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A CENTRAL LIMIT THEOREM FOR PIECEWISE CONVEX TRANSFORMATIONS OF THE UNIT INTERVAL

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1. Introduction. In [1], Boyarsky and Scarowsky have shown that a central limit theorem holds for a class of piecewise C^2 transformations from an interval into itself with slopes of absolute value greater than 1 and taking partition points into partition points. A central limit theorem was given also by Tran Vinh-Hien [10] for a class of Rényi's transformations and by Ishitani [3] for a class of piecewise linear transformations. The purpose of our paper is to prove a central limit theorem for piecewise convex transformations which not necessarily satisfy the assumptions of the theorem given in [1].

In Section 2 we state the main result and in Section 3 we give preparatory lemmas for the proof of the theorem and we prove the theorem.

2. A central limit theorem. Let $([0, 1], \Sigma, m)$ be a probability space with Lebesgue measure m and let $\tau: [0, 1] \to [0, 1]$ be a transformation of the unit interval into itself satisfying the following conditions:

(a) There exists a partition $0 = a_0 < a_1 < \ldots < a_p = 1$ of the unit interval such that for each integer i ($i = 1, 2, \ldots, p$) the restriction τ_i of τ to the open interval (a_{i-1}, a_i) is a C^2-function which can be extended to the closed interval $[a_{i-1}, a_i]$ as a C^2-function.

(b) $\tau_i = \tau|_{[a_{i-1}, a_i]}$ ($i = 1, 2, \ldots, p$) are convex.

(c) $\tau_i(a_{i-1}) = 0$ ($i = 1, 2, \ldots, p$).

(d) $\tau([a_0, a_1]) = [0, 1]$.

(e) The inequality

$$s = \sup_{i,x} \left| \frac{\varphi_i''(x)}{\varphi_i'(x)} \right| + \sup_{i,x} |\varphi_i'(x)| \left(1 + \sum_{i=1}^{p} \delta_i\right) < 1$$

holds, where $\delta_i = 2 - \operatorname{card}\{\{\tau(a_i+), \tau(a_i-)\} \cap \{0, 1\}\}$ and $\varphi_i = \tau_i^{-1}$.

It is well known (see [4] and [7]) that for such a transformation there exists a unique probabilistic, absolutely continuous τ-invariant measure μ with density f_0 satisfying the following condition:

$$0 < 1/c \leqslant f_0 \leqslant c \quad \text{for some } c > 0. \tag{1}$$

Under the above assumptions on the transformation $\tau\colon [0, 1] \to [0, 1]$ we shall prove the following

THEOREM 1. *If either f is a function of bounded variation over $[0, 1]$ or f is Hölder continuous, then*

$$\sigma^2 = \mathrm{E}_\mu(f-\mathrm{E}_\mu f)^2 + 2\sum_{j=1}^{\infty} \mathrm{E}_\mu[(f-\mathrm{E}_\mu f)(f\circ\tau^j - \mathrm{E}_\mu f)] < \infty, \tag{2}$$

$$\lim_{n\to\infty} \mu\left\{\frac{1}{\sqrt{n}}\sum_{j=0}^{n-1}(f\circ\tau^j - \mathrm{E}_\mu f) < z\right\} = \Phi(z), \tag{3}$$

$$\lim_{n\to\infty} m\left\{\frac{1}{\sqrt{n}}\sum_{j=0}^{n-1}\left(f\circ\tau^j - \mathrm{E}_m(f\circ\tau^j)\right) < z\right\} = \Phi(z), \tag{4}$$

where $\mathrm{E}_\mu f = \int_0^1 f d\mu$, $\mathrm{E}_m(f\circ\tau^j) = \int_0^1 f\circ\tau^j dm$, *and*

$$\Phi(z) = \begin{cases} \dfrac{1}{\sqrt{2\pi}\sigma}\displaystyle\int_{-\infty}^{z} \exp\left(\frac{-t^2}{2\sigma^2}\right)dt & \text{if } \sigma > 0,\\[2ex] \begin{cases} 0 & \text{if } z < 0\\ 1 & \text{if } z \geqslant 0\end{cases} & \text{if } \sigma = 0.\end{cases}$$

