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SCHAUDER DECOMPOSITIONS IN DUAL AND BIDUAL SPACES

BY

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1. Introduction. In this paper we throughout assume that $(\chi, \mathscr{F})$ is a Hausdorff locally convex topological vector space (l.c. TVS) equipped with the topology $\mathscr{F}$. Let χ^* be the topological dual of $(\chi, \mathscr{F})$ and χ^{**} be the topological dual of $(\chi^*, \beta(\chi^*, \chi))$, where $\beta(\chi^*, \chi)$ is the strong topology on χ^* generated by the polars

$$A^0 = \{f\colon f\in \chi^*, |f(x)| \leqslant 1\} \quad \text{for all } x\in A,$$

where A is a $\sigma(\chi, \chi^*)$-bounded subset of χ. Our aim is to interrelate Schauder decompositions in χ, χ^* and χ^{**} equipped with various locally convex topologies. Motivations for investigating these results are spelled out in Sections 3 and 4 of this paper.

Let $\{M_i\}$ be a sequence of non-trivial subspaces of χ. We say that $\{M_i\}$ is an *$\mathscr{F}$-basis* of subspaces (that is, $\mathscr{F}$-bos) or an *$\mathscr{F}$-decomposition* if to each $x\in \chi$ there corresponds a unique sequence $\{x_i\}$, $x_i\in M_i$, such that

$$x = \lim_{n\to\infty} \sum_{i=1}^{n} x_i, \tag{1.1}$$

the convergence of the infinite series being with respect to the topology $\mathscr{F}$ on χ. Corresponding to a given $\mathscr{F}$-bos $\{M_i\}$ in $(\chi, \mathscr{F})$, there exists a sequence $\{P_i\}$ of orthogonal projections on χ defined by $P_i(x) = x_i$, where x is given by (1.1). Moreover, $P_i(\chi) = M_i = R(P_i)$, the range of P_i, $i \geqslant 1$. In case each P_i is $\mathscr{F}$-continuous, the $\mathscr{F}$-decomposition $\{M_i\}$ or, more explicitly, $\{R(P_i), P_i\}$ is usually referred to as an $\mathscr{F}$-Schauder decomposition or merely an $\mathscr{F}$-Sbos.

2. Adjoints on χ^*. In the sequel we will need various results concerning properties of adjoint operators on χ^*. To begin with let us recall the canonical embedding $J\colon (\chi, \mathscr{F}) \to (\chi^{**}, \beta(\chi^{**}, \chi^*))$ defined by $J(x)(f) = f(x)$ for $x\in \chi$ and $f\in \chi^*$. If $(\chi, \mathscr{F})$ is infrabarrelled ([2], p. 217), it is known ([2], p. 229) that J is a topological isomorphism from $(\chi, \mathscr{F})$

onto $(J(\chi), \beta(J(\chi), \chi^*))$. The proof of the following simple proposition is omitted:

PROPOSITION 2.1. *The map* J: $(\chi, \sigma(\chi, \chi^*)) \to (J(\chi), \sigma(J(\chi), \chi^*))$ *is a topological isomorphism.*

For an l.c. TVS $(\chi, \mathscr{F})$ we now recall the following ([3], Propositions 2.3 and 2.6):

PROPOSITION 2.2. *The adjoint map* T^* *of a* $\beta(\chi^*, \chi)$-$\beta(\chi^*, \chi)$ *continuous linear operator* T *on* χ^* *into itself takes* χ^{**} *into* χ^{**} *and is* $\beta(\chi^{**}, \chi^*)$-$\beta(\chi^{**}, \chi^*)$ *continuous.*

PROPOSITION 2.3. *Let* T *be a* $\sigma(\chi^*, \chi)$*-continuous linear operator on* χ^* *and let* T^* *be the adjoint of* T *defined on the algebraic dual of* χ^*. *Then* $T^*(J(\chi)) \subset J(\chi)$.

