COLLOQUIUM MATHEMATICUM

VOL. XXVII 1973 FASC. 2

ABSOLUTE CONTINUITY OF VECTOR MEASURES

BY

K. MUSIAŁ (WROCŁAW)

It is known [1] that if μ is a measure with values in a Banach space, then there exists positive finite measure ν on the same σ-algebra with respect to which μ is absolutely continuous. For a class of measures with values in locally convex vector spaces (l.c.v.s.), a similar problem was considered in [2].

The aim of this note is to give a characterization of those measures with values in l.c.v.s. for which an analogue of the above theorems holds. Our proof is different from that given in [1] and [2].

Let $\mathscr{A}$ be a σ-ring of subsets of a non empty set S. If X is an abelian topological group and $\mu\colon \mathscr{A}\to X$ is a measure, then by $\mathscr{N}(\mu)$ we denote the σ-ideal of all $A\in\mathscr{A}$ such that, for every $B\in\mathscr{A}\cap A$, we have $\mu(B) = 0$.

Definition 1. If ν and μ are arbitrary measures taking values in groups X_1 and X_2, respectively, and $\mathscr{N}(\mu)\subset\mathscr{N}(\nu)$, then we say that ν is *absolutely continuous with respect to* μ, and we write $\nu\ll\mu$.

Definition 2. A measure $\mu\colon\mathscr{A}\to X$ satisfies the *countable chain condition* (C.C.C.) if each family of pairwise disjoint sets of non-zero measure is at most countable.

LEMMA. *Assume that, for two measures $\mu\colon\mathscr{A}\to X_2$ and $\nu\colon\mathscr{A}\to X_1$, the relation $\nu\ll\mu$ holds. If μ satisfies* C.C.C., *then for every $A\in\mathscr{A}-\mathscr{N}(\nu)$ there exists $B\in\mathscr{A}-\mathscr{N}(\nu)$ such that $B\subset A$ and $\mu\ll\nu$ on the σ-ring $\mathscr{A}\cap B$.*

Proof. Suppose that there exists $A\in\mathscr{A}-\mathscr{N}(\nu)$ such that, for every $B\in\mathscr{A}\cap A-\mathscr{N}(\nu)$, there exists $D\in B\cap\mathscr{N}(\nu)-\mathscr{N}(\mu)$. The lemma of Kuratowski-Zorn and C.C.C. give the existence of at most countable maximal family $\{D_n\}$ of pairwise disjoint sets $D_n\in A\cap\mathscr{N}(\nu)-\mathscr{N}(\mu)$. Since $\bigcup_n D_n\in\mathscr{N}(\nu)$ and $A\notin\mathscr{N}(\nu)$, so $A-\bigcup_n D_n\notin\mathscr{N}(\nu)$, but this contradicts the maximality of $\{D_n\}$.

THEOREM 1. *Let $\mu\colon\mathscr{A}\to X$ be a measure and let $K\neq\emptyset$ be a family of measures $\mu_a\colon\mathscr{A}\to X_a$. If $\mu_a\ll\mu$ for every a, and if μ satisfies* C.C.C., *then there exists at most countable family $\{\mu_n\}\subset K$ with the property*

$$\bigcap_n\mathscr{N}(\mu_n) = \bigcap_a\mathscr{N}(\mu_a).$$

Proof. It follows from the Lemma that, for every a, there exists $B_a \in \mathscr{A} - \mathscr{N}(\mu_a)$ such that $\mu \ll \mu_a$ on $\mathscr{A} \cap B_a$. The lemma of Kuratowski-Zorn and C.C.C. give the existence of at most countable maximal family $\{B_n\}$ of disjoint sets B_n corresponding to measures μ_n.

Let

$$S_1 = \bigcup_n B_n.$$

If $A \in \mathscr{A}$ and $A \cap S_1 = \emptyset$, then it is easy to see that $A \in \mathscr{N}(\mu_a)$ for every a. Take an arbitrary $A \in \mathscr{N}(\mu_n)$, $n = 1, \ldots$ We have $A \cap B_n \in \mathscr{N}(\mu)$ for all n, and hence $A \cap S_1 \in \mathscr{N}(\mu)$. The assumption $\mu_a \ll \mu$ yields now

$$A = (A - S_1) \cup (A \cap S_1) \in \mathscr{N}(\mu_a) \quad \text{for every } a.$$

Corollary 1. *Let X be a locally convex vector space and let $\mu\colon \mathscr{A} \to X$ be a measure satisfying* C.C.C. *Then there exists a sequence of functionals $\{x_n^*\} \subset X^*$ such that*

$$\mathscr{N}(\mu) = \bigcap_n \mathscr{N}(x_n^* \mu).$$

Proof. Let us observe that

$$\mathscr{N}(\mu) = \bigcap_{x^* \in X^*} \mathscr{N}(x^* \mu),$$

so that, by putting $K = \{x^* \mu\colon x^* \in X^*\}$ in Theorem 1, we get the desired result.

Theorem 2. *Let X be a locally convex vector space and let $\mu\colon \mathscr{A} \to X$ be a measure. In order that there exists a positive finite measure ν on $\mathscr{A}$ equivalent to μ (i.e. $\mu \ll \nu$ and $\nu \ll \mu$), it is necessary and sufficient that μ satisfies* C.C.C.

Proof. Necessity is obvious. Putting

$$\nu(A) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|x_n^* \mu|(A)}{1 + |x_n^* \mu|(S)},$$

where x_n^* are taken from Corollary 1, and $|\lambda|$ is the variation of a measure λ, we have also the proof of sufficiency.

Corollary 2. *If X is a metrizable locally convex space and $\mu\colon \mathscr{A} \to X$ is a measure, then there exists a positive finite measure ν on $\mathscr{A}$ equivalent to μ.*

Using the definition of perfectness (see [5]) of a measure, we obtain the following

Corollary 3. *Let X, $\mathscr{A}$ and μ be the same as in Corollary 1. Then the measure μ is perfect (compact) if and only if, for every $x^* \in X^*$, the measure $x^* \mu$ is perfect (compact). In particular, a measure with values in a metrizable locally convex vector space X, being perfect for every $x^* \in X^*$, is perfect.*

Added in proof. For alternative proofs of Corollary 2 the reader is referred to [3] and [4]. Theorem 2 has been also obtained (independently) by L. Drewnowski.

REFERENCES

[1] R. G. Bartle, N. Dunford and J. Schwartz, *Weak compactness and vector measures*, Canadian Journal of Mathematics 7 (1955), p. 289-305.

[2] H. G. Garnir et J. Schmets, *Théorie de l'intégration par rapport à une mesure dans un espace linéaire à semi-normes*, Bulletin de la Société Royale des Sciences de Liège 33 (1964), p. 381-410.

[3] J. Hoffman-Jørgensen, *Vector measures*, Mathematica Scandinavica 28 (1971), p. 5-32.

[4] I. Labuda, *Sur le théorème de Bartle-Dunford-Schwartz*, Bulletin de l'Académie Polonaise des Sciences, Série des sciences mathématiques, astronomiques et physiques, 20 (1972), p. 549-553.

[5] K. Musiał, *Inverse limits of measure spaces*, ibidem 19 (1971), p. 437-440.

INSTITUTE OF MATHEMATICS
UNIVERSITY OF WROCŁAW

Reçu par la Rédaction le 16. 3. 1972