3. Auxiliary lemmas and the proof of Theorem 1. Let $([0, 1], \Sigma, \nu)$ be a measure space with σ-finite measure ν and let $L^1([0, 1], \Sigma, \nu)$ be the space of all integrable functions defined on $[0, 1]$. For a nonsingular transformation $\tau\colon [0, 1] \to [0, 1]$ $(\nu(\tau^{-1}(A)) = 0$ whenever $\nu(A) = 0)$ we define the Frobenius-Perron operator

$$P_\tau\colon L^1([0, 1], \Sigma, \nu) \to L^1([0, 1], \Sigma, \nu)$$

by the formula

$$\int_A P_\tau f d\nu = \int_{\tau^{-1}(A)} f d\nu,$$

which is valid for each measurable set $A \subset [0, 1]$. It is well known that the operator P_τ is linear and continuous and, in particular, satisfies the condition

$$\int_0^1 (P_\tau f)(x) g(x) dv = \int_0^1 f(x) g(\tau(x)) dv \tag{5}$$

for any $f \in L^1([0, 1], \Sigma, v)$ and any bounded g.

In the sequel we denote by P_τ and $\bar{P}_\tau$ the Frobenius-Perron operators defined on $L^1([0, 1], \Sigma, \mu)$ and $L^1([0, 1], \Sigma, m)$, respectively.

Now we are able to state the following

LEMMA 1. *If τ satisfies the assumptions* (a)-(e), *then for any function $f \geqslant 0$ of bounded variation we have*

$$|(P_\tau^n f)(x) - \|f\|_{L^1(\mu)}| \leqslant s^n [M_1 \|f\|_{L^1(\mu)} + M_2 \bigvee_0^1 f], \tag{6}$$

$$|(\bar{P}_\tau^n f)(x) - f_0(x) \|f\|_{L^1(m)}| \leqslant s^n [\bigvee_0^1 f + \bigvee_0^1 f_0 \|f\|_{L^1(m)}], \tag{7}$$

where $\bigvee_0^1 f$ denotes the variation of f over the interval $[0, 1]$, $M_1 = 2c \bigvee_0^1 f_0$, $M_2 = (\bigvee_0^1 f_0 + c)c$, *and c is defined in* (1).

The proof of this lemma is given in [4].

We shall prove the following

LEMMA 2. *If the transformation τ satisfies the assumptions* (a)-(e), *then for any function g such that $0 \leqslant g \leqslant D < \infty$ for some D and any function $f \geqslant 0$ of bounded variation we have*

$$\left|\int_0^1 g(\tau^n(x)) f(x) d\mu - \int_0^1 g d\mu \int_0^1 f d\mu\right| \leqslant s^n (M_1 \|f\|_{L^1(\mu)} + M_2 \bigvee_0^1 f) \int_0^1 g d\mu \tag{8}$$

and, consequently, for any measurable set B and any union $A = \bigcup_{i=1}^r A_i$ of intervals A_i

$$\left|\mu\big((\tau^{-n}(B)) \cap A\big) - \mu(A)\mu(B)\right| \leqslant s^n (M_1 + M_2 \cdot 2r) \mu(B). \tag{9}$$

Proof. The first part of the thesis is a simple consequence of (5) and Lemma 1. The second part can be obtained from (8) setting $g = \chi_B$ and $f = \chi_A$, where χ_B and χ_A denote the indicator functions of the sets B and A, respectively.

Let χ_n be a process on the probability space $([0, 1], \Sigma, \mu)$ given by the formula

$$\chi_n = \chi(\tau^n),$$

where

$$\chi = \sum_{i=1}^p \alpha_i \chi_{[a_{i-1}, a_i]} \quad \text{with } \alpha_i \neq \alpha_j \text{ for } i \neq j.$$

Denote by $\mathfrak{M}_k^l$ the σ-field generated by the sets of the form

$$\{x \in [0, 1]\colon (\chi_k(x), \ldots, \chi_l(x)) \in A\},$$

where $A \subset R^{l-k+1}$ is an $(l-k+1)$-dimensional cube.

It is easy to see that $\mathfrak{M}_0^0$ is generated by the set of intervals $\{(a_{i-1}, a_i)\}_{i=1}^p$.

We have the following

LEMMA 3. *If τ satisfies the assumptions of Theorem* 1, *then χ_n is a stationary process with the strong mixing coefficients*

$$\alpha(k) = \sup_{A \in \mathfrak{M}_0^0} \sup_{B \in \mathfrak{M}_k^\infty} |\mu(A \cap B) - \mu(A)\mu(B)|$$

satisfying the inequalities

$$\alpha(k) \leqslant s^k(M_1 + 2pM_2), \tag{10}$$

where M_1 and M_2 appear in Lemma 1 *and p is given in condition* (a).