PROPOSITION 2.4. *Let* $(\chi, \mathscr{F})$ *be as before. Suppose* T: $\chi^* \to \chi^*$ *is a linear operator such that* T *is* $\sigma(\chi^*, \chi)$-$\sigma(\chi^*, \chi)$ *continuous. Then* T *is* $\beta(\chi^*, \chi)$-$\beta(\chi^*, \chi)$ *continuous.*

PROPOSITION 2.5. *Let* $(\chi, \mathscr{F})$ *be an infrabarrelled space and* T *be a linear operator on* χ^* *such that* T *is* $\sigma(\chi^*, \chi)$-$\sigma(\chi^*, \chi)$ *continuous. Then* $E = J^{-1}T^*J$ *is an* $\mathscr{F}$-$\mathscr{F}$ *continuous linear operator.*

Remark. Following Proposition 2.5 one can easily see that if T is a projection on χ^*, then E defined as above is also a projection on χ.

3. $\sigma(\chi^{**}, \chi^*)$**-Sbos for** χ^*. In an earlier paper [3], two of us have proved the following results:

THEOREM 3.1. (i) *Let* $(\chi, \mathscr{F})$ *be an l.c. TVS. If* $\{M_i, E_i\}$ *is an* $\mathscr{F}$*-Sbos for* χ, *then* $\{R(E_i^*), E_i^*\}$ *is a* $\sigma(\chi^*, \chi)$*-Sbos for* χ^*.

(ii) *Let* $(\chi, \mathscr{F})$ *be a complete barrelled space. If* $\{N_i, P_i\}$ *is a* $\sigma(\chi^*, \chi)$*-Sbos for* χ^*, *then* $\{R(E_i), E_i\}$, *where* $E_i = J^{-1}P_i^*J$ *is an* $\mathscr{F}$*-Sbos for* χ.

THEOREM 3.2. *Let* $(\chi, \mathscr{F})$ *be a complete barrelled space such that* $\{N_i, P_i\}$ *is a* $\sigma(\chi^*, \chi)$*-Sbos for* χ^*. *Then* $\{N_i, P_i\}$ *is a* $\beta(\chi^*, \chi)$*-Sbos for* $[\bigcup_{i=1}^{\infty} N_i]$, *the* $\beta(\chi^*, \chi)$*-closure of the space generated by* $\bigcup_{i=1}^{\infty} N_i$.

In view of Theorem 3.1 there arise two natural questions, namely, if $\{M_i, E_i\}$ is a $\sigma(\chi^{**}, \chi^*)$-Sbos for χ^{**}, then

(i) under what circumstances $\{M_i, E_i\}$ is $\beta(\chi^{**}, \chi^*)$-Sbos for $[\bigcup_{i=1}^{\infty} M_i]$, the $\beta(\chi^{**}, \chi^*)$-closure of the space generated by $\bigcup_{i=1}^{\infty} M_i$, and

(ii) under what circumstances $J(\chi) = [\bigcup_{i=1}^{\infty} M_i]$?

A partial answer to these problems is contained in the following two theorems for which we need a few more definitions.

A space $(\chi, \mathscr{F})$ is said to be *bornological* if every balanced, convex and bornivorous (i.e., a set which absorbs every $\mathscr{F}$-bounded set) subset is $\mathscr{F}$-neighbourhood of 0 in χ. A space $(\chi, \mathscr{F})$ is said to be *distinguished* if every $\sigma(\chi^{**}, \chi^*)$-bounded subset in χ^{**} is contained in $\sigma(\chi^{**}, \chi^*)$-closure of some $\mathscr{F}$-bounded subset of χ. An $\mathscr{F}$-Sbos $\{N_i, D_i\}$ in $(\chi, \mathscr{F})$ is called *shrinking* if $\{R(D_i^*)\}$ is a $\beta(\chi^*, \chi)$-Sbos in χ^*. We now state and prove the following result:

