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NUMBER OF POLYNOMIALS IN ORDERED ALGEBRAS

BY

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Recently much work has been done in describing the number of polynomials of a given arity in abstract algebras (see papers by members of the seminar of Professor G. Grätzer in Winnipeg and of Professor E. Marczewski in Wrocław). Our paper* deals with this topic for ordered algebras, especially with the following question: Given an ordered algebra $\mathfrak{A}$, what can be said about the set $S(\mathfrak{A})$ of those n for which there exists an n-ary polynomial in $\mathfrak{A}$ depending on all variables.

It is proved (Theorem 3.5) that in the case where $\mathfrak{A}$ is a bidirected algebra, this set is one of the following types: $\{0, 1, \ldots, n\}$, $\{1, 2, \ldots, n\}$, $\{0, 1, \ldots, n, \ldots\}$, $\{1, 2, \ldots, n, \ldots\}$, $\{1, 3, 4, 5, \ldots, n, \ldots\}$. In proving this, great use is made of Urbanik's paper [6] describing the sets $S(\mathfrak{A})$ for idempotent algebras. Last section of our paper deals with linearly ordered idempotent algebras without constants.

The supposition that algebras are bidirected is essential in our results. If only directedness is assumed, then there is no restriction for the type of $S(\mathfrak{A})$ (this result is a consequence of considerations about ordering some algebras constructed by G. Grätzer, J. Płonka and A. Sekanina; the proof will be given in a paper which is prepared in collaboration with A. Sekanina).

I. GENERAL DEFINITIONS AND STATEMENTS

1.1. Definition. Let $\mathfrak{A} = (A; F)$ be an algebra in the sense of [3]. Let A be ordered by some order $\leqslant$ (*order* is considered as a reflexive, antisymmetric and transitive relation). Let every $f \in F$ be *isotone* in every variable, i.e. $x_1, \ldots, x_n, x'_{i_0} \in A$, $x_{i_0} \leqslant x'_{i_0}$ imply $f(x_1, \ldots, x_{i_0}, \ldots, x_n) \leqslant f(x_1, x_2, \ldots, x'_{i_0}, \ldots, x_n)$.

Then $\mathfrak{A}$ is called an *ordered algebra.*

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We shall be mostly interested in the case where $(A, \leqslant)$ is a *bidirected set*, i.e. for every $a, b \in A$ there exist c, d such that $c \leqslant a, b \leqslant d$. Then $\mathfrak{A}$ is called a *bidirected algebra*. A special case is if $(A; \leqslant)$ is a *chain*, i.e. every two elements are comparable.

1.2. PROPOSITION. *Let $\mathfrak{A}$ be an ordered algebra. Then every polynomial over $\mathfrak{A}$ is isotone in all variables.*

Proof is clear.

1.3. Definition. Let $\mathfrak{A}$ be an algebra. Then $p_n(\mathfrak{A})$ denotes the number of all essentially n-ary polynomials in $\mathfrak{A}$. $S(\mathfrak{A})$ is the set of all those natural numbers (0 is considered as a natural number) for which $p_n(\mathfrak{A}) \neq 0$.

Let us emphasize that the identity mapping is considered as an essentially unary polynomial for $\operatorname{card} A \geqslant 2$.

II. $S(\mathfrak{A})$ FOR BIDIRECTED ALGEBRAS

2.1. PROPOSITION. *Let $\mathfrak{A}$ be a bidirected algebra and $f(x_1, \ldots, x_n)$ be a polynomial over $\mathfrak{A}$ depending on a variable x_i. Let $1 \leqslant i_1 < i_2 < \ldots < i_k \leqslant n$. Then $f(x_1, \ldots, x, \ldots, x, \ldots, x_n)$ (where x stands in place of $x_{i_1}, \ldots, x_{i_k}, x_i$) depends on x.*

Proof. Since $f(x_1, \ldots, x_n)$ depends on x_i, there exist $a_1, \ldots, a_n, a_i'$ such that $f(a_1, \ldots, a_i, \ldots, a_n) \neq f(a_1, \ldots, a_i', \ldots, a_n)$. Choose the notation so that $f(a_1, \ldots, a_i, \ldots, a_n) < f(a_1, \ldots, a_i', \ldots, a_n)$ or $f(a_1, \ldots, a_i, \ldots, a_n) \| f(a_1, \ldots, a_i', \ldots, a_n)$ ($a \| b$ means that the elements a and b are *incomparable*, i.e. neither $a \leqslant b$ nor $b \leqslant a$ holds). As A is bidirected, there exist b and c such that $b \leqslant a_{i_1}, \ldots, a_{i_k}, a_i, a_i' \leqslant c$. Then $f(a_1, \ldots, a_{i_1}, \ldots, a_i, \ldots, a_{i_k}, \ldots, a_n) \geqslant f(a_1, \ldots, b, \ldots, b, \ldots, a_n)$ (b stands for $a_{i_1}, \ldots, a_{i_k}, a_i$) $f(a_1, \ldots, a_{i_1}, \ldots, a_i', \ldots, a_{i_k}, \ldots, a_n) \leqslant f(a_1, \ldots, c, \ldots, c, \ldots, a_n)$. So, clearly, $f(a_1, \ldots, c, \ldots, c, \ldots, a_n) \neq f(a_1, \ldots, b, \ldots, b, \ldots, a_n)$.

