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SOME ALGEBRAIC ASPECTS OF TOURNAMENT THEORY

BY

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0. Introduction. A *tournament* $\mathfrak{T} = \langle T; \leqslant \rangle$ consists of a non-empty set T and a binary relation $\leqslant$ on T such that

(1) $\leqslant$ is reflexive;

(2) $\leqslant$ is antisymmetric;

(3) for any $x, y \in T$, either $x \leqslant y$ or $y \leqslant x$.

In graph theory [3], a tournament is defined as a non-trivial complete asymmetric digraph ($x \leqslant y$ is denoted by $x \to y$ and read x beats y or y loses to x). In other words, (1) is not assumed and $|T| > 1$. By postulating (1) we just add a loop at each vertex of the digraph, which does not alter essentially the relational system and enables us to consider $\mathfrak{T}$ as a *trellis*, that is, a pseudo-ordered set any two elements of which have an l.u.b. ($\vee$) and a g.l.b. ($\wedge$) [5]. Consequently, a tournament is a trellis in which any two elements are comparable.

We have

$$x \leqslant y \Leftrightarrow x = x \wedge y \Leftrightarrow y = x \vee y,$$

so that $\mathfrak{T}^* = \langle T; \vee, \wedge \rangle$ is a universal algebra of type $\langle 2, 2 \rangle$ in which any subset is a subalgebra. In what follows we shall always write $\mathfrak{T}$ instead of $\mathfrak{T}^*$.

The notion of convexity is defined in a tournament in the same fashion as in a partially ordered set. Convex subsets are investigated in Section 1. Among them the so-called ideals and filters are of special interest. Roughly speaking, they behave like prime ideals and prime filters in lattices. The existence of principal ideals and filters and the transitivity of the relation $\leqslant$ are shown to be closely linked.

The study of convex subsets is primarily motivated by the following fact: any congruence-class of a tournament is a convex subset and conversely. Hence the dependence of the congruence lattice on the convex subset lattice is total. Congruences constitute the object of Section 2. Subdirectly irreducible tournaments as well as tournaments in which congruences are pairwise permutable are characterized and methods of construction are provided.

In the final section special attention is paid to transitive tournaments. Let us only mention the following two properties: a tournament $\mathfrak{T}$ ($|T| \neq 3$) is transitive if and only if it enjoys the congruence extension property; the transitivity of a finite non-simple tournament is equivalent to the fact that its congruence lattice is Boolean.

The set-theoretical difference of A and B is denoted by $A-B$ and we write $a < b$ when $a \leqslant b$ and $a \neq b$.

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1. Convex subsets. A subset S of T is a *convex subset* of the tournament $\mathfrak{T}$ if $a, b \in S$ and $a \leqslant c \leqslant b$ imply $c \in S$. Equivalently, S is *convex* if, for all $a, b \in S$ and all $c \in T-S$, the inequality $a < c$ implies $b < c$. Of course, T, $\emptyset$ and all singletons are convex. In accordance with [1], a convex subset S will be called *non-trivial* if $S \neq T$ and $|S| > 1$. We denote the set of all convex subsets of $\mathfrak{T}$ and the set of all non-trivial convex subsets of $\mathfrak{T}$ by $\mathscr{C}(\mathfrak{T})$ and $\mathscr{C}^*(\mathfrak{T})$, respectively.

The following proposition is quite elementary but useful in the sequel:

THEOREM 1.1. *For any $A, B \in \mathscr{C}(\mathfrak{T})$, we have*

(1) $A \cap B \in \mathscr{C}(\mathfrak{T})$;

(2) *if $A \cap B \neq \emptyset$, then $A \cup B \in \mathscr{C}(\mathfrak{T})$;*

(3) *if $A \cap B \neq \emptyset$ and A, B are incomparable (in symbols, $A \| B$), then $A-B \in \mathscr{C}(\mathfrak{T})$, $B-A \in \mathscr{C}(\mathfrak{T})$ and $(A-B) \cup (B-A) \notin \mathscr{C}(\mathfrak{T})$.*

Proof. The first two statements are obvious. Let us now assume $A \cap B \neq \emptyset$ and $A \| B$. Let us pick a, b in $A-B$, c in $B-A$ and d in $A \cap B$. If $d < a$, then $c < a$ (B convex), $c < b$ (A convex) and $d < b$ (B convex), which shows that $A-B \in \mathscr{C}(\mathfrak{T})$. We also have $c < d$ (A convex). Since $d < a$, we may claim that $(A-B) \cup (B-A) \notin \mathscr{C}(\mathfrak{T})$.

