COLLOQUIUM MATHEMATICUM

VOL. LVII 1989 FASC. 2

UNIVERSAL EMBEDDINGS OF $l^1(\alpha)$ INTO THE SPACE OF CONTINUOUS FUNCTIONS ON A PRODUCT SPACE

BY

N. KALAMIDAS AND TH. ZACHARIADES (ATHENS)

Preliminaries. The ordinals are defined in such a way that an *ordinal* is the set of smaller ordinals. A *cardinal* is an ordinal not in one-to-one correspondence with any smaller ordinal. The *cofinality* of a cardinal α, denoted by $\mathrm{cf}(\alpha)$, is the least cardinal β such that α is a cardinal sum of β-many cardinals, each smaller than α.

A cardinal α is *regular* if $\alpha = \mathrm{cf}(\alpha)$, and *singular* otherwise. We denote by α^+ the least cardinal which is strictly greater than α, and by ω the first infinite cardinal.

If α and β are cardinals, we denote by $\alpha^{\underline{\beta}}$ the cardinal sum $\sum_{p<\beta} \alpha^p$.

The cardinality of a set A is denoted by $|A|$. We denote by $\mathscr{P}_\alpha(A)$ the set of those subsets of A that have cardinality less than α, and by $\mathscr{P}(A)$ the set of all subsets of A.

Let α and $\varkappa$ be cardinals. We say that α is *strongly $\varkappa$-inaccessible* if $\beta^\lambda < \alpha$ for every $\beta < \alpha$ and $\lambda < \varkappa$. If, in addition, $\alpha > \varkappa$, we write $\alpha \gg \varkappa$.

For set-theoretic background we refer the reader to [4].

0.1. THEOREM (Erdös–Rado). *Let $\alpha > \omega$ be a regular cardinal and*

$$\{N_\xi\colon \xi < \alpha\} \subset \mathscr{P}_\omega(\alpha).$$

Then there are $A \subset \alpha$, with $|A| = \alpha$, and $N \subset \alpha$ such that

$$N_{\xi_1} \cap N_{\xi_2} = N \quad \text{for } \xi_1, \xi_2 \in A,\ \xi_1 \neq \xi_2.$$

0.2. PROPOSITION. *Let α, β be cardinals, $\alpha > \beta \geqslant \omega^+$, let*

$$\{J_\xi\colon \xi < \alpha\} \subset \mathscr{P}_\beta(\alpha) \quad \text{and} \quad \{N_\xi\colon \xi < \alpha\} \subset \mathscr{P}(\alpha)$$

with $N_\xi \cap N_\eta = \emptyset$ for every $\xi < \eta < \alpha$. Then there exists $A \subset \alpha$, with $|A| = \alpha$, such that

$$N_\xi \cap J_\eta = \emptyset \quad \text{for every } \xi, \eta \in A,\ \xi \neq \eta.$$

The above proposition in the case $\beta = \omega^+$ is contained in [1]. To prove it we use the Hajnal free set theorem.

All topological spaces in this paper are assumed to be infinite and Hausdorff. Let X be a topological space and α an infinite cardinal. We say that X has *precaliber* α if every family $\{U_\xi\colon \xi < \alpha\}$ of non-empty open subsets of X contains a subfamily with the same cardinality and the finite intersection property.

If X has precaliber α, then $\mathrm{cf}(\alpha) > \omega$ (Corollary 2.25 in [5]). X is said to have *caliber* α if every family $\{U_\xi\colon \xi < \alpha\}$ of non-empty open subsets of X contains a subfamily of the same cardinality with non-empty intersection.

The *Souslin number* $S(X)$ of X is defined to be the smallest cardinal α such that there is no family of α-many pairwise disjoint non-empty open subsets of X. By the Erdös–Tarski theorem, $S(X)$ is an uncountable regular cardinal.

X is called *pseudo-α-compact* if for every family of α-many non-empty open subsets of X there exists $x \in X$ such that every neighbourhood of x meets infinitely many sets from the family.

We set

$$r(X) = \min\{\alpha\colon X \text{ is pseudo-}\alpha\text{-compact}\},$$

$$\mathrm{ca}(X) = \min\{\alpha\colon X \text{ has caliber } \alpha\}.$$

Then it is clear that

$$r(X) \leqslant \mathrm{ca}(X) \leqslant d(X)^+,$$

where $d(X)$ is the density character of X.