2.2. Definition. Function with the property given in 2.1 will be called *dependence preserving*.

Thus all polynomials in a bidirected algebra are dependence preserving. The supposition of bidirectedness is essential as the following simple example shows.

Let $(A, \leqslant)$ be a three-element semilattice with $a \leqslant c, b \leqslant c, a \| b$. Put $f(a, b, a) = a$ and $f(x, y, z) = c$ otherwise. Then $(A; f)$ is ordered, $f(x, y, z)$ depends on all three variables, but $f(x, x, y) = c$.

Note. Almost everything in the sequel is a consequence of the fact that all polynomials in a bidirected algebra are dependence preserving. Nevertheless, there exist algebras $(A; F)$ in which every algebraic function is dependence preserving but which are not orderable by directed order.

For example, take a group G of type p^∞ and take as the algebra $(A; F)$ the reduct of G with the fundamental operation $g(x, y) = x+y$ (so neither taking the opposite element nor zero is a polynomial over $(A; F)$). Then, clearly, every polynomial over $(A; F)$ is a sum of type $x_1+\ldots$ $\ldots+x_1+x_2+\ldots+x_2+\ldots+x_n+\ldots+x_n$ and so dependence preserving, but 0 cannot be comparable with any other element since every element of G has a finite order. So the only ordering for the algebra $(A; F)$ is the trivial order (every two distinct elements are incomparable).

2.3. COROLLARY OF 2.1. *Let $f(x_1, \ldots, x_n)$ be an essentially n-ary polynomial, $n \geqslant 1$, over a bidirected algebra. Then $f(x, \ldots, x)$ is not constant.*

2.4. Definition. Let $\mathfrak{A}$ be an algebra, f a polynomial over $\mathfrak{A}$. Then $I(f)$ will denote the system of all polynomials over $\mathfrak{A}$ which are obtainable from f by some identifications of variables. For example, if $f = f(x_1, x_2, x_3, x_4)$, then $f(x, x, y, z)$, $f(y, y, x, x)$, $f(x, x, x, x)$ etc. belong to $I(f)$. By definition of algebraic operation, functions belonging to $I(f)$ are polynomials over $\mathfrak{A}$.

2.5. LEMMA. *Let $\mathfrak{A} = (A; F)$ be a bidirected algebra, f an essentially n-ary polynomial, $n \geqslant 4$. Then there exists an essentially $(n-1)$-ary polynomial in $I(f)$.*

Proof. Let us identify two variables (say, x_i, x_j) in $f(x_1, \ldots, x_n)$. We get a polynomial from $I(f)$ and denote it by $f_{i,j}$. By 2.1, $f_{i,j}$ is not constant.

Assume that all functions $f_{i,j}$ (under all possible choice of indices i, j) depend only on one (i.e. identified) variable. Then $f_{1,2}(x_1) = f(x_1, x_1, x_3, x_4, \ldots, x_n) = f(x_1, x_1, x_3, \ldots, x_n, x_n) = f_{n-1,n}(x_n)$, a contradiction.

Therefore, let us choose our notation so that $f(x, x, x_3, \ldots, x_n)$ depends on $x, x_k, x_{k+1}, \ldots, x_n$ and every $f_{i,j}$ depends on $n-k+2$ or less variables. If $k = 3$, the proof is finished. Suppose $k > 3$. Then:

$$f(\underbrace{x, x, \ldots, x}_{k-1\text{ times}}, x_k, \ldots, x_n) \text{ depends on } x, x_k, \ldots, x_n; \tag{1}$$

$$f(x_1, x_2, x_2, x_4, \ldots, x_k, \ldots, x_n) \text{ does not depend on } x_1 \tag{2}$$

(by 2.1 and the assumption on k).

The function $f(x_1, x_2, x_k, x_4, \ldots, x_{k-1}, x_k, x_{k+1}, \ldots, x_n)$ depends on $x_k, x_{k+1}, \ldots, x_n$. In fact, dependence of the function $f(x, x, x_3, \ldots, x_{k-1}, x_k, \ldots, x_n)$ on $x_k, \ldots, x_n$ and independence of the same function on x_3 imply dependence of $f(x_1, x_2, x_k, x_4, \ldots, x_{k-1}, x_k, \ldots, x_n)$ on $x_k, \ldots, x_n$. So $f(x_1, x_2, x_k, x_4, \ldots, x_{k-1}, x_k, \ldots, x_n)$ can depend at most on one of the variables x_1, x_2 (by choice of k).

Suppose our notation is chosen so that it does not depend on x_2. Then we have

$$\begin{aligned} f(x, x, x_3, \ldots, x_{k-1}, x_k, \ldots, x_n) &= f(x, x, x_k, x_4, \ldots, x_{k-1}, x_k, \ldots, x_n) \\ &= f(x, x_k, x_k, x_4, \ldots, x_{k-1}, x_k, \ldots, x_n) \\ &= f(x_k, x_k, x_k, x_4, \ldots, x_{k-1}, x_k, \ldots, x_n) \end{aligned}$$

(the first equality because $f(x, x, x_3, \ldots, x_n)$ does not depend on x_3, the second — as $f(x, x_2, x_k, x_4, \ldots, x_{k-1}, x_k, \ldots, x_n)$ does not depend on x_2, the third — by (2)). Thus $f(x, x, x_3, \ldots, x_n)$ does not depend on x, a contradiction.