COROLLARY 1.1. *$\mathscr{C}(\mathfrak{T})$, ordered by inclusion, is a complete atomistic lattice. If $\emptyset \neq X \in \mathscr{C}(\mathfrak{T})$, then*

$$[X) = \{Y\colon\ Y \in \mathscr{C}(\mathfrak{T}),\ Y \supseteq X\}$$

is infinitely distributive. Every 4-element sublattice $\{A, B, A \vee B, A \wedge B \neq \emptyset\}$ is contained in a 7-element sublattice including $\emptyset$.

Proof. T and $\emptyset$ are the bounds of $\mathscr{C}(\mathfrak{T})$ and every element distinct from $\emptyset$ is the supremum of the atoms, i.e. the singletons, which it contains. In $[X)$, when $X \neq \emptyset$, $\wedge$ and $\vee$ coincide with $\cap$ and $\cup$, respectively. Hence

$$A \wedge \Big(\bigvee_{i \in I} B_i\Big) = \bigvee_{i \in I} (A \wedge B_i)$$

and dually. Finally, if $A \| B$ and $A \wedge B \neq \emptyset$, the 7 elements, i.e. A, B, $A \wedge B\ (= A \cap B)$, $A \vee B\ (= A \cup B)$, $A - B$, $B - A$ and $\emptyset$, form a sublattice, since the proof of Theorem 1.1 (3) shows that every convex subset which contains $A - B$ and $B - A$ must contain $A \cup B$.

When dealing with three convex subsets of $\mathfrak{T}$, combinatorial problems arise. As a sample we furnish the following statement:

THEOREM 1.2. *Let A, B and C be three convex subsets of $\mathfrak{T}$ such that $A \| B$, $(A \cap B) - C \neq \emptyset$, $A \| C$, $(A \cap C) - B \neq \emptyset$ and $B \cap C \neq \emptyset$. Then $B \cap C \subseteq A \subseteq B \cup C$.*

Proof. Let $x \in (A \cap B) - C$, $z \in (A \cap C) - B$ and $y \in B \cap C$. Without loss of generality we may suppose $x < y$. Taking successively in account the convexity of C, B and A, we obtain $x < z$, $y < z$ and $y \in A$, whence $B \cap C \subseteq A$.

Since $A \| B$ and $A \cap B \neq \emptyset$, we have $A - B$ convex; similarly, $A - C$ is convex. If $(A - B) \cap (A - C) \neq \emptyset$, then

$$(A - B) \cup (A - C) = A \cap \complement (B \cap C)$$

is convex. Since x and z both belong to $(A - B) \cup (A - C)$ and $x < y < z$, we have $y \in (A - B) \cup (A - C)$, a contradiction owing to $y \in B \cap C$. In conclusion,

$$(A - B) \cap (A - C) = A \cap \complement (B \cup C) = \emptyset \qquad \text{and} \qquad A \subseteq B \cup C,$$

which completes the proof.

When there is a binary partition $\{A, B\}$ of T such that, for every $a \in A$ and every $b \in B$, we have $a \leqslant b$, the tournament $\mathfrak{T}$ is said to be *reducible* (otherwise it is *irreducible*). In accordance with [5], we call A an ideal of $\mathfrak{T}$ and B a filter (or dual ideal) of $\mathfrak{T}$. Let us formulate this notion clearly.

An *ideal* I of a tournament $\mathfrak{T}$ is a non-empty subset of $\mathfrak{T}$ such that, for every $i \in I$, $x \leqslant i$ implies $x \in I$ (or, equivalently, $i \in I$ and $y \notin I$ imply $i < y$). An ideal I of $\mathfrak{T}$ can also be defined as a non-empty subset of T which contains $a \vee b$ if and only if it contains a and b.

The ideals and filters of a tournament play a role analogous to those of prime ideals and prime filters in lattices. Firstly, the complement of an ideal is a filter. Secondly, if two convex subsets form a partition of T, one is an ideal and the other a filter.