THEOREM 3.3. *Let $(\chi, \mathscr{F})$ be a distinguished bornological space and χ^{**} be its bidual. Suppose χ^{**} has a $\sigma(\chi^{**}, \chi^*)$-Sbos $\{M_i, E_i\}$ and let E_i' be the restriction of E_i to $J(\chi)$. If $\{R(E_i')\}$ is a Sbos for $J(\chi)$ with respect to the topology induced on $J(\chi)$ by $\beta(\chi^{**}, \chi^*)$, then $\{R(J^{-1}E_i'J)\}$ is a shrinking Sbos for χ.*

Proof. Let J_*: $\chi^* \to \chi^{***}$ be the canonical embedding, where χ^{***} is the dual of χ^{**} with respect to $\beta(\chi^{**}, \chi^*)$. As χ is a distinguished bornological space, $\big(\chi^*, \beta(\chi^*, \chi)\big)$ is barrelled ([2], p. 288) and complete ([2], p. 223). Now $\{M_i, E_i\}$ is a $\sigma(\chi^{**}, \chi^*)$-Sbos for χ^{**}. It follows, therefore, by Theorem 3.1 (ii) that $\{R(J_*^{-1}E_i^*J_*)\}$ is $\beta(\chi^*, \chi)$-Sbos for χ^{**}. From the fact that $\{R(E_i')\}$ is Sbos for $J(\chi)$ with respect to the topology induced by $\beta(\chi^{**}, \chi^*)$, $\{R(J^{-1}E_i'J)\}$ can easily be shown to form an $\mathscr{F}$-Sbos for χ.

Now, in order to complete the proof, we need to show that $(J^{-1}E_i'J)^* = J_*^{-1}E_i^*J_*$. Since $\{R(E_i')\}$ is a $\beta(\chi^{**}, \chi^*)$-Sbos for $J(\chi)$, $E_i'J(\chi) \subset J(\chi)$ for each $i \geqslant 1$. Therefore, for each $x \in \chi$, there exists a $y_i \in \chi$ such that $E_i'J(x) = J(y_i)$. By Proposition 2.3, $E_i^*J_*(\chi^*) = J_*(\chi^*)$ for each $i \geqslant 1$ implies that for $f \in \chi^*$ there is $g_i \in \chi^*$ such that $E_i^*J_*(f) = J_*(g_i)$. Hence, for $x \in \chi$,

$$\begin{aligned}
[(J^{-1}E_i'J)^*(f)](x) &= f[(J^{-1}E_i'J)(x)] \\
&= f(y_i) = J(y_i)(f) = J_*(f)\big(J(y_i)\big) = J_*(f)[E_i'J(x)] \\
&= J_*(f)[E_iJ(x)] = (E_i^*J_*)(f)\big(J(x)\big) = J_*(g_i)\big(J(x)\big) \\
&= J(x)(g_i) = g_i(x) = [(J_*^{-1}E_i^*J_*)(f)](x)
\end{aligned}$$

and this implies

$$(J^{-1}E_i'J)^* = J_*^{-1}E_i'J_*.$$

This completes the proof.

THEOREM 3.4. *If $(\chi, \mathscr{F})$ is an infrabarrelled space having a shrinking Sbos $\{N_i, D_i\}$, then $\{R(D_i^{**})\}$ is a $\sigma(\chi^{**}, \chi^*)$-Sbos for χ^{**}. Moreover, if P_i is the restriction of D_i^{**} to $J(\chi)$, then $\{R(P_i)\}$ is a Sbos for $J(\chi)$ in the topology induced by $\beta(\chi^{**}, \chi^*)$.*

Proof. Since $\{N_i, D_i\}$ is a shrinking Sbos for $(\chi, \mathscr{F})$, it follows that $\{R(D_i^*)\}$ is a $\beta(\chi^*, \chi)$-Sbos for χ^*. Hence, by Theorem 3.1 (i), $\{R(D_i^{**})\}$ is a $\sigma(\chi^{**}, \chi^*)$-Sbos for χ^{**}.