2.6. COROLLARY. *Let $\mathfrak{A}$ be a bidirected algebra containing an essentially n-ary polynomial for some $n \geqslant 3$. Then $\{3, 4, \ldots, n\} \subset S(\mathfrak{A})$.*

2.7. LEMMA. *Let $2 \notin S(\mathfrak{A})$, $3 \in S(\mathfrak{A})$ for some bidirected algebra $\mathfrak{A}$. Then $n \in S(\mathfrak{A})$ for all $n \geqslant 3$.*

Proof. Let $f(x_1, x_2, x_3)$ be some essentially ternary polynomial over $\mathfrak{A}$. Write $g(x) = f(x, x, x)$. By 2.3, $g(x)$ is not constant. We have

$$f(x, x, y) = f(x, y, x) = f(y, x, x) = g(x). \tag{3}$$

First we prove

$$f\big(x_1, x_2, f(x_3, x_4, x_5)\big) \text{ depends on } x_1, x_2. \tag{4}$$

Assume that $f\big(x_1, x_2, f(x_3, x_4, x_5)\big)$ does not depend on x_1; then

$$f\big(x_1, x_2, f(x_3, x_4, x_5)\big) = f\big(f(x_3, x_4, x_5), x_2, f(x_3, x_4, x_5)\big) = g\big(f(x_3, x_4, x_5)\big).$$

So $f\big(x_1, x_2, f(x_3, x_4, x_5)\big)$ does not depend on x_2, but then $f\big(x_1, x_2, f(x_3, x_4, x_5)\big) = f\big(x_1, x_1, f(x_3, x_4, x_5)\big) = g(x_1)$, a contradiction.

Similarly, the dependence on x_2 of $f\big(x_1, x_2, f(x_3, x_4, x_5)\big)$ can be proved.

Having proved (4) we deduce that

$$f\big(x_1, x_2, f(x_1, x_1, x_1)\big) \text{ depends on } x_1 \text{ and so not on } x_2. \tag{5}$$

Thus

$$g(x_1) = f\big(x_1, x_1, f(x_1, x_1, x_1)\big) = f\big(x_1, f(x_1, x_1, x_1), f(x_1, x_1, x_1)\big) = g\big(g(x_1)\big).$$

Hence

$$g(x) = g\big(g(x)\big). \tag{6}$$

Take now a complete iteration (see [4]) of f:

$$\begin{aligned} (7)\quad & f^n(x_1, \ldots, x_{3^n}) \\ &= f\Big(f\big(f(\ldots), f(\ldots), f(\ldots)\big), f\big(f(\ldots), f(\ldots), f(\ldots)\big), f\big(f(\ldots), f(\ldots), f(\ldots)\big)\Big). \end{aligned}$$

We shall prove that $f^n(x_1, \ldots, x_{3^n})$ depends on all variables.

In the first place, $f^n(x_1, \ldots, x_{3^n})$ is not constant as $f^n(x, \ldots, x) = g\big(g(g(\ldots))\big) = g(x)$ by (6). Let $f^n(x_1, \ldots, x_{3^n})$ depend on some variable which occurs in $f(x_k, x_{k+1}, x_{k+2})$ contained in expression (7) for $f^n(x_1, \ldots, x_{3^n})$.

We shall show that $f^n(x_1, \ldots, x_{3^n})$ depends on all variables x_k, x_{k+1}, x_{k+2}.

Assume it depends only on x_k. Then $g(x_{k+1}) = f(x_k, x_{k+1}, x_{k+1})$ and after the substitution of x_{k+1} for x_{k+2} in the right-hand side of (7) (which we can do since $f^n(x_1, \ldots, x_{3^n})$ does not depend on x_{k+2}), it follows that $f^n(x_1, \ldots, x_{3^n})$ does not depend on x_k.

Assume $f^n(x_1, \ldots, x_{3^n})$ depends on x_k, x_{k+1}, but not on x_{k+1}. Then $f(x_k, x_{k+1}, x_{k+1}) = g(x_{k+1})$, and substituting this in the right-hand side of (7) for $f(x_k, x_{k+1}, x_{k+2})$ we see that $f^n(x_1, \ldots, x_{3^n})$ does not depend on x_k.

Formally we can write (7) as

$$(8)\quad f\Big(f\big(\ldots f\big(f^k(x_1, x_2, \ldots, x_{3^k}), f^k(x_{3^k+1}, \ldots, x_{2\cdot 3^k}), f^k(x_{2\cdot 3^k+1}, \ldots, x_{3\cdot 3^k})\big), \\ f\big(f^k(x_{3\cdot 3^k+1}, \ldots, x_{4\cdot 3^k}), \ldots\big), \ldots, f^k(x_{(3^{n-k}-1)\cdot 3^k+1}, \ldots, x_{3^n})\big)\Big),$$

where $1 \leqslant k \leqslant n$.