THEOREM 1.3. *The proper ideals and filters of $\mathfrak{T}$ form two disjoint maximal chains of $\mathscr{C}(\mathfrak{T})$.*

Proof. Clearly, ideals and filters are convex subsets. Moreover, a convex subset C which contains an ideal I is an ideal. In fact, if $I \subset C \subset T$ and C is not an ideal, there are $x \in T - C$ and $y \in C - I$ such that $x < y$. But then, for any $z \in I$, $z < x < y$ and, owing to the convexity of C, we have $x \in C$, a contradiction. Finally, two ideals I and J are always comparable, since $a \in I - J$ and $b \in J - I$ would imply $a \leqslant b$ and $b \leqslant a$, whence $a = b$.

COROLLARY 1.2. *If A is an ideal of $\mathfrak{T}$, then $[A)$ is a chain of $\mathscr{C}(\mathfrak{T})$.*

We leave as an exercise for the reader the following proposition:

THEOREM 1.4. *Let A be an ideal of $\mathfrak{T}$, $B \in \mathscr{C}(\mathfrak{T})$ and $A \| B$. Then*

(1) *$A - B$ is also an ideal;*

(2) *$B - A$ is a filter if and only if $A \vee B = T$.*

An ideal I is *principal* if it is generated by a single element a, that is, if it contains an element a such that $x \leqslant a$ for any $x \in I$. Such an ideal will be denoted by $(a]$. Hence, when $I = (a]$, we have $x \leqslant a$ if and only if $x \in I$ and $(a] = \{x: x \in T, x \leqslant a\}$. A direct consequence of this definition is the following: if $I = (a]$, then $I - \{a\}$ is an ideal or $\emptyset$, $[a)$ is a principal filter F and $F - \{a\}$ is a filter or $\emptyset$.

In trellis theory, the following terminology is used:

An element a is *right transitive* if

$$(1) \qquad (a \leqslant x \leqslant y) \Rightarrow (a \leqslant y);$$

left transitive if

$$(2) \qquad (x \leqslant y \leqslant a) \Rightarrow (x \leqslant a);$$

middle transitive if

$$(3) \qquad (x \leqslant a \leqslant y) \Rightarrow (x \leqslant y);$$

transitive if it satisfies (1), (2) and (3).

It is easy to see that in a tournament conditions (1), (2) and (3) are equivalent and, consequently, any of them can help to define the transitivity of the element a. It is also immediate that an element a of a tournament is transitive if and only if no triple including a is cyclic.

An element a is *distributive* if, for any triple including a, each of the operations $\vee$ and $\wedge$ is distributive with respect to the other. Finally, an element a is *associative* if any triple (x, y, z) including a is both $\vee$-associative and $\wedge$-associative, that is

$$(x \vee y) \vee z = x \vee (y \vee z) \quad \text{and} \quad (x \wedge y) \wedge z = x \wedge (y \wedge z).$$

The equivalence of these notions in a tournament is established in the following statement:

THEOREM 1.5. *For an element a of a tournament $\mathfrak{T}$, the following conditions are equivalent:*

(1) *a generates a principal ideal I;*

(1′) *a generates a principal filter F;*

(2) *a is transitive;*

(3) *a is distributive;*

(4) *a is associative.*

Proof. Clearly, (1) and (1') are equivalent. (1) implies (2) since, if $I = (a]$, a cannot belong to a cyclic triple. An easy computation shows that (2) implies (3), while Theorem 15 of [5] proves that (3) implies (4). Finally, if a does not generate a principal ideal, $A = \{x: x \leqslant a\}$ is not an ideal and there exist $x \in A$ and $y \notin A$ such that $x < a < y$ and $y < x$. Hence

$$y = (x \vee a) \vee y \neq x \vee (a \vee y) = x$$

and a is not associative, which completes the proof.

An *endomorphism* φ of $\mathfrak{T}$ is a mapping of T into T satisfying

$$x \leqslant y \Rightarrow x\varphi \leqslant y\varphi$$

or, equivalently,

$$(x \vee y)\varphi = x\varphi \vee y\varphi \quad \text{and} \quad (x \wedge y)\varphi = x\varphi \wedge y\varphi.$$

We shall denote by $\mathscr{E}(\mathfrak{T})$, $\mathscr{E}_0(\mathfrak{T})$ and $\mathscr{A}(\mathfrak{T})$ the semigroup of endomorphisms, the semigroup of ependomorphisms and the group of automorphisms of $\mathfrak{T}$, respectively.