Proof. This lemma is a simple consequence of Lemma 2 and the fact that the measure μ is invariant under τ.

Proof of Theorem 1. By Lemma 3, to show (2) and (3) it is sufficient to prove the condition imposed in Theorem 18.6.2 of [2]. Thus, we have only to prove

$$\mathrm{E}_\mu |f|^{2+\delta} < \infty, \tag{11}$$

$$\sum_{k=1}^{\infty} [\mathrm{E}_\mu |f - \mathrm{E}_\mu \{f \mid \mathfrak{M}_0^k\}|^{(2+\delta)/(1+\delta)}]^{(1+\delta)/(2+\delta)} < \infty, \tag{12}$$

where $\mathrm{E}_\mu \{f \mid \mathfrak{M}_0^k\}$ is the conditional expectation of f given a σ-field $\mathfrak{M}_0^k$, and

$$\sum_{n=1}^{\infty} (\alpha(n))^{\delta/(2+\delta)} < \infty \tag{13}$$

for some $\delta > 0$ and for any f satisfying the assumptions of Theorem 1.

Since f is of bounded variation and $\alpha(n)$ satisfies (10), inequalities (11) and (13) are obvious. Therefore, it remains only to prove (12). It is easy to verify that $\mathfrak{M}_0^k$ is generated by the intervals of the form

$$\bigcap_{i=1}^{k} \tau^{-i}(a_{j_i - 1}, a_{j_i}), \tag{14}$$

where a_{j_i} occur in condition (a).

Denote by Q_k the set of intervals of the form (14). It is obvious that the length of each interval from Q_k is not greater than $(\inf|\tau'|)^{-k}$, that is

$$m(A) \leqslant (\inf|\tau'|)^{-k} \tag{15}$$

for every $A \in Q_k$, where $(\inf|\tau'|)^{-1} \leqslant s < 1$. For any function f of bounded variation, by (1) and (15), we have

$$\begin{aligned}
\mathrm{E}_\mu|f-\mathrm{E}_\mu\{f|\mathfrak{M}_0^k\}|^2 &\leqslant \sum_{A\in Q_k}\int_A\left[f-\frac{1}{\mu(A)}\int_A f\,d\mu\right]^2 d\mu \\
&\leqslant \sum_{A\in Q_k}\int_A(\bigvee_A f)^2\,d\mu \leqslant (\bigvee_0^1 f)\sum_{A\in Q_k}\int_A(\bigvee_A f)\,d\mu \\
&\leqslant (\bigvee_0^1 f)\sum_{A\in Q_k}(\bigvee_A f)\sup_{A\in Q_k}\mu(A) \\
&\leqslant (\bigvee_0^1 f)^2 c(\inf|\tau'|)^{-k}.
\end{aligned}$$

Hence, since $L^\theta \subset L^2$ for $\theta = (2+\delta)/(1+\delta) < 2$, we obtain (12).

Now, let f be Hölder continuous. We have

$$\begin{aligned}
\max_{[0,1]}|f-\mathrm{E}_\mu\{f|\mathfrak{M}_0^k\}| &\leqslant \max_{A\in Q_k}\sup_A\left|f-\frac{1}{\mu(A)}\int_A f\,d\mu\right| \\
&\leqslant \max_{A\in Q_k}\sup_{x,y\in A}|f(x)-f(y)| \leqslant K(m(A))^\alpha \leqslant K((\inf|\tau'|)^\alpha)^{-k}
\end{aligned}$$

for some $K > 0$ and $\alpha > 0$. This yields (12) and completes the proof of (2) and (3).

Now we prove (4). Let

$$z_n^1 = \frac{1}{\sqrt{n}}\sum_{k=0}^{n-1}(f\circ\tau^k - \mathrm{E}_\mu f)$$

and

$$z_n^2 = \frac{1}{\sqrt{n}}\sum_{k=0}^{n-1}(f\circ\tau^k - \mathrm{E}_m(f\circ\tau^k)).$$

