (and hence a congruence on $\Psi(A)$), then $\Phi(\Psi(A)/\theta) = A/\bar{\theta}$, where $\bar{\theta}$ is the congruence on A generated by θ. In particular, $\Phi\Psi$ is the identity functor.

LEMMA 3. *Ψ preserves a_σ-up-directed colimits over A.*

Proof. It is enough to show that, given homomorphisms $h_{\alpha\beta} : E_\alpha \to E_\beta$ over A for all $\alpha \leqslant \beta$ in the partially ordered a_σ-up-directed set Λ, if E is the colimit over A of this diagram, with colimit maps $h_\alpha : E_\alpha \to E$, then

$$\Psi(E) = \bigcup_{\alpha\in\Lambda} \Psi(h_\alpha)(\Psi(E_\alpha))$$

and

$$\mathrm{RKer}(\Psi(h_\alpha)) = \bigcup_{\beta\geqslant\alpha} \mathrm{RKer}(\Psi(h_{\alpha\beta})) \quad \text{for all } \alpha \in \Lambda.$$

The first equality is a trivial consequence of the fact that $E = \bigcup h_\alpha(E_\alpha)$, since Ψ preserves underlying sets. Also, it is clear that

$$\bigcup_{\beta\geqslant\alpha} \mathrm{RKer}(\Psi(h_{\alpha\beta})) \subseteq \mathrm{RKer}(\Psi(h_\alpha)),$$

since $\Psi(h_\beta)\Psi(h_{\alpha\beta}) = \Psi(h_\alpha)$ for all $\beta \geqslant \alpha$, and the inclusion

$$\mathrm{RKer}(\Psi(h_\alpha)) \cap \Big(\Psi(E_\alpha)^2 \amalg \coprod_{i\in I} \Psi(E_\alpha)^{\lambda_i}\Big) \subseteq \bigcup_{\beta\geqslant\alpha} \mathrm{RKer}(\Psi(h_{\alpha\beta}))$$

is an immediate consequence of the fact that

$$\mathrm{RKer}(h_\alpha) \subseteq \bigcup_{\beta\geqslant\alpha} \mathrm{RKer}(h_{\alpha\beta}).$$

If $\xi \in \mathrm{RKer}(\Psi(h_\alpha)) \cap \Psi(E_\alpha)^{\lambda_j}$ for $j \in J$, then $h_\alpha\xi$ is an element of the j-th relation of $\Psi(E)$, and hence $h_\alpha\xi(\mu_j) = f_j(h_\alpha\xi|\mu_j)$. This implies that $h_\alpha\xi(\mu_j) = h_\alpha(f_j(\xi|\mu_j))$, since h_α is an S_σ-homomorphism, and thus $)\xi(\mu_j), f_j(\xi|\mu_j)) \in \mathrm{Ker}(h_\alpha)$. Since

$$\mathrm{Ker}(h_\alpha) \subseteq \bigcup_{\beta\geqslant\alpha} \mathrm{Ker}(h_{\alpha\beta}),$$

there exists $\beta \geqslant \alpha$ with $h_{\alpha\beta}(\xi(\mu_j)) = h_{\alpha\beta}(f_j(\xi \mid \mu_j)) = f_j(h_{\alpha\beta}\xi \mid \mu_j)$, and, consequently, $h_{\alpha\beta}\xi$ is in the j-th relation on $\Psi(E_\beta)$, i.e. $\xi \in \mathrm{RKer}(\Psi(h_{\alpha\beta}))$. It follows that

$$\mathrm{RKer}(\Psi(h_\alpha)) \subseteq \bigcup_{\beta \geqslant \alpha} \mathrm{RKer}(\Psi(h_{\alpha\beta}))$$

and this completes the proof.

PROPOSITION 2. *A homomorphism $f: A \to B$ in $\boldsymbol{S}_\sigma$ is pure iff $\Psi(f): \Psi(A) \to \Psi(B)$ is pure.*

Proof. The necessity follows from Lemmas 2 and 3. The sufficiency follows from the fact that Φ, being a left adjoint functor, preserves all colimits, and hence, in particular, a_σ-up-directed ones, and from the fact that $\Phi\Psi$ is the identity functor.

The following proposition was noted for finitary structures by Taylor [6]:

PROPOSITION 3. *$A \in \boldsymbol{S}_\sigma$ is atomic compact iff $\Psi(A)$ is atomic compact.*

Proof. The atomic compactness of A implies the atomic compactness of $\Psi(A)$ since Ψ is a right adjoint functor between categories of structures with the same characteristic (see Banaschewski [1]).