THEOREM 1.6. *Let $\varphi \in \mathscr{E}(\mathfrak{T})$ and let $S \subseteq T$.*

(1) *If S is convex in $\mathfrak{T}$, then $S\varphi^{-1}$ (the complete inverse image of S) is convex in $\mathfrak{T}$.*

(2) *If S is an ideal of $\mathfrak{T}$, then $S\varphi^{-1}$ is either empty or an ideal of $\mathfrak{T}$.*

Proof. (1) If $S\varphi^{-1}$ is not convex, there are in T three distinct elements a, b, c such that $a\varphi \notin S$, $b\varphi \in S$, $c\varphi \in S$ and $b < a < c$. Since $\varphi \in \mathscr{E}(\mathfrak{T})$, we have $b\varphi < a\varphi < c\varphi$, contradicting the convexity of S.

(2) If $S\varphi^{-1}$ is neither an empty set nor an ideal of $\mathfrak{T}$, there exist $a \in S\varphi^{-1}$ and $b \notin S\varphi^{-1}$ such that $b < a$. Hence $b\varphi < a\varphi$ with $b\varphi \notin S$ and $a\varphi \in S$, a contradiction.

If the endomorphism φ is onto, Theorem 1.6 can be completed as follows:

THEOREM 1.7. *Let $\varphi \in \mathscr{E}_0(\mathfrak{T})$ and let $S \subseteq T$.*

(1) *If S is convex in $\mathfrak{T}$, then $S\varphi$ is convex in $\mathfrak{T}$.*

(2) *If S is an ideal of $\mathfrak{T}$, then $S\varphi$ is an ideal of $\mathfrak{T}$.*

(3) *If, moreover, φ is injective (whence an automorphism), then $S\varphi = S$ for any finite ideal S of $\mathfrak{T}$.*

Proof. (1) Let us suppose $S\varphi$ is not convex whereas S is. There are a, b, c in T such that $b \in S\varphi$, $c \in S\varphi$, $a \notin S\varphi$ and $b < a < c$. Since φ is onto, one can find x, y, z satisfying $x \notin S$, $y \in S$, $z \in S$ and $y < x < z$, contradicting the convexity of S.

(2) is obvious.

(3) Since $S\varphi$ is a finite ideal and $|S\varphi| = |S|$, Theorem 1.3 enables us to conclude that $S\varphi = S$.

2. Congruences. A congruence Θ of $\mathfrak{T} = \langle T;\ \vee,\ \wedge\rangle$ is an equivalence relation such that if $(x, y) \in \Theta$, then

$$(x \wedge z, y \wedge z) \in \Theta \quad \text{and} \quad (x \vee z, y \vee z) \in \Theta \quad \text{for any } z \in T.$$

When the only congruences of $\mathfrak{T}$ are the equality ω and the all congruence ι, $\mathfrak{T}$ is simple. It is well known that there is a simple tournament of any cardinality but 4. The congruence lattice $\mathscr{K}(\mathfrak{T})$ of $\mathfrak{T}$ is distributive ([5], Theorem 46).

We first enumerate without a proof a few elementary properties of congruences in a tournament.

Every congruence-class of $\mathfrak{T}$ is convex; conversely, every convex subset C is class of at least one congruence, denoted by Θ_C, which is the least congruence collapsing the elements of C: $(x, y) \in \Theta_C$ if and only if $x \in C$ and $y \in C$, or $x = y$. If the family of non-trivial classes of a congruence Θ is $\{C_i\}_{i \in I}$ (that is, $C_i \in \mathscr{C}^*(\mathfrak{T})$ for any $i \in I$), then

$$\Theta = \bigvee_{i \in I} \Theta_{C_i}.$$

Hence, *if a congruence is $\vee$-irreducible, then it has the form Θ_C for some $C \in \mathscr{C}(\mathfrak{T})$.*

All congruences of $\mathfrak{T}$ are *nuclear*, that is, kernel congruences. If the Θ-classes are C_i $(i \in I)$, choose in every C_i an arbitrary element c_i and define the endomorphism φ as follows: $x\varphi = c_i$ if and only if $(x, c_i) \in \Theta$; then $\ker\varphi = \Theta$.

For a tournament $\mathfrak{T}$, the following conditions are equivalent:

(1) *$\mathfrak{T}$ is simple;*

(2) *$\mathscr{C}^*(\mathfrak{T})$ is empty;*

(3) *the only endomorphisms of $\mathfrak{T}$ are injective endomorphisms and constant maps;*

(4) *$\mathfrak{T}$ is a regular algebra (i.e., if two congruences have a class in common, they are equal).*

In [2] subdirectly irreducible tournaments are investigated but not characterized. Before filling the gap, we notice that we do not consider a one-element algebra as a subdirectly irreducible algebra.

