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ON LOCALLY CONNECTED PLANE ONE-TO-ONE CONTINUOUS IMAGES OF $[0, \infty)$

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Let X be a metric space satisfying the following condition:

$(*)$ $X = \bigcup_{j=1}^{\infty} [a, a_j]$, where $[a, a_j]$ denotes an arc with non-cut points a and a_j, $[a, a_j] \subset [a, a_{j+1}]$, $a_j \neq a_{j+1}$ for each j, and $a = \lim_{j\to\infty} a_j$.

B. Knaster has asked (cf. [6]): If X is in the plane and is locally connected, then must X be a simple closed curve?

Weron [6] showed that the answer is yes if, in addition, it is assumed that X is hereditarily locally connected. Our main purpose here is to characterize all locally connected plane one-to-one continuous images of $[0, \infty)$. As a consequence, we answer Knaster's question above with no additional assumptions.

We begin with the following simple observation:

LEMMA. *If X is a metric space satisfying $(*)$, then there is a one-to-one continuous function f from $[0, \infty)$ onto X such that $f(0) = a$ and $f(j) = a_j$ for each $j = 1, 2, \ldots$*

Proof. Let f_1: $[0, 1] \to [a, a_1]$ be a homeomorphism such that $f_1(0) = a$ and $f_1(1) = a_1$. Assume inductively that we have defined f_n and let

$$A_{n+1} = ([a, a_{n+1}] - [a, a_n]) \cup \{a_n\}.$$

From $(*)$ we see that A_{n+1} is an arc with non-cut points a_n and a_{n+1}. Let f_{n+1}: $[n, n+1] \to A_{n+1}$ be a homeomorphism such that $f_{n+1}(n) = a_n$ and $f_{n+1}(n+1) = a_{n+1}$. Clearly, the function f: $[0, \infty) \to X$ given by

$$f(t) = f_n(t) \quad \text{for } t \in [n-1, n]$$

is the desired mapping.

In [4] (see also Theorem 7.1 of [5], p. 69), we determined the structure of all locally compact one-to-one continuous images of $[0, \infty)$, and showed that all such images are planar. The problem of characterizing the planar one-to-one continuous images of $[0, \infty)$ appears to be very difficult, but with the condition of local connectivity we may use a theorem of Jones [1] to obtain the following result:

THEOREM. *If M is a locally connected planar one-to-one continuous image of $[0, \infty)$, then M must be one of the following*:

(i) *a half-line*;

(ii) *a simple closed curve*;

(iii) *a simple closed curve with a sticker, i.e., homeomorphic to*

$$\{(x, y) \in R^2\colon\ x^2+y^2 = 1\} \cup \{(x, y) \in R^2\colon\ x = 0 \text{ and } 1 \leqslant y \leqslant 2\}.$$

Proof. Assume that $f\colon [0, \infty) \to M$ is a one-to-one continuous function onto M where M is locally connected and in the plane. We also assume, without loss of generality, that M is bounded. Let $N = M - \{f(0)\}$. Then N is a locally connected one-to-one continuous plane image of a line. Hence, by Theorem 2 of [1], N is either (a) a line, (b) a figure eight, (c) a dumbbell, (d) a theta curve, or (e) an (open ended) noose. Since N cannot be compact (because $f(0)$ is a limit point of N), N cannot be (b), (c), or (d). Assume that N is (e). Let $r \in (0, \infty)$ be such that $f(r)$ is the point of N of order three. It follows easily that, since $f([0, r])$ is an arc, $f\big([r, \infty)\big)$ is the simple closed curve in N. Therefore, since

$$M = f([0, r]) \cup f\big([r, \infty)\big),$$

M is (iii). Next, assume that N is (a). Hence,

(1) if $t_n \to \infty$ and $f(t_n) \to x \in M$, then $x = f(0)$.

Assume that M is not (i). Then, of course, f is not a homeomorphism. This implies (using (1))

(2) there are $s_n \to \infty$ such that $f(s_n) \to x \in M$ and $x = f(0)$.

Hence, letting $K = \mathrm{cl}\big[f\big([2, \infty)\big)\big]$, we see that K is a continuum (recall that M is bounded, so K is compact) and $f([0, 1])$ is an arc such that $f([0, 1]) \cap K = \{f(0)\}$. By well-known techniques (or by Theorem 44 of [3], p. 195, directly) it can be shown that there is an arc a in R^2 such that

$$a \cap \big[f([0, 1]) \cup K\big] = \{f(0)\}.$$

We assume, modifying a if necessary, that $a \cap f([1, 2]) = \emptyset$ and that $f(0)$ is one of the non-cut points of a. Let p be the other non-cut point of a and let

$$Z = [a \cup M] - \{p\}.$$

It is easy to see that Z is a locally connected one-to-one continuous planar image of a line. Hence, again by Theorem 2 of [1], Z is either (a), (b), (c), (d), or (e). From the fact that Z is not compact and from (2) we conclude that Z must be (e). It follows that M is (ii).

The following corollary answers Knaster's question affirmatively.

COROLLARY. *If X is a locally connected set in the plane satisfying condition $(*)$, then X is a simple closed curve.*

Proof. The Corollary is a simple consequence of the Lemma and the Theorem. In particular, X must be (i), (ii), or (iii). Since

$$a = \lim_{j\to\infty} a_j, \quad \text{where } a_j = f(j) \text{ for each } j = 1, 2, \ldots,$$

X is not (i) or (iii) (the latter because $f(0) = a$). Hence, X is (ii).

We mention that there are locally connected spaces in R^3 satisfying $(*)$ which are not simple closed curves. Such an example can be obtained from Example 0 of [2]. Hence, the condition of being planar is important.

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