Suppose we have proved

(9) If $f^n(x_1, \ldots, x_{3^n})$ depends on one variable from $x_{l\cdot 3^k+1}, \ldots, x_{(l+1)\cdot 3^k}$, then $f^n(x_1, \ldots, x_{3^n})$ depends on all of them.

For $k = 1$ this assertion is the same as that just proved above. So let us prove (9) for $k+1$.

Assume for example that $f^n(x_1, \ldots, x_{3^n})$ depends on $x_1, \ldots, x_{3^k}$ and not on $x_{3^k+1}, \ldots, x_{2\cdot 3^k}$. Then put $x_{3^k+1} = x_{2\cdot 3^k+1}, \ldots, x_{2\cdot 3^k} = x_{3\cdot 3^k}$ and substitute this in the right-hand side of (8) (we can do this without a change of the left-hand side). But then

$$f\big(f^k(x_1, \ldots, x_{3^k}), f^k(x_{2\cdot 3^k+1}, \ldots, x_{3\cdot 3^k}), \; f^k(x_{2\cdot 3^k+1}, \ldots, x_{3\cdot 3^k})\big) \\ = g\big(f^k(x_{2\cdot 3^k+1}, \ldots, x_{3\cdot 3^k})\big)$$

so $f^n(x_1, \ldots, x_{3^n})$ does not depend on $x_1, \ldots, x_{3^k}$, a contradiction.

It follows by induction that $f^n(x_1, \ldots, x_{3^n})$ depends on all variables. Our assertion 2.7 is now a consequence of 2.6.

2.8. Corollary of 2.6 and 2.7. *Therefore* $S(\mathfrak{A})$ *must be one of the following types*:

$$(10)\qquad S(\mathfrak{A}) = \{0, 1, 2, \ldots, n\};$$

$$(11)\qquad S(\mathfrak{A}) = \{1, 2, \ldots, n\};$$

$$(12)\qquad S(\mathfrak{A}) = \{0, 1, 2, \ldots\};$$

$$(13)\qquad S(\mathfrak{A}) = \{1, 2, \ldots\};$$

$$(14)\qquad S(\mathfrak{A}) = \{0, 1, 3, 4, \ldots\};$$

$$(15)\qquad S(\mathfrak{A}) = \{1, 3, 4, \ldots\}.$$

Assume that a bidirected algebra $\mathfrak{A}$ fulfils (14). Let $f(x, y, z)$ be an essentially ternary polynomial of $\mathfrak{A}$ and a an algebraic constant of $\mathfrak{A}$. Then $f(x, y, a)$ is a polynomial in $\mathfrak{A}$. Assume it does not depend on x. Then (because $f(x, y, z)$ is dependence preserving) $f(y, y, y) = f(y, y, a) = f(a, y, a) = f(a, a, a)$, a contradiction to 2.3. Similarly dependence on y can be proved.

We shall prove in the next section that (10)-(13) and (15) can really occur.

III. IDEMPOTENT ALGEBRAS

An algebra $\mathfrak{A}$ is called *idempotent* if for every polynomial $f(x_1, \ldots, x_n)$ over $\mathfrak{A}$, which is not a constant, $f(x, \ldots, x) = x$.

In Urbanik's paper [6] the $S(\mathfrak{A})$ are described for idempotent algebras without constants (notice that Urbanik does not include 1 in $S(\mathfrak{A})$).

In our notation Urbanik's theorem 1 reads as follows:

For each idempotent algebra $\mathfrak{A}$ one of the following cases holds:

(i) $$S(\mathfrak{A}) = \{1\};$$

(ii) $$S(\mathfrak{A}) = \{1, 3, 5, \ldots, 2n+1, \ldots\};$$

(ii) $$S(\mathfrak{A}) = \{1, 2, 3, \ldots, n\}, \quad n \geqslant 2;$$

(iv) $$S(\mathfrak{A}) = \{1, m, m+1, m+2, \ldots\}, \quad m \geqslant 2;$$

(v) $$S(\mathfrak{A}) = \{1, 3, 5, \ldots, 2n+1, \ldots\} \cup \{m, m+1, \ldots\}, \quad m \geqslant 5.$$

(vi) $$S(\mathfrak{A}) = \{1, 2, 3, \ldots, n\} \cup \{m, m+1, m+2, \ldots\}, \quad m > n+1, \quad n \geqslant 2.$$

In our case, in view of 2.8, we are only interested in the cases (i), (iii) and (iv) of Urbanik's theorem. From (iv) there remain for us only cases where $m \leqslant 3$.

Case (i) concerns with a trivial algebra. Every such algebra can be considered as a chain.

By Theorem 2.2 in [6], case (iii) concerns with diagonal algebras. The next assertion describes all ordered diagonal algebras.