THEOREM 2.1. *A tournament $\mathfrak{T}$ is subdirectly irreducible if and only if $K = \bigcap \{C_i\colon C_i \in \mathscr{C}^*(\mathfrak{T})\}$ has at least two elements.*

Proof. Sufficiency. Since $|K| \geqslant 2$, we have $\Theta \geqslant \Theta_K$ for every $\Theta \neq \omega$, and $\mathfrak{T}$ is subdirectly irreducible.

Necessity. In case $\mathscr{C}^*(\mathfrak{T})$ is empty, $K = T$ with $|T| > 1$. Otherwise, if $|K| = 0$ or 1, then

$$\bigwedge_{i \in I} \{\Theta_{C_i}\colon C_i \in \mathscr{C}^*(\mathfrak{T})\} = \omega$$

and $\mathfrak{T}$ is subdirectly reducible.

THEOREM 2.2. *If $\mathfrak{T}$ is a subdirectly irreducible tournament for which K is finite, then Θ_K is fully characteristic. If, furthermore, T is finite, then Θ_K is fully invariant.*

Proof. Let us notice that the property is trivial if $\mathfrak{T}$ is simple. In consequence, let K be a proper subset of T. First we have to show that, for any $\xi \in \mathscr{A}(\mathfrak{T})$, $(a, b) \in \Theta_K$ implies $(a\xi, b\xi) \in \Theta_K$ or, equivalently, $a, b \in K$ implies $a\xi, b\xi \in K$ or $a\xi = b\xi$. Since ξ is an automorphism, $K\xi = K$. In fact, by Theorem 1.7, $K\xi$ is a non-trivial convex subset, whence $K\xi \supseteq K$ and, since $|K\xi| = |K|$ and K is finite, $K\xi = K$.

Let us now consider T finite and $\varphi \in \mathscr{E}(\mathfrak{T})$. If φ is injective, $\varphi \in \mathscr{A}(\mathfrak{T})$ and, by virtue of the first part, $K\varphi = K$. If φ is not injective, there exist x, y in T such that $x \neq y$ and $x\varphi = y\varphi$, whence $(x, y) \in \Theta_\varphi \neq \omega$ and $\Theta_K \leqslant \Theta_\varphi$. Then $(a, b) \in \Theta_K$ implies $(a, b) \in \Theta_\varphi$, i.e. $a\varphi = b\varphi$.

Let us now focus our attention on tournaments in which congruences are pairwise permutable. Such tournaments can be characterized as follows:

THEOREM 2.3. *Congruences of $\mathfrak{T}$ are pairwise permutable if and only if any two convex subsets of $\mathfrak{T}$ are either disjoint or comparable.*

Proof. Sufficiency. We have to show that, for any Θ, $\Phi \in \mathscr{K}(\mathfrak{T})$, $(a, b) \in \Theta$ and $(b, c) \in \Phi$ imply $(a, d) \in \Phi$ and $(d, c) \in \Theta$ for some $d \in T$. Let us denote $[a]\Theta$ and $[c]\Phi$ by C_1 and C_2, respectively. Since C_1 and C_2 both contain b, we have either $C_1 \subseteq C_2$ or $C_2 \subseteq C_1$. It suffices to take $d = c$ in the first case and $d = a$ in the second.

Necessity. Let us suppose that there exist C_1, C_2 in $\mathscr{C}(\mathfrak{T})$ such that $C_1 \cap C_2 \neq \emptyset$ and yet C_1 and C_2 are incomparable. Then it is easy to see that Θ_{C_1} and Θ_{C_2} are not permutable by considering the elements a, b, c chosen as follows: $a \in C_1 - C_2$, $b \in C_1 \cap C_2$ and $c \in C_2 - C_1$.

Remark. Other formulations of the condition in Theorem 2.3 are the following: $A \wedge B \neq \emptyset$ $(A, B \in \mathscr{C}(\mathfrak{T}))$ implies $A \wedge B = A$ or $A \wedge B = B$; all elements of $\mathscr{C}(\mathfrak{T}) - \emptyset$ are $\wedge$-irreducible.