Clearly, $\{R(JD_iJ^{-1})\}$ is a Sbos for $J(\chi)$. Thus we need to show that $JD_iJ^{-1} = P_i$ for each $i \geqslant 1$. So, let $x \in \chi$ and $f \in \chi^*$. Then

$$\big(P_iJ(x)\big)(f) = \big(D_i^{**}J(x)\big)(f) = J(x)\big(D_i^*(f)\big) = \big(D_i^*(f)\big)(x) = f\big(D_i(x)\big)$$
$$= \big(J\big(D_i(x)\big)\big)(f) = \big[(JD_iJ^{-1})\big(J(x)\big)\big](f)$$

implies

$$P_i = JD_iJ^{-1},$$

thereby completing the proof.

4. $\mu(\chi^*, \chi)$-decompositions in χ^*. In this section we investigate Schauder decompositions in χ^* with respect to the strongest locally convex topology $\mu(\chi^*, \chi)$ on χ^*, which coincides with $\sigma(\chi^*, \chi)$ on χ^* on every equicontinuous subset of χ^*. For the sake of completeness we include a short description of the topology $\mu(\chi^*, \chi)$ leading to the formation of its neighbourhood system at the origin in χ^*. Let, therefore, $\lambda(\chi^*, \chi)$ be the locally convex topology on χ^* generated by polars of all precompact subsets of $(\chi, \mathscr{F})$. It is known ([2], p. 235, Proposition 8) that the topologies induced on an equicontinuous subset of χ^* by $\lambda(\chi^*, \chi)$ and $\sigma(\chi^*, \chi)$ are equivalent. Further, suppose that $\nu(\chi^*, \chi)$ is the strongest topology on χ^* which induces on every equicontinuous subset the topology induced thereon by $\sigma(\chi^*, \chi)$. The topology need not be a linear topology ([4], p. 160). We, therefore, introduce the topology $\mu(\chi^*, \chi)$ defined as above. If $\varkappa(\chi^*, \chi)$ (Arens [1] writes $c(\chi^*, \chi)$ for $\varkappa(\chi^*, \chi)$) denotes the topology on χ^* generated by the polars of all balanced, convex and $\mathscr{F}$-compact subsets in $(\chi, \mathscr{F})$, then it is clear that

$$\sigma(\chi^*, \chi) \subset \varkappa(\chi^*, \chi) \subset \tau(\chi^*, \chi), \tag{4.1}$$

where $\tau(\chi^*, \chi)$ is the Mackey topology on χ^*.

If $(\chi, \mathscr{F})$ is metrizable, it follows from Banach-Dieudonné Theorem ([2], p. 245) that

$$\tau(\chi^*, \chi) = \nu(\chi^*, \chi) = \mu(\chi^*, \chi), \tag{4.2}$$

and a neighbourhood system at the origin in χ^* in the topology $\mu(\chi^*, \chi)$ is given by the polars of subsets of χ consisting of points of a sequence converging to zero in $(\chi, \mathscr{F})$ (see, for instance, [2], p. 247, Exercise 1(b)).

Let us observe that if $(\chi, \mathscr{F})$ is quasi-complete, then

$$\lambda(\chi^*, \chi) = \varkappa(\chi^*, \chi). \tag{4.3}$$

Thus, if $(\chi, \mathscr{F})$ is a quasi-complete metrizable space, it follows from (4.1) through (4.3) that

$$\sigma(\chi^*, \chi) \subset \mu(\chi^*, \chi) \subset \tau(\chi^*, \chi). \tag{4.4)}$$

Since $\mu(\chi^*, \chi)$ is locally convex, it turns out from Mackey-Arens Theorem ([2], p. 205; [1], p. 793) that $\mu(\chi^*, \chi)$ is compatible ([2], p. 198) with the duality of χ and χ^*, that is, we have

PROPOSITION 4.1. *If $(\chi, \mathcal{F})$ is quasi-complete and metrizable, then a linear functional φ on χ^* is $\sigma(\chi^*, \chi)$-continuous if and only if it is $\mu(\chi^*, \chi)$--continuous.*