All Banach spaces in this paper are assumed to be real. Let X be a topological space. By $C^*(X)$ we denote the Banach space of all real-valued bounded continuous functions on X with the supremum norm. If Y is a Banach space and α a cardinal, we say that $l_1(\alpha)$ *embeds universally* in Y if for any closed subspace Z of Y, with $\dim Z = \alpha$, there exists an isomorphic embedding of $l_1(\alpha)$ into Z. It is clear that an isomorphic embedding of $l_1(\alpha)$ into Z exists iff there exist a uniformly bounded family $\{z_\xi\colon \xi < \alpha\} \subset Z$ and a constant $M > 0$ such that

$$\Big\| \sum_{i=1}^{n} c_i z_{\xi_i} \Big\| \geqslant M \sum_{i=1}^{n} |c_i|$$

for all $c_1, \ldots, c_n \in \mathbf{R}$, $\xi_1, \ldots, \xi_n < \alpha$ pairwise different and for each $n \in \mathbf{N}$. Such a family is said to be *equivalent* to the usual basis $\{e_\xi\colon \xi < \alpha\}$ of $l_1(\alpha)$, where $e_\xi(\eta) = 0$ for $\xi \neq \eta$ and $e_\xi(\xi) = 1$.

Let X be a set and $\{(A_i, B_i)\colon i \in I\}$ be a family of subsets of X with the property $A_i \cap B_i = \varnothing$ for $i \in I$. The above family is called *independent* if for every pair of finite disjoint subsets K, F of I we have

$$\Big(\bigcap_{i \in K} A_i\Big) \cap \Big(\bigcap_{i \in F} B_i\Big) \neq \varnothing.$$

The connection between independent families of sets and the isomorphic embedding of $l_1(\alpha)$ into subspaces of spaces of the form $C^*(X)$ is described by the following lemma, due to Rosenthal [9].

0.3. LEMMA. *Let X be a set and $\{f_i\colon i \in I\}$ a family of uniformly bounded real functions on X. Let also r, δ be real numbers with $\delta > 0$ such that if*

$$A_i = \{x \in X\colon f_i(x) > r + \delta\}, \quad B_i = \{x \in X\colon f_i(x) < r\},$$

then the family $\{(A_i, B_i)\colon i \in I\}$ is independent. Then the following inequality is valid:

$$\Big\|\sum_{j=1}^{n} c_j f_{i_j}\Big\| \geqslant \frac{\delta}{2}\sum_{j=1}^{n} |c_j|$$

for every $c_1, \ldots, c_n \in \mathbf{R}$, $i_1, \ldots, i_n \in I$ pairwise different and each $n \in N$.

If $(X_i)_{i\in I}$ is a family of topological spaces and $f \in C^*(\prod_{i\in I} X_i)$, we say that f *depends only on* $J \subset I$ if for each $x, y \in \prod_{i\in I} X_i$ with $\mathrm{pr}_J(x) = \mathrm{pr}_J(y)$ we have $f(x) = f(y)$.

If $J \subset I$, it is clear that $C^*(\prod_{i\in J} X_i)$ embeds isometrically in $C^*(\prod_{i\in I} X_i)$. Let $(Z_i)_{i\in I}$ be a family of Banach spaces. By $(\sum_{i\in I} \oplus Z_i)_\infty$ we denote the Banach space

$$\{z = (z_i)_{i\in I}\colon z_i \in Z_i \text{ and } \sup_{i\in I} \|z_i\| < +\infty\}$$

with the norm

$$\|z\| = \sup_{i\in I} \|z_i\|.$$

1.1. DEFINITION. Let $(X_i)_{i\in I}$ be a family of topological spaces and

$$X = \prod_{i\in I} X_i.$$

For every cardinal α let

$C_\alpha^*(X) = \{f \in C^*(X)$: there exists $J \in \mathcal{P}_\alpha(I)$ such that f depends only on $J\}$.

We set

$$l(X) = \min\{\alpha\colon \overline{C_\alpha^*(X)} = C^*(X)\}, \quad k(X) = \min\{\alpha\colon C_\alpha^*(X) = C^*(X)\}.$$

1.2. Remarks. (i) It is clear that

$$l(X) \leqslant k(X) \leqslant l(X)^+,$$

and if $\mathrm{cf}(l(X)) > \omega$, then $k(X) = l(X)$.