COROLLARY 2.1. *Congruences of a subdirectly irreducible tournament $\mathfrak{T}$ are pairwise permutable if and only if $\mathscr{C}^*(\mathfrak{T})$ is a chain, possibly empty.*

THEOREM 2.4. *If congruences of $\mathfrak{T}$ are pairwise permutable, then $\mathscr{K}(\mathfrak{T})$ is isomorphic to a direct product of chains with an additional greatest element ι.*

Proof. Elements of $\mathscr{C}^*(\mathfrak{T})$ can be partitioned into chains γ_i and every congruence has at most one element of each γ_i as a class. The trivial congruences ω and ι are obtained by refusing any element of $\mathscr{C}^*(\mathfrak{T})$ as a class. Let us notice that

$$|\mathscr{K}(\mathfrak{T})| = \prod_i (|\gamma_i|+1)+1.$$

COROLLARY 2.2. *If the non-trivial convex subsets of* $\mathfrak{T}$ *are mutually disjoint, then* $\mathscr{K}(\mathfrak{T})$ *is isomorphic to a Boolean lattice with an additional greatest element* ι.

Theorem 2.3 shows that tournaments in which non-trivial convex subsets are mutually disjoint, as well as tournaments in which non-trivial convex subsets form a chain, are worth mentioning. The existence of the first is proved by the following proposition (in fact, they abound):

THEOREM 2.5. *Let* $\mathscr{P} = \{S_i\}_{i\in I}$ *be a partition of a set* E. *There exists an irreducible tournament* $\mathfrak{T}$ *whose vertex set is* E *and whose convex subsets are exactly* E, $\emptyset$, *all* S_i *and the singletons of* E *if and only if* $|\mathscr{P}| \notin \{2, 4\}$ *and* $|S_i| \neq 4$ *for every* i.

Proof. Sufficiency. Since $|S_i| \neq 4$, each block S_i can be made into a simple tournament $S_i = \langle S_i; \leqslant_{S_i}\rangle$. As $|\mathscr{P}| \neq 4$, we can construct a simple tournament $\mathscr{P} = \langle \mathscr{P}; \leqslant_{\mathscr{P}}\rangle$. Combining these two constructions, we obtain a tournament on E, which we denote by $\mathfrak{T} = \langle E; \leqslant\rangle$, where $\leqslant$ has the following meaning: $x \leqslant y$ in $\mathfrak{T}$ if either x and y lie in the same block S_i and $x \leqslant_{S_i} y$, or x and y lie in S_i and S_j $(i \neq j)$, respectively, with $S_i <_{\mathscr{P}} S_j$.

We want to show that $\mathfrak{T}$ has the required properties. Of course, every S_i is a convex subset of $\mathfrak{T}$; moreover, no other proper subset of $\mathfrak{T}$, not a singleton, is convex.

First, if A convex in $\mathfrak{T}$ meets some S_i, then $|A \cap S_i| = 1$ or $A \supseteq S_i$, since S_i is simple ($A \cap S_i$ convex in $\mathfrak{T}$ implies $A \cap S_i$ convex in S_i).

Second, if A convex in $\mathfrak{T}$ meets two distinct blocks S_i and S_j, then A meets all blocks. Indeed, let

$$P = \bigcup\{S_i\colon\ S_i \cap A \neq \emptyset\}.$$

Clearly, P is convex in $\mathfrak{T}$, whence $P = T$ as $\{S_i\}$ forms a simple tournament.

Finally, let $x \notin A$ and denote by S_x the block of x. Since A is convex, for all $y \notin S_x$ either $x < y$ or $y < x$, that is $S_x <_{\mathscr{P}} S_y$ or $S_y <_{\mathscr{P}} S_x$. In other terms, $\bigcup\{S_i\colon\ S_i \neq S_x\}$ is convex, which is impossible since $|\mathscr{P}| \neq 2$.

Necessity. The necessity of the condition $|\mathscr{P}| \neq 2$ follows from the irreducibility of $\mathfrak{T}$. Since there is no simple tournament of cardinality 4, the other two conditions are also necessary.

Now we turn our attention to the tournaments in which the non-trivial convex subsets form a chain.