3.1. PROPOSITION. *Let $(A; f)$ be an ordered n-dimensional diagonal algebra, $A = B_1 \times \ldots \times B_n$. Then there exist orders on $B_1, \ldots, B_n$ such that A is the cardinal product* (see [1], p. 55) *of $B_1, \ldots, B_n$. Conversely, given arbitrary orders on $B_1, \ldots, B_n$, and taking A as the cardinal product of $B_1, \ldots, B_n$, $(A; f)$ is an ordered algebra.*

Proof. Let $(A; f)$ be an ordered n-dimensional diagonal algebra. Let $\leqslant$ denote the order of A. Let

(16) $$\langle x_1, \ldots, x_n \rangle \leqslant \langle y_1, \ldots, y_n \rangle .$$

Then for arbitrary $\langle x_1^1, \ldots, x_n^1\rangle, \ldots, \langle x_1^{n-1}, \ldots, x_n^{n-1}\rangle \in A$ we have

$$f(\langle x_1^1, \ldots, x_n^1\rangle, \ldots, \langle x_1^{n-1}, \ldots, x_n^{n-1}\rangle, \langle x_1, \ldots, x_n\rangle) \leqslant f(\langle x_1^1, \ldots, x_n^1\rangle, \ldots, \langle x_1^{n-1}, \ldots, x_n^{n-1}\rangle, \langle y_1, \ldots, y_n\rangle),$$

so

$$\langle x_1^1, \ldots, x_{n-1}^{n-1}, x_n\rangle \leqslant \langle x_1^1, \ldots, x_{n-1}^{n-1}, y_n\rangle. \tag{17}$$

Thus (16) implies (17) for arbitrary $x_1^1 \in B_1, \ldots, x_{n-1}^{n-1} \in B_{n-1}$. If (17) is true we write $x_n \leqslant_n y_n$.

Since $\leqslant$ is an order on A, $\leqslant_n$ is an order on B_n. Similarly, $\leqslant_i$ is defined for $i = 1, 2, \ldots, n-1$. By definition, (16) implies $x_i \leqslant_i y_i$ for $i = 1, \ldots, n$.

Conversely, let $x_i \leqslant_i y_i$ for $i = 1, \ldots, n$. Let $z_i \in B_i$, $i = 1, \ldots, n$, be quite arbitrary. Then

$$\langle z_1, \ldots, z_{i-1}, x_i, z_{i+1}, \ldots, z_n\rangle \leqslant \langle z_1, \ldots, z_{i-1}, y_i, z_{i+1}, \ldots, z_n\rangle. \tag{18}$$

Then substituting in the operation f the right-hand sides and left-hand sides of (18) for $i = 1, \ldots, n$, we get $\langle x_1, \ldots, x_n\rangle \leqslant \langle y_1, \ldots, y_n\rangle$.

Conversely, let $\precsim_i$ be some order of B_i, $i = 1, \ldots, n$. Let A be the cardinal product of B_i. Denote the resulting order on A as $\leqslant$. Assume

$$\langle x_1^i, \ldots, x_n^i\rangle \leqslant \langle y_1^i, \ldots, y_n^i\rangle, \quad i = 1, \ldots, n. \tag{19}$$

(19) implies $x_i^i \precsim_i y_i^i$ for $i = 1, \ldots, n$. Therefore $\langle x_1^1, \ldots, x_n^n\rangle \leqslant \langle y_1^1, \ldots, y_n^n\rangle$. But

$$f(\langle x_1^1, \ldots, x_n^1\rangle, \ldots, \langle x_1^n, \ldots, x_n^n\rangle) = \langle x_1^1, \ldots, x_n^n\rangle,$$
$$f(\langle y_1^1, \ldots, y_n^1\rangle, \ldots, \langle y_1^n, \ldots, y_n^n\rangle) = \langle y_1^1, \ldots, y_n^n\rangle.$$

This shows

$$f(\langle x_1^1, \ldots, x_n^1\rangle, \ldots, \langle x_1^n, \ldots, x_n^n\rangle) \leqslant f(\langle y_1^1, \ldots, y_n^1\rangle, \ldots, \langle y_1^n, \ldots, y_n^n\rangle).$$

Therefore $(A; f)$ is an ordered algebra.

Note. Taking $\leqslant_i$ in the first part of the proof as $\precsim_i$ we get the original order on A.

For the order $\leqslant$, defined in the second part of the proof, $\leqslant_i$ coincides with $\precsim_i$.

3.2. COROLLARY. *Let $\mathfrak{A} = (A; F)$ be an n-dimensional diagonal algebra, $n \geqslant 2$. Then $\mathfrak{A}$ cannot be linearly ordered (i.e. ordered in a chain).*

Case (iv) in Urbanik's theorem for $m = 2$ is a general one. An example is a chain having at least two elements with $\max(x, y)$ as a fundamental operation. We shall now be interested in the case where $S(\mathfrak{A}) = \{1, 3, 4, 5, \ldots\}$.

3.3. PROPOSITION. *Let S be a lattice containing at least two elements. Let* $f(x, y, z) = (x \vee y) \wedge (x \vee z) \wedge (y \vee z)$ *and* $\mathfrak{A} = (S; f)$. *Then* $S(\mathfrak{A}) = \{1, 3, 4, 5, \ldots\}$.