(ii) If X_i is a compact topological space for every $i \in I$, then, by the Stone–Weierstrass theorem, $l(X) = \omega$.

(iii) From [3] we have $k(X) \leqslant r(X)^+$.

1.3. Theorem. *Let α be a cardinal with $\alpha > \omega^+$. Let $(X_i)_{i\in I}$ be a family of topological spaces and*

$$X = \prod_{i\in I} X_i.$$

We suppose that X has precaliber α and $\alpha > k(X)$. If Z is a closed subspace of $C^*(X)$ and if, for every $J \in \mathscr{P}_\alpha(I)$, Z is not contained in $C^*(\prod_{i\in J} X_i)$, then Z contains isomorphically a copy of $l_1(\alpha)$.

Proof. For every $f \in Z$ there is a $J_f \in \mathscr{P}_{k(X)}(I)$ such that f depends only on J_f. Without loss of generality we may assume that

$$I = \bigcup_{f\in Z} J_f.$$

By transfinite induction, we may construct a family

$$\{f_\xi\colon \xi < \alpha\} \subset Z,$$

with $\|f_\xi\| = 1$, such that if we set

$$J_{f_\xi} = J_\xi \quad \text{and} \quad T_\eta = \bigcup_{\xi<\eta} J_\xi,$$

then f_η does not depend only on T_η, hence $J_\eta \not\subset T_\eta$. We distinguish two cases:

Case I. There exists an $A \subset \alpha$, with $|A| = \alpha$, such that

$$T_\eta \cap J_\eta \neq \emptyset \quad \text{for all } \eta \in A.$$

Since X has precaliber α, we have $\mathrm{cf}(\alpha) > \omega$. Hence there exist $A_1 \subset A$, with $|A_1| = \alpha$, and rational numbers r, δ with $\delta > 0$ such that the set

$$\Lambda_\eta = \Big\{x \in \prod_{i\in J'_\eta} X_i\colon \sup\{f_\eta(z)\colon z \in \{x\} \times \prod_{i\in I\setminus J'_\eta} X_i\} > r+\delta > r > \inf\{f_\eta(z)\colon z \in \{x\} \times \prod_{i\in I\setminus J'_\eta} X_i\}\Big\}$$

is non-empty for every $\eta \in A_1$, where $J'_\eta = T_\eta \cap J_\eta$. It is clear that Λ_η is an open subset of $\prod_{i\in J'_\eta} X_i$. We set $N_\eta = J_\eta \setminus T_\eta$. Clearly, $N_\eta \cap N_\xi = \emptyset$ for every η, $\xi \in A_1$. By Proposition 0.2 there exists $A_2 \subset A_1$, with $|A_2| = \alpha$, such that $J_\xi \cap N_\eta = \emptyset$ for every ξ, $\eta \in A_2$ with $\xi \neq \eta$. We set

$$M_\eta = \Lambda_\eta \times \prod_{i\in I\setminus J'_\eta} X_i.$$

Since X has precaliber α, there exists $A_3 \subset A_2$, with $|A_3| = \alpha$, such that the family $\{M_\eta\colon \eta \in A_3\}$ has the finite intersection property. Let

$$B_\eta = \{x \in X\colon f_\eta(x) > r+\delta\}, \qquad C_\eta = \{x \in X\colon f_\eta(x) < r\}$$

for every $\eta \in A_3$. Clearly, B_η and C_η are not empty. We will see that the above family is independent.

Let F and G be finite disjoint subsets of A_3. Then there is

$$z \in \bigcap_{\eta \in F \cup G} M_\eta.$$

We set $z_\eta = \mathrm{pr}_{J'_\eta}(z_\eta)$. There exists

$$y_\eta \in \prod_{i \in I \setminus J'_\eta} X_i$$

such that $f_\eta(z_\eta, y_\eta) > r+\delta$ if $\eta \in F$ and $f_\eta(z_\eta, y_\eta) < r$ if $\eta \in G$. We consider $x \in \prod_{i \in I} X_i$ such that

$$\mathrm{pr}_{J'_\eta}(x) = z_\eta \quad \text{and} \quad \mathrm{pr}_{N_\eta}(x) = \mathrm{pr}_{N_\eta}(y_\eta)$$

for every $\eta \in F \cup G$ (such an x exists, since $J_\eta \cap N_\xi = \emptyset$ for every $\xi, \eta \in A_3$ with $\xi \neq \eta$). Since $f_\eta(x) = f_\eta(z_\eta, y_\eta)$ for $\eta \in F \cup G$, we have

$$x \in \Big(\bigcap_{\eta \in F} B_\eta\Big) \cap \Big(\bigcap_{\eta \in G} C_\eta\Big).$$

Thus by Lemma 0.3 the family $\{f_\eta\colon \eta \in A_3\}$ is equivalent to the usual basis of $l_1(\alpha)$.