THEOREM 2.6. *If* $\mathscr{C}^*(\mathfrak{T})$, *ordered by inclusion, is a chain, then* $\mathfrak{T}$ *is always subdirectly irreducible; it is irreducible if and only if*

$$\left|T - \bigcup\{C_i\colon\ C_i \in \mathscr{C}^*(\mathfrak{T})\}\right| \neq 1.$$

Moreover, $\mathscr{C}^*(\mathfrak{T})$ *is totally ordered if and only if* $\mathscr{K}(\mathfrak{T})$ *is totally ordered.*

Proof. The first part is a direct consequence of Theorem 2.1. Then let us observe that if $\mathscr{C}^*(\mathfrak{T})$ is a chain, $A = \bigcup\{C_i\colon C_i \in \mathscr{C}^*(\mathfrak{T})\}$ is a convex subset. If $A = T$, then $\mathfrak{T}$ is, clearly, irreducible. If $|T-A| = 1$, then $T-A$ is convex and $\mathfrak{T}$ is reducible. If $|T-A| > 2$, then $\mathfrak{T}$ is irreducible, since, denoting by B one of the two complementary convex subsets of $\mathfrak{T}$, one can easily verify that $B \cap A = \emptyset$ and $B \cap A \neq \emptyset$ are equally impossible.

If $\mathscr{C}^*(\mathfrak{T}) = \{C_i\}$ is totally ordered, the only non-trivial congruences of $\mathfrak{T}$ are Θ_{C_i}, and $\mathscr{K}(\mathfrak{T})$ is a chain. Conversely, if in $\mathscr{C}^*(\mathfrak{T})$ there exist C_1 and C_2 which are not comparable, then $\Theta_{C_1} \| \Theta_{C_2}$, and $\mathscr{K}(\mathfrak{T})$ is not totally ordered.

3. Transitive tournaments. A tournament $\mathfrak{T} = \langle T; \leqslant\rangle$ is *transitive* if the binary relation $\leqslant$ is transitive. By virtue of Theorem 1.5, a tournament is transitive if and only if every of its elements generates a principal ideal. Many characterizations of the transitivity of a tournament are known, at least in the finite case ([4], p. 15). Other characterizations in terms of convex subsets or congruences will be provided. The first one is, in fact, an application of a more general property of trellises ([5], Theorem 40). Its direct proof is very easy and omitted.

THEOREM 3.1. *A tournament $\mathfrak{T}$ is transitive if and only if any two elements of $\mathfrak{T}$ can be separated by an ideal.*

Since the word "ideal" can be replaced by "filter", the following statement is rather natural:

THEOREM 3.2. *A tournament $\mathfrak{T}$ is transitive if and only if every convex subset of $\mathfrak{T}$ is the intersection of an ideal and a filter.*

Proof. In a transitive tournament, every convex subset is, clearly, the intersection of the ideal and the filter it generates. If a tournament is not transitive, it contains a cyclic triple, say $a < b < c < a$. Any ideal (respectively, any filter) which contains a must also contain b and c, and hence the convex set $\{a\}$ cannot be the intersection of an ideal and a filter.

We cannot substitute the words "non-trivial convex subset" for "ideal" in Theorem 1, as shown by the tournament whose elements are a_i ($i = 1, 2, 3$), b_j ($j = 1, 2, 3$) and c, and where the a_i form a transitive subtournament, the b_j form a transitive subtournament and $a_i < b_j < c < a_i$ for every i and j. Nevertheless, we have the following result:

THEOREM 3.3. *A tournament $\mathfrak{T}$ of cardinality at least 3 is transitive if and only if, for any three distinct elements, there exists a convex subset containing two of them but not the third.*

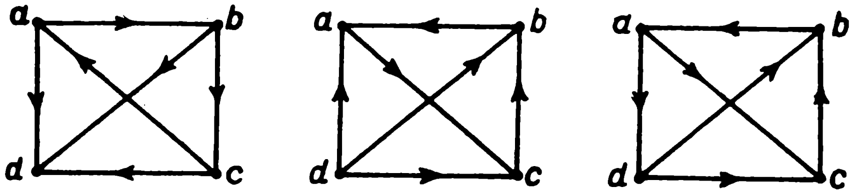
Proof. Sufficiency. Let us suppose $\mathfrak{T}$ is not transitive. There exists a cyclic triple, and any convex subset containing two of its elements must contain the third, which contradicts the hypothesis.

Necessity. Let $a < b < c$ in the transitive tournament $\mathfrak{T}$. The ideal generated by b contains a and b, but not c.

We remind the reader that a class of algebras is said to have the *congruence extension property* if, for any algebra $\mathfrak{A}$ of this class and any subalgebra $\mathfrak{S}$ of $\mathfrak{A}$, whenever Θ is a congruence on $\mathfrak{S}$, there exists a congruence Φ on $\mathfrak{A}$ such that $\Phi|\mathfrak{S} = \Theta$.