Proof. $f(x, y, z)$ is clearly essentially ternary, idempotent and isotone in each variable. By 2.8, it is sufficient to prove that $2 \notin S(\mathfrak{A})$. Assume that a polynomial symbol $F(x_1, \ldots, x_n)$ (for the notion of polynomial symbol see [2]) for $\mathfrak{A}$ gives a function which is essentially binary. Let the corresponding function depend on x_1, x_2. Calculation of $F(x_1, \ldots, x_n)$ is obtained by calculations of expressions of the following types:

$$f(f(y_1, y_2, y_3), y_4, y_5),$$
$$f(f(y_1, y_2, y_3), f(y_4, y_5, y_6), y_7),$$
$$f(f(y_1, y_2, y_3), f(y_4, y_5, y_6), f(y_7, y_8, y_9))$$

and obvious modifications of these. But keeping in mind that we have a function depending only on x_1, x_2, we ascertain that every one of these expressions is equal to $f(x_1, x_1, x_2)$ or $f(x_1, x_2, x_1)$ or $f(x_2, x_1, x_1)$ or $f(x_1, x_1, x_1)$ or to an expression obtained from these after the interchanging of x_2 for x_1. So as a final result we get x_1 or x_2, a contradiction. Therefore $S(\mathfrak{A}) = \{1, 3, 4, \ldots\}$.

For chains we shall prove later the statement converse to 3.3. Recall that f defined in 3.3 is called a *median* on S (see [1]).

Now in order to prove that the types of $S(\mathfrak{A})$ given by (10)-(13) and (15) can really occur, it remains to prove the following proposition:

3.4. LEMMA. *Let $\mathfrak{A}$ be an algebra which does not contain an essentially m-ary polynomial with $m > n_0$ (n_0 a given integer). Add to $\mathfrak{A}$ as a new polynomial a constant a. There are no polynomials of arity $m > n_0$ in the new algebra $\mathfrak{B}$.*

Proof. We shall prove:

Every polynomial $f(x_1, \ldots, x_n)$ over $\mathfrak{B}$ is of type $g(x_1, \ldots, x_n, a)$, where g is a polynomial over $\mathfrak{A}$.

Let $F(x_1, \ldots, x_n)$ be a polynomial symbol giving $f(x_1, \ldots, x_n)$. For every occurrence of a put a new variable x_{n+1}. In this way we obtain a new polynomial symbol $G(x_1, \ldots, x_{n+1})$ over $\mathfrak{A}$. Let $g(x_1, \ldots, x_{n+1})$ be determined by $G(x_1, \ldots, x_{n+1})$. Then $g(x_1, \ldots, x_n, a) = f(x_1, \ldots, x_n)$.

Thus, by 3.4 and the representability of (11) by diagonal algebras, we get the representability of (10). Let us formulate our result in the following statement:

3.5. MAIN THEOREM. *The types of $S(\mathfrak{A})$ given by* (10)-(13) *and* (15) *are the only possible types and for each of them there exists a bidirected algebra $\mathfrak{A}$ with $S(\mathfrak{A})$ of that form.*

IV. ON IDEMPOTENT ALGEBRAS $\mathfrak{A}$ WITHOUT CONSTANTS WHICH ARE LINEARLY ORDERED AND FOR WHICH $S(\mathfrak{A}) = \{1, 3, 4, \ldots\}$

4.1. LEMMA. *Let f be the median on a chain S. Then $f(x_1, x_2, x_3)$ is the element from x_1, x_2, x_3 which is between (in the sense of $\leqslant$) the other two.*

We shall prove

4.2. PROPOSITION. *Let $\mathfrak{A} = (A; F)$ be a linearly ordered idempotent algebra without constants for which $S(\mathfrak{A}) = \{1, 3, 4, \ldots\}$. Let f be the median on A. Then $\mathfrak{A} = (A; f)$.*

The proof will be accomplished by means of several lemmas. In all of them $\mathfrak{A}$ fulfils the suppositions of 4.2.

4.3. LEMMA. *Let $g(x, y, z)$ be an essentially ternary polynomial over $\mathfrak{A}$. Then $g(x, y, z) = f(x, y, z)$.*

Proof. Let $a < b < c$. Then $a \leqslant g(a, b, c) \leqslant g(b, b, c) = b$, $c \geqslant g(a, b, c) \geqslant g(a, b, b) = b$. So $g(a, b, c) = b$. Similar results can be obtained for other permutations of a, b, c.

Further, $g(x, x, x) = g(x, x, y) = g(x, y, x) = g(y, x, x) = x$, so indeed $f = g$.

4.4. LEMMA. *Let g be a polynomial over $\mathfrak{A}$, $g(a_1, a_2, \ldots, a_n) = a_1$, where $a_1 \leqslant a_2 \leqslant \ldots \leqslant a_n$. Let j be the first index for which $a_j > a_1$. Then $g(\underbrace{x, \ldots, x}_{j-1 \text{ times}}, x_j, \ldots, x_n)$ does not depend on $x_j, \ldots, x_n$ (so $g(x, \ldots, x, x_j, \ldots$ $\ldots, x_n) = x$).*

Proof. Assume that $g(\underbrace{x, \ldots, x}_{j-1 \text{ times}}, x_j, \ldots, x_n)$ depends on a variable from $x_j, \ldots, x_n$. Then

$$g(\underbrace{x, \ldots, x}_{j-1 \text{ times}}, y, \ldots, y) = y.$$

But $a_1 = g(a_1, \ldots, a_1, a_j, \ldots, a_n) \geqslant g(a_1, \ldots, a_1, a_j, \ldots, a_j) = a_j$, a contradiction.