We now prove

PROPOSITION 4.2. *Let $(\chi, \mathcal{F})$ be quasi-complete and metrizable and let T be a linear operator on χ^*. Then T is $\sigma(\chi^*, \chi)$-$\sigma(\chi^*, \chi)$ continuous if and only if it is $\mu(\chi^*, \chi)$-$\mu(\chi^*, \chi)$ continuous.*

Proof. Suppose first T is $\sigma(\chi^*, \chi)$-$\sigma(\chi^*, \chi)$ continuous. Let U be a 0-neighbourhood in $\mu(\chi^*, \chi)$. We may assume that

$$U = \{g\colon\ g \in \chi^*,\ |g(x_i)| \leqslant 1\},$$

where $\{x_i\}$ is a sequence in $(\chi, \mathcal{F})$ with $x_i \to 0$ in $\mathcal{F}$. From Proposition 2.3, $T^* J(\chi) \subset J(\chi)$; moreover, $(\chi, \mathcal{F})$ being metrizable, is infrabarrelled. Therefore, by Proposition 2.5, $J^{-1}T^*J$ is an $\mathcal{F}$-$\mathcal{F}$ continuous linear operator from χ into χ which implies $J^{-1}T^*J(x_i) \to 0$ in $(\chi, \mathcal{F})$. Hence

$$V = \big\{f\colon\ f \in \chi^*,\ \big|f\big(J^{-1}T^*J(x_i)\big)\big| \leqslant 1\big\}$$

defines a neighbourhood at origin in χ^* with respect to $\mu(\chi^*, \chi)$. The fact that $T^*J(\chi) \subset J(\chi)$ implies that for each x_i there is a $y_i \in \chi$ with $T^*J(x_i) = J(y_i)$. Thus $f \in V$ and

$$\begin{aligned} |T(f)(x_i)| &= \big|J(x_i)\big(T(f)\big)\big| = |T^*J(x_i)(f)| \\ &= |J(y_i)(f)| = |f(y_i)| = \big|f\big(J^{-1}T^*J(x_i)\big)\big| \leqslant 1, \end{aligned}$$

that is, $T(f) \in U$ and so T is $\mu(\chi^*, \chi)$-$\mu(\chi^*, \chi)$ continuous.

For converse, let T be $\mu(\chi^*, \chi)$-$\mu(\chi^*, \chi)$ continuous and let U be any $\sigma(\chi^*, \chi)$-neighbourhood of 0 in χ^*. For some $\varepsilon > 0$ we may take

$$U = \{f\colon\ |f(x_i)| < \varepsilon,\ x_i \in \chi,\ i = 1, 2, \ldots, n\}.$$

For each $i \geqslant 1$ it easily follows that $J(x_i)T$ is a $\mu(\chi^*, \chi)$-continuous linear functional on χ^*. Indeed, $f_n \to 0$ in $\mu(\chi^*, \chi)$ implies $T(f_n) \to 0$ in $\mu(\chi^*, \chi)$, whence $Tf_n(x_i) \to 0$ for each x_i ($i = 1, 2, \ldots, n$) and, consequently, $J(x_i)\big(T(f_n)\big) \to 0$ for $1 \leqslant i \leqslant n$. Thus $J(x_i)T$ is $\mu(\chi^*, \chi)$-continuous. Therefore, by Proposition 4.1, $J(x_i)T$ is a $\sigma(\chi^*, \chi)$-continuous linear functional on χ^*.

Thus, for $\varepsilon > 0$ defining U and for each fixed i, $1 \leqslant i \leqslant n$, there is a $\sigma(\chi^*, \chi)$-neighbourhood U_i such that $f \in U_i$ implies

$$|T(f)(x_i)| = \big|J(x_i)\big(T(f)\big)\big| < \varepsilon.$$

Let

$$V = \bigcap_{i=1}^{n} U_i.$$

Then $f \in V$ implies $|T(f)(x_i)| < \varepsilon$ for $1 \leqslant i \leqslant n$, i.e., $T(f) \in U$. Hence T is $\sigma(\chi^*, \chi)$-$\sigma(\chi^*, \chi)$ continuous. The proof is completed.