THEOREM 3.4. *A tournament $\mathfrak{T}$ ($|T| \neq 3$) is transitive if and only if it enjoys the congruence extension property.*

Proof. Sufficiency. Since tournaments with one or two elements are transitive, we may restrict ourselves to a tournament $\mathfrak{T}$ such that $|T| > 3$. If $\mathfrak{T}$ is not transitive, it has at least one cyclic triple. Since there are three non-isomorphic tournaments of cardinality 4 with at least one cyclic triple ([4], p. 91), $\mathfrak{T}$ has a subset isomorphic to one of the following tournaments:



It is easy to verify that, in every case, $\Theta = \{\{a, d\}, \{c\}\}$ is a congruence on the subalgebra $\{a, c, d\}$. This congruence cannot be extended to the whole tournament $\mathfrak{T}$, since the class containing a and d must then contain b and also c, due to the fact that $\{a, b, c\}$ is a cyclic triple.

Necessity. Let Θ be a congruence on the subset S of the transitive tournament $\mathfrak{T}$. Define the relation Φ as follows:

for any $x, y \in T$, $(x, y) \in \Phi$ if there are $u, v \in S$ such that $u \leqslant x \leqslant y \leqslant v$ and $(u, v) \in \Theta$, or if $x = y$.

Clearly, Φ is a congruence on T extending Θ.

Let us introduce the last definition: an element a of a tournament $\mathfrak{T}$ will be said *isolated* if no non-trivial convex subset of $\mathfrak{T}$ contains a.

THEOREM 3.5. *Let $\mathfrak{T}$ be a finite, non-simple tournament. Then $\mathfrak{T}$ is transitive if and only if $\mathscr{K}(\mathfrak{T})$ is a Boolean lattice.*

Proof. Sufficiency. Let us begin with a few elementary remarks.

First, $|\mathscr{K}(\mathfrak{T})| > 2$, since $\mathfrak{T}$ is not simple.

Second, $\mathfrak{T}$ has no isolated element, since otherwise $\mathscr{K}(\mathfrak{T}) - \{\iota\}$ would have a greatest element and $\mathscr{K}(\mathfrak{T})$ would cease to be complemented.

Third, for any $C \in \mathscr{C}^*(\mathfrak{T})$ and any $x \notin C$, there exists $C' \in \mathscr{C}^*(\mathfrak{T})$ such that $C' \ni x$ and $|C \cap C'| = 1$. Indeed, if Φ denotes the complement of Θ_C in $\mathscr{K}(\mathfrak{T})$, $\Theta_C \vee \Phi = \iota$ requires the existence of $C' \in \mathscr{C}^*(\mathfrak{T})$ with $C' \ni x$, $C' \cap C \neq \emptyset$, whereas $\Theta_C \wedge \Phi = \omega$ yields $|C \cap C'| = 1$.

Now, let us consider an arbitrary triple $\{x, y, z\}$ of T and denote by C a non-trivial convex subset of $\mathfrak{T}$ containing x.

If C contains one of the elements y, z but not the other, the conclusion follows from Theorem 3.3.

If $y \notin C$ and $z \notin C$, there is $C' \in \mathscr{C}^*(\mathfrak{T})$ with $C' \ni y$ and $|C' \cap C| = 1$. If $C' \not\ni z$, then $C \cup C'$ is the desired convex subset. If $C' \ni z$ and $C' \not\ni x$, then C' separates $\{x, y, z\}$ adequately. If $C' \ni z$ and $C' \ni x$, then $C' - C$ behaves as required by Theorem 3.3.

The case where $y \in C$ and $z \in C$ is somewhat harder. Since $C \in \mathscr{C}^*(\mathfrak{T})$, there is $C_1 \in \mathscr{C}^*(\mathfrak{T})$ such that $|C_1 \cap C| = 1$. If C_1 contains one of the elements x, y, z, then $C - C_1$ has the required separation property. Otherwise, since $C - C_1$ is convex, there is C_2 such that $|(C - C_1) \cap C_2| = 1$. Eventually, after finitely many steps, one can find C_k containing one of the elements x, y, z, and the convex subset $C - (C_1 \cup C_2 \cup \ldots \cup C_k)$ is adequate. One may conclude that $\mathfrak{T}$ is transitive.

Necessity. When $\mathfrak{T}$ is a finite transitive tournament, any congruence has in $\mathscr{K}(\mathfrak{T})$ a complement whose description is analogous to those of a finite chain considered as a lattice.

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