4.5. LEMMA. *Let g be a polynomial over $\mathfrak{A}$. Then*

$$g(a_1, \ldots, a_n) \in \{a_1, \ldots, a_n\}.$$

Proof. This is clearly true if g is unary. Suppose it is true for all m less than given $n > 1$. For that reason we need deal only with the case where $a_1, \ldots, a_n$ are different. We can suppose that $a_1 < \ldots < a_n$ and that g is essentially n-ary. Then

$$a_1 = g(a_1, \ldots, a_1) \leqslant g(a_1, \ldots, a_n) \leqslant g(a_n, \ldots, a_n) = a_n. \tag{20}$$

Put $b = g(a_1, \ldots, a_n)$. Suppose our assertion is not true. Then there exists a j such that $a_j < b < a_{j+1}$.

Assume $j \geqslant 2$.

Take the function $g(\underbrace{x, \dots, x}_{j \text{ times}}, x_{j+1}, \dots, x_n)$. Then

$$g(a_1, \dots, a_1, a_{j+1}, \dots, a_n) \leqslant b \leqslant g(a_j, \dots, a_j, a_{j+1}, \dots, a_n).$$

As the arity of $g(x, x, \dots, x, x_{j+1}, \dots, x_n)$ is less than n, we have

$$g(a_1, \dots, a_1, a_{j+1}, \dots, a_n) = a_1, \tag{21}$$

$$g(a_j, \dots, a_j, a_{j+1}, \dots, a_n) \geqslant a_{j+1}. \tag{22}$$

(21) implies, in view of 4.4, $g(x, \dots, x, x_{j+1}, \dots, x_n) = x$. This contradicts (22).

If $j = 1$, we proceed dually.

4.6. LEMMA. *Let $g(x_1, \dots, x_n)$ be a polynomial over $\mathfrak{A}$. Let $a_1 \leqslant a_2 \leqslant \dots \leqslant a_n$, $b_1 \leqslant b_2 \leqslant \dots \leqslant b_n$ and let any equality $a_i = a_j$ imply $b_i = b_j$. Then $g(a_1, \dots, a_n) = a_k$ implies $g(b_1, \dots, b_n) = b_k$.*

Proof. In view of 4.4, the assertion is true if $k = 1$ or $k = n$. Let us suppose $1 < k < n$. The assertion is trivial for unary functions. Suppose $g(x_1, \dots, x_n)$ is essentially n-ary, $n > 1$, and $a_1 < a_2 < \dots < a_n$ (otherwise we can deal with a function of less arity).

We shall prove that $g(\underbrace{x, x, \dots, x}_{k-1 \text{ times}}, y, z, \dots, z)$ depends on x, y, z.

Because $g(x_1, x_2, \dots, x_n)$ is dependence preserving, $g(\underbrace{x, \dots, x}_{k-1 \text{ times}}, y, z, \dots, z)$ depends on x or z.

Let it depend only on x. Then $g(x, \dots, x, y, z, \dots, z) = x$, but $g(a_{k-1}, \dots, a_{k-1}, a_k, a_n, \dots, a_n) \geqslant a_k > a_{k-1}$. So $g(x, \dots, x, y, z, \dots, z)$ depends on x, y, z. The proof is similar for dependence on z.

Suppose $g(b_1, \dots, b_n) = b_l \neq b_k$. Suppose e.g. $k < l$ (the second case is dual). Then $g(b_{k-1}, \dots, b_{k-1}, b_k, b_n, \dots, b_n) \geqslant b_l$, so, by 4.5, $g(b_{k-1}, \dots, b_{k-1}, b_k, b_n, \dots, b_n) = b_n$. In view of the theorem dual to 4.4, $g(x_1, \dots, x_k, z, \dots, z) = z$, a contradiction to the fact that $g(\underbrace{x, \dots, x}_{k-1 \text{ times}}, y, z, \dots, z)$ depends on x, y, z.

4.7. LEMMA. *Let g be a polynomial over $\mathfrak{A}$. Let $a_1 \leqslant a_2 \leqslant \dots \leqslant a_n$, $b_1 \geqslant b_2 \geqslant \dots \geqslant b_n$ and let any equality $a_i = a_j$ imply $b_i = b_j$. Then $g(a_1, \dots, a_n) = a_k$ implies $g(b_1, \dots, b_n) = b_k$.*

Proof. Just as above we can suppose that $g(x_1, \dots, x_n)$ is essentially n-ary, $a_1 < \dots < a_n$ and $1 \neq k \neq n$. By the same method as in 4.6, $g(\underbrace{x, x, \dots, x}_{k-1 \text{ times}}, y, z, \dots, z)$ is essentially ternary.

Assume $g(b_1, \dots, b_n) = b_l \neq b_k$. Suppose again $k < l$. Then $g(b_{k-1}, \dots, b_{k-1}, b_k, b_n, \dots, b_n) = b_n$, which gives a contradiction as in the end of the proof of 4.6.

Polynomials satisfying 4.5 are called *quasitrivial* ([3]), polynomials satisfying 4.6 will be called *transferable* and polynomials with the property given in 4.7 will be called *convertible*.

