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ON SOME NUMERICAL CHARACTERIZATION OF BOOLEAN ALGEBRAS

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Let A be a Boolean algebra. A *measure* on A is a map m from A into the closed interval $[0, 1]$ such that $m(\bigvee_A) = 1$ and $m(a\cup b) = m(a)+ +m(b)$ for $a\cap b = \bigwedge_A$ ($\bigwedge_A$ and $\bigvee_A$ denote the zero and the unit elements of A, respectively). A set X of measures on A is said to be *full* if $m(a) \leqslant m(b)$ for all $m\in X$ implies $a\leqslant b$. We then have $a\neq b$ if and only if there is $m\in X$ such that $m(a)\neq m(b)$. Every Boolean algebra admits a full set of measures. Namely, if A is a Boolean algebra and M the set of all maximal filters in A, then, for each $I\in M$, the characteristic function m_I of I is a measure on A, and the set $\{m_I\colon I\in M\}$ is full (see [3]). These measures are two-valued, but clearly there are full sets of measures which are not two-valued. For example, if $A = \{\bigwedge, a, a', \bigvee\}$ is a four-element Boolean algebra, then the measures m_1 and m_2 defined by $m_1(a) = \alpha$ and $m_2(a) = \beta$, respectively, for any $0<\alpha<1/2$ and $1/2<\beta<1$, form a full set of measures on A.

If X is a non-empty set, $[0,1]^X$ denotes the set of all functions from X into $[0, 1]$. For any $a, b\in[0,1]^X$, $a+b$ and $a-b$ denote the sum and the difference of the functions a and b, respectively. For simplicity, 0 will denote the zero function and 1 the function equal to 1 for all $x\in X$. For $a, b\in[0,1]^X$, $a\leqslant b$ means that $a(x)\leqslant b(x)$ for all $x\in X$.

We now adopt the following definition:

Definition. Let $A\subset[0,1]^X$ be a set of functions from $X\neq\emptyset$ into $[0,1]$. We say that A is a *numerical Boolean algebra* if A is a Boolean algebra with respect to the natural ordering $a\subset b \Leftrightarrow a\leqslant b$ in A with the complementation $a' = 1-a$, and $a\cup b = a+b$ for $a\cap b = \bigwedge_A$.

If A is a numerical Boolean algebra, then we have $a' = 1-a$ and $a\cap a' = \bigwedge_A$ for $a\in A$; hence $\bigvee_A = a\cup a' = a+a' = a+(1-a) = 1$, and $\bigwedge_A = \bigvee_A' = 1' = 1-1 = 0$. Thus the zero and the unit functions belong to A and are the zero and the unit elements of A, respectively. Since in a Boolean algebra $a\cap b = \bigwedge$ is equivalent to $a\subset b'$, in a numerical Boolean algebra $a \cap b = \bigwedge$ is equivalent to $a \leqslant 1-b$, i.e. to $a+b \leqslant 1$. Hence, in a numerical Boolean algebra $a+b \leqslant 1$ implies $a \cup b = a+b$.

Every Boolean algebra can be isomorphically represented as a numerical Boolean algebra and each full set of measures on a Boolean algebra induces such a representation. Namely, let A be a Boolean algebra with a full set X of measures. For each $a \in A$, let $\bar{a}$ denote the function from X into $[0, 1]$ defined by $\bar{a}(x) = x(a)$ for all $x \in X$. We have $A_X = \{\bar{a}: a \in A\} \subset [0, 1]^X$, and it is easy to see that A_X is a numerical Boolean algebra isomorphic to A under the correspondence $a \Leftrightarrow \bar{a}$. If we take X to be the set of all two-valued measures induced by maximal filters in A, then X can be interpreted as the Stone space of A, and A_X is the set of characteristic functions of open-closed subsets of X. Thus we see that the Stone representation is a special case of numerical representation, namely such that all functions in this representation are two-valued.

If $A \subset [0, 1]^X$ is a numerical Boolean algebra, we can easily obtain a full set of measures on A by setting, for each $x \in X$, $m_x(a) = a(x)$ for all $a \in A$. Then m_x is a measure on A and the set $\bar{X} = \{m_x: x \in X\}$ is full. Hence, every numerical Boolean algebra arises essentially from a full set of measures on a Boolean algebra in the manner described above.

The following theorem provides a full characterization of numerical Boolean algebras in the family of all partially ordered subsets of $[0, 1]^X$.

THEOREM. *Let X be a non-empty set and let $A \subset [0, 1]^X$ be a set of functions from X into $[0, 1]$. Let us say that a sequence (a_1, a_2, a_3) of members of A is a triangle if $a_i + a_j \leqslant 1$ for $i \neq j$. Then A is a numerical Boolean algebra if and only if the following conditions are satisfied:*

1° *The zero function $a \equiv 0$ belongs to A.*

2° *For every $a \in A$, $1-a \in A$.*

3° *For every triangle (a_1, a_2, a_3), $a_i \in A$, $i = 1, 2, 3$, we have $a_1 + a_2 + a_3 \in A$.*

4° *For every pair $a, b \in A$ there is a triangle (c_1, c_2, c_3), $c_i \in A$, $i = 1, 2, 3$, such that $a = c_1 + c_2$ and $b = c_2 + c_3$.*

Proof. Let A be a numerical Boolean algebra. It follows from the definition that A satisfies conditions 1° and 2°. Let (a_1, a_2, a_3) be a triangle. We have, in A, $a \cap b = \bigwedge_A$ equivalent to $a+b \leqslant 1$, so $a_i \cap a_j = \bigwedge_A$ for $i \neq j$. From the definition of a numerical Boolean algebra it then follows that $a_1 \cup a_2 = a_1 + a_2$. Since

$$(a_1 \cup a_2) \cap a_3 = (a_1 \cap a_3) \cup (a_2 \cap a_3) = \bigwedge_A,$$

we obtain

$$(a_1 \cup a_2) \cup a_3 = (a_1 \cup a_2) + a_3 = a_1 + a_2 + a_3 \in A.$$

Thus condition 3° holds. For $a, b \in A$, let $c_2 = a \cap b$, $c_1 = a \cap c_2'$ and $c_3 = b \cap c_2'$. It follows from the distributive laws that $c_i \cap c_j = \bigwedge_A$, i.e.