Case II. Case I does not hold. Then there exists $A \subset \alpha$, $|A| = \alpha$, such that $J_\eta \cap T_\eta = \emptyset$ for all $\eta \in A$. Then, as in case I, we can find an $A_1 \subset A$, with $|A_1| = \alpha$, such that $\{f_\eta\colon \eta \in A_1\}$ is equivalent to the usual basis of $l_1(\alpha)$.

1.4. THEOREM. *Let $(X_i)_{i \in I}$ be a family of topological spaces, let*

$$X = \prod_{i \in I} X_i \quad \textit{and} \quad l(X) = \omega.$$

We suppose that, for each $i \in I$, X_i has precaliber ω^+ and $C^*(\prod_{i \in F} X_i)$ is separable for every $F \in \mathscr{P}_\omega(I)$. Then, if $C^*(X)$ is non-separable, $l_1(\omega^+)$ embeds universally in $C^*(X)$.

Proof. Let Z be a closed subspace of $C^*(X)$ with $\dim Z = \omega^+$, and $0 < \vartheta < 1$. There exists a family $\{f_\xi\colon \xi < \omega^+\} \subset Z$ with $\|f_\xi\| = 1$ and $\|f_\xi - f_\eta\| \geq \vartheta$ for every $\xi < \eta < \omega^+$.

For every $\xi < \omega^+$ there is a $g_\xi \in C^*(X)$ such that

$$\|g_\xi - f_\xi\| < \vartheta/20$$

and g_ξ depends only on a set J_ξ with $|J_\xi| < \omega$. It is clear that

$$\|g_\xi - g_\eta\| > 9\vartheta/10 \quad \text{for every } \xi < \eta < \omega^+.$$

By Theorem 0.1 there is an $A \subset \omega^+$, with $|A| = \omega^+$, and a $J \subset I$ such that

$$J_\xi \cap J_\eta = J \quad \text{for every } \xi, \eta \in A \text{ with } \xi \neq \eta.$$

We have that $C^*(\prod_{i \in J} X_i)$ is separable. Hence we may suppose that $J \neq J_\xi$ for every $\xi \in A$. We distinguish two cases.

Case I. Let $J \neq \emptyset$. Then there exist $A_1 \subset A$, with $|A_1| = \omega^+$, and $r \in \boldsymbol{R}$ such that $A_\xi \neq \emptyset$ for every $\xi \in A_1$, where

$$A_\xi = \Big\{x \in \prod_{i\in J} X_i\colon \sup\{g_\xi(z)\colon z \in \{x\} \times \prod_{i\in I\setminus J} X_i\} > r + \vartheta/8 > r > \inf\{g_\xi(z)\colon z \in \{x\} \times \prod_{i\in I\setminus J} X_i\}\Big\}.$$

It is clear that A_ξ is an open subset of $\prod_{i\in J} X_i$. Since X_i has precaliber ω^+, it is easy to see that $\prod_{i\in J} X_i$ has precaliber ω^+, hence there is an $A_2 \subset A_1$, with $|A_2| = \omega^+$, such that the family $\{A_\xi\colon \xi \in A_2\}$ has the finite intersection property. We set

$$B_\xi = \{x \in X\colon g_\xi(x) > r + \vartheta/8\}, \qquad C_\xi = \{x \in X\colon g_\xi(x) < r\}$$

for every $\xi \in A$. It is easy to see that the family $\{(B_\xi, C_\xi)\colon \xi \in A_2\}$ is independent, and so by Lemma 0.3 we have

$$\Big\|\sum_{i=1}^{n} c_i g_{\xi_i}\Big\| \geqslant (\vartheta/16) \sum_{i=1}^{n} |c_i|$$