We have

$$\begin{aligned}
&|\mathrm{E}_\mu(\exp(i\xi z_n^1)) - \mathrm{E}_m(\exp(i\xi z_n^2))| \\
&\leqslant |\mathrm{E}_\mu(\exp(i\xi z_n^1)) - \mathrm{E}_m(\exp(i\xi z_n^1))| + |\mathrm{E}_m(\exp(i\xi z_n^1)) - \mathrm{E}_m(\exp(i\xi z_n^2))| \\
&\leqslant \mathrm{E}_\mu\left|1-\exp\left\{\frac{i\xi}{\sqrt{n}}\sum_{k=0}^{r}(f\circ\tau^k - \mathrm{E}_\mu f)\right\}\right| + \mathrm{E}_m\left|1-\exp\left\{\frac{i\xi}{\sqrt{n}}\sum_{k=0}^{r}(f\circ\tau^k - \mathrm{E}_\mu f)\right\}\right| + \\
&\quad + \left|(\mathrm{E}_\mu - \mathrm{E}_m)\exp\left\{\frac{i\xi}{\sqrt{n}}\sum_{k=r+1}^{n-1}(f\circ\tau^k - \mathrm{E}_\mu f)\right\}\right| +
\end{aligned}$$

$$+\mathrm{E}_m\left|1-\exp\left\{\frac{i\xi}{\sqrt{n}}\sum_{k=0}^{r}(f\circ\tau^k-\mathrm{E}_\mu f)\right\}\right|+$$

$$+\mathrm{E}_m\left|1-\exp\left\{\frac{i\xi}{\sqrt{n}}\sum_{k=0}^{r}(f\circ\tau^k-\mathrm{E}_m(f\circ\tau^k))\right\}\right|+$$

$$+\mathrm{E}_m\left|1-\exp\left\{\frac{i\xi}{\sqrt{n}}\sum_{k=r+1}^{n-1}(\mathrm{E}_\mu f-\mathrm{E}_m(f\circ\tau^k))\right\}\right|.$$

Hence, setting $r = [\log n]$, by (7) we obtain

$$\lim_{n\to\infty}|\mathrm{E}_\mu(\exp(i\xi z_n^1))-\mathrm{E}_m(\exp(i\xi z_n^2))| = 0$$

uniformly on any bounded interval. This gives us (4) and completes the proof of Theorem 1.

Under the same assumptions as in Theorem 1, without any difficulties we may prove the following

THEOREM 2. *If* $\sigma > 0$, *then*

$$\lim_{n\to\infty}\nu\left\{\frac{1}{\sqrt{n}}\sum_{k=0}^{n-1}(f\circ\tau^k-\mathrm{E}_\nu(f\circ\tau^k)) < z\right\}=\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{z}\exp\left(-\frac{t^2}{2\sigma^2}\right)dt$$

whenever

$$0 < \inf\frac{d\nu}{dm} \quad \textit{and} \quad \bigvee_0^1\frac{d\nu}{dm} < \infty.$$

REFERENCES

[1] A. Boyarsky and M. Scarowsky, *On a class of transformations which have unique absolutely continuous invariant measures*, Transactions of the American Mathematical Society 255 (1979), p. 243-262.

[2] I. A. Ibragimov and Yu. V. Linnik, *Independent and stationary sequences of random variables*, Wolters-Noordhoff, 1971.

[3] H. Ishitani, *The central limit theorem for piecewise linear transformations*, Kyoto University, Research Institute for Mathematical Sciences, Publications, 11 (1975/76), p. 281-296.

[4] M. Jabłoński and J. Malczak, *The rate of convergence of iterates of the Frobenius-Perron operator for piecewise monotonic transformations*, this fascicle, p. 67-72.

[5] K. Krzyżewski and W. Szlenk, *On invariant measures for expanding differentiable mappings*, Studia Mathematica 33 (1969), p. 83-92.

[6] A. Lasota, *A fixed point theorem and its application in ergodic theory* (to appear).

[7] — and J. A. Yorke, *On the existence of invariant measures for piecewise monotonic transformations*, Transactions of the American Mathematical Society 186 (1973), p. 481-488.

[8] A. Rényi, *Representation for real numbers and their ergodic properties*, Acta Mathematica Academiae Scientiarum Hungaricae 8 (1957), p. 477-493.

[9] V. A. Rohlin, *Exact endomorphisms of a Lebesgue space*, Izvestija Akademii Nauk SSSR, Serija Matematičeskaja, 25 (1961), p. 499-530.

[10] Tran Vinh-Hien, *A central limit theorem for stationary processes arising from number-theoretic endomorphisms*, Vestnik Moskovskogo Universiteta, Serija I – Matematika, Mehanika, 5 (1963), p. 28-34.

[11] S. Wong, *A central limit theorem for piecewise monotonic mappings of the unit interval*, Ann. Probability 7 (1979), p. 500-514.

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