Conversely, if $\Psi(A)$ is atomic compact and $f: A \to B$ in $\boldsymbol{S}_\sigma$ is pure, then $\Psi(f): \Psi(A) \to \Psi(B)$ is pure, and hence there is a retraction $h: \Psi(B) \to \Psi(A)$. Since Ψ is full, $h = \Psi(g)$ for $g: B \to A$, and then g is the desired retraction of f. Thus A is a pure-absolute retract, and hence, by Proposition 1, it is atomic compact.

PROPOSITION 4. *A homeomorphism $f: A \to B$ in $\boldsymbol{S}_\sigma$ is pure-essential iff $\Psi(f): \Psi(A) \to \Psi(B)$ is pure-essential.*

Proof. If $\Psi(f)$ is pure-essential, then, by Proposition 2, f is pure. If gf is pure, then so is $\Psi(g)\Psi(f)$, and hence $\Psi(g)$ is an embedding. It is easy to see that this implies that g is an embedding, and hence f is pure-essential.

Conversely, if f is pure-essential, then Proposition 2 implies that $\Psi(f)$ is pure. If $g: \Psi(B) \to C$ is such that $g\Psi(f)$ is pure, then $\Phi(g)\Phi\Psi(f) = \Phi(g)f$ is pure (since Φ preserves all colimits, and hence, in particular, a_σ-up-directed ones), and hence $\Phi(g)$ is an embedding. But $\Psi\Phi(g) = \eta_C g$, where $\eta_C: C \to \Psi\Phi(C)$ is the front adjunction. It is easy to see that Ψ preserves embeddings, and hence $\eta_C g$ is an embedding, and this implies that g is an embedding. It follows that $\Psi(f)$ is pure-essential.

COROLLARY 1. *$B \supseteq A$ in $\boldsymbol{S}_\sigma$ is the atomic compact hull of A iff $\Psi(B)$ is the atomic compact hull of $\Psi(A)$.*

COROLLARY 2. *If σ is finitary, then Ψ reflects the existence of atomic compact hulls, i.e. if $\Psi(A)$ has an atomic compact hull in $\boldsymbol{R}_\sigma$, then A has an atomic compact hull in $\boldsymbol{S}_\sigma$.*

Proof. If $B \supseteq \Psi(A)$ is an atomic-compact pure-essential extension, then, for all pure-essential extensions $E \supseteq A$, $\Psi(E)$ is a pure-essential extension of $\Psi(A)$, and hence, by the pure-injectivity of B, $\Psi(E)$ can be embedded in B. Thus $|E| = |\Psi(E)| \leqslant |B|$, and this implies that A has only a set (up to isomorphism) of pure-essential extensions. But, for finitary structures, this is enough to guarantee the existence of an atomic compact hull of A. (The proof of this proposition for finitary algebras in Banaschewski and Nelson [2] can be modified to produce a proof for finitary structures.) Note that this proof does not work for infinitary structures; the two-element $\aleph_0$-Boolean algebra has no proper pure-essential extensions in the class of all $\aleph_0$-Boolean algebras, but has no atomic-compact hull (see Nelson [5]).

Corollary 2 is rather surprising in view of the fact that Ψ does not reflect the existence of atomic compact extensions. Namely, every relational structure (finitary or infinitary) has a one-point extension which is atomic compact: if $\big(A, (R_i)_{i\in I}\big)$ is a relational structure, then it is easy to see that the relational structure $\big(A\cup\{0\}, (T_i)_{i\in I}\big)$, where $0\notin A$ and

$$T_i = R_i \cup \big\{a \in (A\cup\{0\})^{\lambda_i} \,|\, a(\mu) = 0 \text{ for some } \mu\big\}$$

is atomic compact. However, it is well known that not every structure has an atomic compact extension; for instance, there is an example in Węglorz [9] of a lattice which has no equationally compact extension.

Final remark. The definition of pure-essential embedding given by Taylor in [7] is not equivalent to the one given here; he calls a pure embedding *h pure-essential* if *gh* pure implies that *g* is one-one. Let us call such a homomorphism *T-pure-essential.* It is clear that T-pure-essential is weaker than pure-essential and that for algebras they are equivalent. If, in the proof of Proposition 4, "embedding" is replaced by "one-one homomorphism" and "pure-essential" by "T-pure-essential", then the result is a proof that F preserves and reflects T-pure-essentialness. Moreover, since a finitary structure has an atomic-compact hull whenever it has any pure atomic-compact extension, it follows that Corollary 2 also holds if "atomic-compact hull" is replaced by "T-atomic-compact hull" (i.e. "T-pure-essential atomic-compact extension").

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