for every $\xi_1, \ldots, \xi_n \in A_2$ pairwise different, $c_1, \ldots, c_n \in \boldsymbol{R}$ and $n \in \boldsymbol{N}$. Thus we have

$$\Big\|\sum_{i=1}^{n} c_i f_{\xi_i}\Big\| \geqslant (\vartheta/20) \sum_{i=1}^{n} |c_i|$$

for every $\xi_1, \ldots, \xi_n \in A_2$ pairwise different, $c_1, \ldots, c_n \in \boldsymbol{R}$ and $n \in \boldsymbol{N}$. Thus the family $\{f_\xi\colon \xi \in A_2\}$ is equivalent to the usual basis of $l_1(\alpha)$.

Case II. Let $J = \emptyset$. Then, as in case I, we find $A_1 \subset A$, with $|A_1| = \alpha$, such that the family $\{f_\xi\colon \xi \in A_1\}$ is equivalent to the usual basis of $l_1(\alpha)$.

1.5. Lemma. *Let $(X_i)_{i\in I}$ be a family of topological spaces and*

$$X = \prod_{i\in I} X_i.$$

Then $C^*(X)$ can be embedded isometrically in

$$\Big(\sum_{A\in\mathscr{P}_\omega(I)} \oplus C^*\big(\prod_{i\in A} X_i\big)\Big)_\infty.$$

Proof. Let $x^\circ = (x_i^\circ)_{i\in I}$ be an element of X and take the function

$$T\colon C^*(X) \to \Big(\sum_{A\in\mathscr{P}_\omega(I)} \oplus C^*\big(\prod_{i\in A} X_i\big)\Big)_\infty$$

such that $T(f) = (f_A)_{A\in\mathscr{P}_\omega(I)}$, where

$$f_A\colon \prod_{i\in A} X_i \to \boldsymbol{R}, \qquad f_A(x) = f(y), \text{ where } y_i = x_i \text{ for } i \in A,$$

and $y_i = x_i^\circ$ for $i \in I\setminus A$. It is easy to see that T is a well-defined linear isometry. Combining Theorems 1.3 and 1.4 yields the following general corollary:

1.6. COROLLARY. *Let α be a cardinal. Let $(X_i)_{i\in I}$ be a family of topological spaces and*

$$X = \prod_{i\in I} X_i.$$

We suppose that X has precaliber α, $\alpha > l(X)$, and $\beta^{\underset{\smile}{l(X)}} < \alpha$ for every $\beta < a$. We also suppose that

$$(*) \qquad \alpha > \sup\{\dim C^*(\prod_{i\in F} X_i)\colon\ F\in\mathscr{P}_\omega(I)\}.$$

If $\dim C^*(X) \geqslant \alpha$, *then $l_1(\alpha)$ embeds universally in $C^*(X)$.*

Proof. Since X has precaliber α, we have $\mathrm{cf}(\alpha) > \omega$.

If $\alpha = \omega^+$, then the corollary reduces to Theorem 1.4. Let $\alpha > \omega^+$ and Z be a closed linear subspace of $C^*(X)$ with $\dim Z \geqslant \alpha$. We will prove that the conditions of Theorem 1.3 are valid.

If $l(X) = \omega$, then $k(X) \leqslant \omega^+ < \alpha$.

If $l(X) > \omega$ and $\mathrm{cf}(l(X)) > \omega$, then $k(X) = l(X) < \alpha$.

If $l(X) > \omega$ and $\mathrm{cf}(l(X)) = \omega$, then

$$\alpha > l(X)^{\underset{\smile}{l(X)}} \geqslant l(X)^{\mathrm{cf}(l(X))} > l(X),$$

so $k(X) < \alpha$.

We suppose that there exists $J\in\mathscr{P}_\alpha(I)$ such that

$$\dim C^*(\prod_{i\in J} X_i) \geqslant \alpha.$$

Then, if $0 < \vartheta < 1$, there exists a family

$$\{f_\xi\colon\ \xi < \alpha\} \subset C^*(\prod_{i\in J} X_i)$$

with $\|f_\xi\| = 1$ and $\|f_\xi - f_\eta\| \geqslant \vartheta$ for every $\xi < \eta < \alpha$. Without loss of generality we may assume that f_ξ depends only on $J_\xi \subset J$ with $|J_\xi| < l(X)$ for all $\xi < \alpha$.