The following assertion is clear:

4.8. LEMMA. *Every subset of $\mathfrak{A}$ is a subalgebra of $\mathfrak{A}$. In every subalgebra of $\mathfrak{A}$ every polynomial is dependence preserving, quasitrivial, transferable and convertible.*

4.9. LEMMA. *Let $\mathfrak{B}$ be a subalgebra of $\mathfrak{A}$. Then $p_2(\mathfrak{B}) = 0$.*

Proof. Let $g(x_1, x_2, \ldots, x_n)$ be an essentially n-ary polynomial over $\mathfrak{A}$ which depends as a function over $\mathfrak{B}$ only on two variables, say on x_1, x_2. Clearly, $n \geqslant 3$.

Take some $a, b \in \mathfrak{B}$. Then $a = g(a, b, a, \ldots, a) = g(a, b, b, \ldots, b) = b$, so card $\mathfrak{B} = 1$. Thus $g(x_1, \ldots, x_n)$ cannot depend on two variables on $\mathfrak{B}$.

Take in the sequel two elements $0, 1 \in A, 0 < 1$ and let $\mathfrak{B}$ be the subalgebra of $\mathfrak{A}$ on $\{0, 1\}$.

Write $0' = 1, 1' = 0$. Then, by 4.7 (adopting suitable notation concerning variables and polynomial symbol), for $x_1, \ldots, x_n \in \{0, 1\}$ we have

$$g(x_1', \ldots, x_n') = \big(g(x_1, \ldots, x_n)\big)',$$
$$g(0, \ldots, 0) = 0$$

for every polynomial g of $\mathfrak{A}$.

Therefore, g is homogeneous on $\mathfrak{B}$ in the sence of paper [5], p. 200. By [5] there are only three non-trivial algebras on $\{0, 1\}$ in which all polynomials are homogeneous: the Post algebras $\mathfrak{P}_*, \mathfrak{P}^*, \mathfrak{P}$. But fundametal operations for $\mathfrak{P}_*$ and $\mathfrak{P}$ are not dependence preserving. Consequently, if we prove that $\mathfrak{B}$ is not trivial, we have $\mathfrak{B} = \mathfrak{P}^*$, so $\mathfrak{B} = (\{0, 1\}; f|\{0, 1\})$ (f is the median on $\mathfrak{A}$). Nontriviality of $\mathfrak{B}$ is an immediate consequence of the following assertion 4.10. For simplicity, if g is a polynomial over $\mathfrak{A}$, then $\bar{g} = g|\{0, 1\}$ means restriction of g to $\{0, 1\}$. In particular, $\bar{f}$ is the median on $\{0, 1\}$.

4.10. LEMMA. *Let g, h be two different polynomials over $\mathfrak{A}$. Then $\bar{g} \neq \bar{h}$.*

Proof. Let $g(a_1, a_2, \ldots, a_n) = a_i \neq a_j = h(a_1, \ldots, a_n)$. We can suppose that $a_1 \leqslant a_2 \leqslant \ldots \leqslant a_n, a_i < a_j$ (so $i < j$) and $a_i \neq a_{i+1}$. Then $g(a_1, \ldots, a_1, a_{i+1}, \ldots, a_{i+1}) \leqslant a_i$, so $g(a_1, \ldots, a_1, a_{i+1}, \ldots, a_{i+1}) = a_1$ and, therefore, $g(\underbrace{x, \ldots, x}_{i \text{ times}}, y, \ldots, y) = x$.

Similarly, $h(\underbrace{a_i, \ldots, a_i}_{i \text{ times}}, a_n, \ldots, a_n) = a_n$, so $h(\underbrace{x, \ldots, x}_{i \text{ times}}, y, \ldots, y) = y$. Thus $g(\underbrace{0, \ldots, 0}_{i \text{ times}}, 1, \ldots, 1) = 0 \neq 1 = h(\underbrace{0, \ldots, 0}_{i \text{ times}}, 1, \ldots, 1)$. So $\bar{g} \neq \bar{h}$.

4.11. Proof of 4.2. Let g be a polynomial over $\mathfrak{A}$. Let $F(x_1, \ldots, x_n)$ be a polynomial symbol in $\bar{f}$ yielding $\bar{g}$. Then the same polynomial symbol in f gives some polynomial h over $\mathfrak{A}$ with $\bar{h} = \bar{g}$. By 4.10, $h = g$. Therefore $\mathfrak{A} = (A; f)$.

PROBLEM. Is 4.2 valid in the case where the order on A is (only) assumed to be that of a distributive lattice? (**P 703**).

Let us formulate the concluding theorem concerning linearly ordered idempotent algebras without constants:

4.12. *Let $\mathfrak{A}$ be a linearly ordered idempotent algebra without constants. There are three possibilities*:

1. $\mathfrak{A}$ *is trivial and* $S(\mathfrak{A}) = \{1\}$.
2. $\mathfrak{A} = (A; f)$, *where f is the median and* $S(\mathfrak{A}) = \{1, 3, 4, 5, \ldots\}$.
3. $S(\mathfrak{A}) = \{1, 2, 3, 4, 5, \ldots\}$.

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