$c_i+c_j \leqslant 1$ for $i \neq j$; then (c_1, c_2, c_3) is a triangle, and $a = c_1+c_2$ and $b = c_2+c_3$. Hence condition 4° also holds.

Assume now that $A \subset [0,1]^X$ satisfies conditions 1°-4°. We consider in A the natural partial order $a \leqslant b \Leftrightarrow a(x) \leqslant b(x)$ for $x \in X$. Let $a' = 1-a$ for $a \in A$. We have to show that A is a Boolean algebra with respect to this order with the complementation a'. First, we show that $(A, \leqslant)$ is a lattice. Let us say that $a, b \in A$ are orthogonal, $a \perp b$, if $a+b \leqslant 1$. We shall show that $a \cup b = a+b$ for $a \perp b$. Observe first that since $(a, b, 0)$ is a triangle, $a+b \in A$ and $a \leqslant a+b$, $b \leqslant a+b$. Let $a \leqslant c$ and $b \leqslant c$ for $c \in A$. This implies $(1-c)+a \leqslant 1$ and $(1-c)+b \leqslant 1$. Hence $(a, b, 1-c)$ is a triangle and, by 3°, $a+b+(1-c) \in A$, i.e. $a+b+(1-c) \leqslant 1$. Consequently, $a+b \leqslant c$ which shows that $a \cup b = a+b$ for $a \perp b$. If (c_1, c_2, c_3) is a triangle, $c_1+c_2+c_3 \leqslant 1$ by 3°, and $(c_1 \cup c_2) \perp c_3$. Hence $c_1 \cup c_2 \cup c_3$ exists. Now, for arbitrary $a, b \in A$, there is a triangle (c_1, c_2, c_3) such that $a = c_1+c_2 = c_1 \cup c_2$ and $b = c_2+c_3 = c_2 \cup c_3$. Consequently, $c_1 \cup c_2 \cup c_3 = (c_1 \cup c_2) \cup (c_2 \cup c_3) = a \cup b$, i.e. $a \cup b$ exists. Since, in A, $a \leqslant b$ is equivalent to $b' \leqslant a'$, and $(a')' = a$, it is evident that, for any $a, b \in A$, $(a' \cup b')'$ exists and equals $a \cap b$. Hence $(A, \leqslant, ')$ is an orthocomplemented lattice. For $a \leqslant b$, $a, b \in A$, we have $(1-b) \perp a$, so $(1-b)+a \in A$, and $c = b-a = 1-[(1-b)+a] \in A$. Thus, for $a \leqslant b$, there is $c \in A$, $c \perp a$, such that $a \cup c = b$, i.e. A is orthomodular (see [2], theorem 29.13). According to definition 36.2 in [2], two elements a and b in an orthomodular lattice are said to *commute*, $a\mathrm{C}b$, if $a = (a \cap b) \cup (a \cap b')$. It follows from 4° that in A any two elements commute. In fact, let $a, b \in A$. Let (c_1, c_2, c_3) be a triangle such that $a = c_1+c_2$, $b = c_2+c_3$, and let $d = (c_1+c_2+c_3)' = 1-(c_1+c_2+c_3)$. In the set $\{c_1, c_2, c_3, d\}$ any two elements are orthogonal. Consequently, we have

$$a' = (c_1 \cup c_2)' = 1-c_1-c_2 = d+c_3$$

and

$$b' = (c_2 \cup c_3)' = d+c_1.$$

Thus $a' \cup b' = c_1+d+c_3$, and $a \cap b = (a' \cup b')' = 1-(c_1+d+c_3) = c_2$. Similarly, we have $a \cap b' = c_1$. Hence $(a \cap b) \cup (a \cap b') = c_1+c_2 = a$, i.e. $a\mathrm{C}b$. It was shown by Foulis [1] (see [2], theorem 36.7) that if a, b and c are elements of an orthomodular lattice such that one of them commutes with the other two, then each of the distributive laws involving these three elements is satisfied. Since, in A, $a\mathrm{C}b$ for all $a, b \in A$, we infer that A is a complemented distributive lattice, i.e. A is a Boolean algebra with respect to the order $\leqslant$ and complementation $'$. In a Boolean algebra, $a \perp b$ is equivalent to $a \cap b = \bigwedge$. We have shown that $a \cup b = a+b$ for $a \perp b$, hence A is a numerical Boolean algebra. This completes the proof of the theorem.

REFERENCES

[1] D. J. Foulis, *A note on orthomodular lattices*, Portugaliae Mathematica 21 (1962), p. 65-72.
[2] F. Maeda and S. Maeda, *Theory of symmetric lattices*, Berlin 1971.
[3] R. Sikorski, *Boolean algebras*, Berlin 1964.

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