There exists a regular cardinal β such that

$$\alpha \geqslant \beta > |J|^{\underset{\smile}{l(X)}},$$

$$\beta > \dim C^*(\prod_{i\in F} X_i) \quad \text{for every } F\in\mathscr{P}_\omega(J) \text{ and } \beta \geqslant l(X).$$

Then there are a $J' \subset J$ and an $A \subset \alpha$, with $|A| = \beta$, such that $J_\xi = J'$ for every $\xi\in A$. Consequently,

$$\dim C^*(\prod_{i\in J'} X_i) \geqslant \beta.$$

From Lemma 1.5 we have

$$\beta \leqslant \dim C^*(\prod_{i\in J'} X_i) \leqslant \sup\{\dim C^*(\prod_{i\in F} X_i)^{|J'|}\colon\ F\in\mathscr{P}_\omega(J')\} < \beta,$$

which is a contradiction. Thus, for Z, the conditions of Theorem 1.3 are valid. Therefore $l_1(\alpha)$ embeds isomorphically into Z.

1.7. Remark. It is clear from the proof of Corollary 1.6 that when α is a regular cardinal, then condition $(*)$ may be weakened to:

$$\alpha > \dim C^*(\prod_{i\in F} X_i) \quad \text{for every } F\in\mathscr{P}_\omega(I).$$

1.8. COROLLARY. *Let α be a cardinal. Let also $(X_i)_{i\in I}$ be a family of compact topological spaces and*

$$X = \prod_{i\in I} X_i.$$

We suppose that X_i has caliber α for every $i\in I$ and

$$\alpha > \sup\{w(X_i)\colon\ i\in I\},$$

where $w(X_i)$ is the topological weight of X_i. If $\dim C^*(X) \geqslant a$, *then $l_1(\alpha)$ embeds universally in $C^*(X)$.*

1.9. Remarks. (i) In the case of a regular cardinal α we need only the condition $\alpha > w(X_i)$ for every $i\in I$, as in Corollary 1.6.

(ii) It is obvious that if α is an infinite cardinal and Y a topological space with $w(Y)\geqslant\alpha$, which is a continuous image of the product of a family $(X_i)_{i\in I}$ of compact topological spaces with the properties of Corollary 1.8, then $l_1(\alpha)$ embeds universally in $C(Y)$.

(iii) Corollary 1.8 extends a result of Argyros and Negrepontis [2] and also contains the result of Hagler [6] for dyadic spaces.

Since $k(X)\leqslant r(X)^+$, Corollary 1.6 gives easily the following

1.10. COROLLARY. *Let α be an uncountable cardinal, $(X_i)_{i\in I}$ be a family of topological spaces and*

$$X = \prod_{i\in I} X_i.$$

We suppose that X has precaliber α and $\beta^{r(X)} < \alpha$ for every $\beta<\alpha$. We also suppose that

$$\alpha > \sup\{\dim C^*(\prod_{i\in F} X_i)\colon\ F\in\mathscr{P}_\omega(I)\}.$$

If $\dim C^*(X)\geqslant\alpha$, *then $l_1(\alpha)$ embeds universally in $C^*(X)$.*

Finally, from Theorem 1.3 we obtain easily the following corollary, which is contained in [7].

1.11. COROLLARY. *Let α be a regular cardinal with $\alpha\geqslant\omega^+$, and $(X_i)_{i\in I}$ a family of topological spaces with $\alpha \gg k(X)$, where*

$$X = \prod_{i\in I} X_i.$$

We suppose that $l_1(\alpha)$ embeds universally into $C^*(\prod_{i\in F} X_i)$ for every $F\in\mathscr{P}_\omega(I)$ such that

$$\dim C^*(\prod_{i\in F} X_i) \geqslant \alpha.$$

Then $l_1(\alpha)$ embeds universally into $C^*(X)$ if $\dim C^*(X) \geqslant \alpha$.