PROPOSITION 4.3. *Let $(\chi, \mathscr{F})$ be quasi-complete and metrizable. Then, for a sequence $\{g_n\}$ in χ^* and a function $f \in \chi^*$, $\lim_n g_n = f$ in $\sigma(\chi^*, \chi)$ if and only if $\lim_n g_n = f$ in $\mu(\chi^*, \chi)$.*

Proof. Since the topology $\mu(\chi^*, \chi)$ is stronger than $\sigma(\chi^*, \chi)$, it follows that $\lim_n g_n = f$ in $\mu(\chi^*, \chi)$ implies $\lim_n g_n = f$ in $\sigma(\chi^*, \chi)$.

To prove the converse, let $\lim_n g_n = f$ in $\sigma(\chi^*, \chi)$. Hence the set $M = \{g_1, g_2, \ldots; f\}$ is $\sigma(\chi^*, \chi)$-bounded. As χ is quasi-complete, it follows that M is $\beta(\chi^*, \chi)$-bounded (see [2], p. 210, Theorem 4). Moreover, χ is metrizable and so it is infrabarrelled. Consequently, M is an equicontinuous subset of χ^* (see [2], p. 217, Proposition 6). Let G be any $\mu(\chi^*, \chi)$-neighbourhood of $0 \in \chi^*$. Then $M \cap (G+f)$ is a neighbourhood of f in M. Hence, by the definition of $\mu(\chi^*, \chi)$, there exists a $\sigma(\chi^*, \chi)$-neighbourhood V of 0 in χ^* such that $M \cap (G+f) = M \cap (V+f)$. In view of the hypothesis, there is an integer $N = N(V)$ such that if $g_n \in f+V$ for all $n \geqslant N$, then $g_n \in f+G$ for all $n \geqslant N$. Hence $\lim_n g_n = f$ in $\mu(\chi^*, \chi)$. This completes the proof.

Remark. Proposition 4.3 is true also for barrelled spaces.

THEOREM 4.4. *If $(\chi, \mathscr{F})$ is a quasi-complete, metrizable l.c. TVS, then a sequence of non-trivial subspaces $\{N_i\}$ in χ^* is a $\sigma(\chi^*, \chi)$-Sbos for χ^* if and only if $\{N_i\}$ is $\mu(\chi^*, \chi)$-Sbos for χ^*.*

Proof. Let $\{N_i\}$ be a $\sigma(\chi^*, \chi)$-Sbos for χ^* and $\{P_i\}$ be the associated sequence of orthogonal projections. Then each P_i is $\sigma(\chi^*, \chi)$-continuous and each $f \in \chi^*$ is uniquely expressed as

$$f = \lim_n \sum_{i=1}^{n} P_i(f) \quad \text{in } \sigma(\chi^*, \chi).$$

By Proposition 4.2, each P_i is $\mu(\chi^*, \chi)$-continuous. Also, by Proposition 4.3,

$$f = \lim_n \sum_{i=1}^{n} P_i(f) \quad \text{in } \mu(\chi^*, \chi),$$

and the expansion is unique. Therefore, $\{N_i\}$ is $\mu(\chi^*, \chi)$-Sbos for χ^*. With the help of Propositions 4.2 and 4.3, the converse can be proved in a similar fashion.

Making use of Theorem 3.1, we can derive the following result as

COROLLARY 4.5. *If* $\{M_i, E_i\}$ *is an* $\mathscr{F}$*-Sbos for a quasi-complete, metrizable l.c. TVS* $(\chi, \mathscr{F})$, *then* $\{R(E_i^*), E_i^*\}$ *is a* $\mu(\chi^*, \chi)$*-Sbos for* χ^*.

Remark. Some of the propositions in this note are generalizations of the results due to Retherford [5].

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