Proof. Let Z be a subspace of $C^*(X)$ with $\dim Z = \alpha$. If, for every $J\in\mathscr{P}_\alpha(I)$,

$$Z \not\subsetneq C^*(\prod_{i\in J} X_i),$$

then the result follows from Theorem 1.3. We suppose that there is $J\in\mathscr{P}_\alpha(I)$ with

$$Z \subsetneq C^*(\prod_{i\in J} X_i).$$

If $0 < \vartheta < 1$, there is $\{f_\xi\colon \xi < \alpha\} \subset Z$ with $\|f_\xi\| = 1$ and $\|f_\xi - f_\eta\| > \vartheta$ for every $\xi < \eta < \alpha$. For every $\xi < \alpha$ there are $J_\xi \subset J$, with $|J_\xi| < k(X)$, and f_ξ depending only on J_ξ. Since α is a regular cardinal, $\alpha \gg k(X)$ and $|J| < \alpha$, we see that there exist $A \subset \alpha$, with $|A| = \alpha$, and $L \subset J$ such that $J_\xi = L$ for every $\xi\in A$. Let Y be the closed subspace generated by $\{f_\xi\colon \xi\in A\}$. Since

$$Y \subsetneq C^*(\prod_{i\in L} X_i),$$

by Lemma 1.5, Y embeds isometrically in $(\sum_{\Lambda\in\mathscr{P}_\omega(L)} \oplus Y_\Lambda)_\infty$, where Y_Λ is the closed image of Y in $C^*(\prod_{i\in\Lambda} X_i)$ through the canonical projection of

$$(\sum_{\Lambda\in\mathscr{P}_\omega(L)} \oplus C^*(\prod_{i\in\Lambda} X_i))_\infty$$

onto

$$C^*(\prod_{i\in\Lambda} X_i).$$

If $\dim Y_\Lambda < \alpha$ for every $\Lambda\in\mathscr{P}_\omega(L)$, then

$$\alpha = \dim Y \leqslant \sup\{(\dim Y_\Lambda)^{|L|}\colon \Lambda\in\mathscr{P}_\omega(L)\} < \alpha.$$

So there is a $\Lambda\in\mathscr{P}_\omega(L)$ such that $\dim Y_\Lambda = \alpha$ and, consequently, $l_1(\alpha)$ embeds isomorphically in Y_Λ, hence in Y, since $\mathrm{cf}(\alpha) > \omega$ (see [8]). Thus $l_1(\alpha)$ embeds isomorphically in Z.

Acknowledgment. We want to thank Professor S. Argyros for his suggestions that helped the final form of the paper.

REFERENCES

[1] S. Argyros, *On non-separable Banach spaces*, Trans. Amer. Math. Soc. 270 (1982), pp. 193–216.

[2] — and S. Negrepontis, *Universal embeddings of $l_1(\alpha)$ into $C(X)$ and $L^\infty(p)$*, pp. 75–128 in: Á. Császár (ed.), *Proceedings of the Budapest Colloquium on Topology, August 1978*, Colloq. Math. Soc. János Bolyai 23, North-Holland, Amsterdam 1980.

[3] W. W. Comfort and S. Negrepontis, *Continuous functions on products with strong topologies*, pp. 89–92 in: J. Novak (ed.), *General Topology and its Relations to Modern Analysis and Algebra. III, Proceedings of the Third Prague Topological Symposium, Prague 1971*, Prague Academia 1972.

[4] — *The theory of ultrafilters*, Grundlehren Math. Wiss. 211, Springer-Verlag, 1974.

[5] — *Chain conditions in topology*, Cambridge Tracts in Math. 79, Cambridge University Press, 1982.

[6] J. Hagler, *On the structure of S and $C(S)$ for S dyadic*, Trans. Amer. Math. Soc. 214 (1975), pp. 415–428.

[7] N. Kalamidas and Th. Zachariades, *On the productivity of $l^1(\Gamma)$ embeddings*, Bull. Soc. Math. Grèce (N. S.) 25 (1984), pp. 65–81.

[8] A. Pełczyński, *On Banach spaces containing $L^1(\mu)$*, Studia Math. 30 (1968), pp. 238–246.

[9] H. P. Rosenthal, *A characterization of Banach spaces containing l^1*, Proc. Nat. Acad. Sci. U.S.A. 71 (1974), pp. 2411–2413.

UNIVERSITY OF ATHENS
DEPARTMENT OF MATHEMATICS
SECTION OF MATHEMATICAL ANALYSIS AND ITS APPLICATIONS
PANEPISTEMIOPOLIS 157 81
ATHENS, GREECE

Reçu par la Rédaction le 20.9.1986;
en version modifiée le 20.1.1988