COLLOQUIUM MATHEMATICUM

VOL. LX/LXI DÉDIÉ À M. ANTONI ZYGMUND 1990

A FUNCTIONAL EQUATION INVOLVING f AND f^{-1}

BY

J. S. RATTI AND Y.-F. LIN (TAMPA, FLORIDA)

1. Introduction. In an issue of the American Mathematical Monthly [1] the following problem was raised:

"Determine all homeomorphisms f from $[0,1]$ to $[0,1]$ that are solutions of the functional equation

$$f(2x - f(x)) = x \qquad \text{for all } x \in [0,1]." \tag{1}$$

In this paper we not only give the solution to the problem, but generalize the problem in two different directions. Moreover, we investigate equation (1) in the case where f is a linear transformation from a vector space to itself.

We first introduce the following notations. We define $f^0(x) = x$ and for any positive integer k, $f^k(x) = f[f^{k-1}(x)]$ and $f^{-k}(x) = f^{-1}[f^{-(k-1)}(x)]$. Since f is a homeomorphism, (1) can be written in the equivalent form

$$f(x) + f^{-1}(x) = 2x \qquad \text{for all } x \in [0,1]. \tag{2}$$

Replacing x by $f(x)$ in (2) yields another equivalent form:

$$f^2(x) - 2f(x) + x = 0 \qquad \text{for all } x \in [0,1]. \tag{3}$$

In the following theorem we neither require the continuity of f, nor of f^{-1}, and the set $[0,1]$ is replaced by any compact interval $[a,b]$, $a < b$.

2. Results and generalizations

THEOREM 1. *Let $f : [a,b] \to [a,b]$ be any bijection that satisfies*

$$f^2(x) - 2f(x) + x = 0 \qquad \textit{for all } x \in [a,b].$$

Then $f(x) = x$ for all $x \in [a,b]$.

Proof. We rewrite (3) as

$$f^2(x) - f(x) = f(x) - x \qquad \text{for all } x \in [a,b]. \tag{4}$$

Replacing x by $f(x)$ in (4) gives

$$f^3(x) - f^2(x) = f^2(x) - f(x) = f(x) - x,$$

and inductively, for any positive integer k,

$$f^{k+2}(x) - f^{k+1}(x) = f(x) - x \qquad \text{for all } x \in [a,b]. \tag{5}$$

Hence, for any positive integer m, we have

$$f^{m+2}(x) - f(x) = \sum_{k=0}^{m}[f^{k+2}(x) - f^{k+1}(x)] = (m+1)[f(x) - x]. \tag{6}$$

Therefore

$$|f(x) - x| = \frac{|f^{m+2}(x) - f(x)|}{m+1} \leq \frac{b-a}{m+1}\,. \tag{7}$$

By letting $m \to \infty$ in (7) we obtain

$$f(x) = x \qquad \text{for all } x \in [a,b]. \tag{8}$$

We now generalize this result in the following

THEOREM 2. *Let $f : [a,b] \to [a,b]$ be a bijection such that, for some n,*

$$\sum_{k=0}^{n}(-1)^k \binom{n}{k} f^{n-k}(x) = 0 \qquad \textit{for all } x \in [a,b]. \tag{9}$$

Then $f(x) = x$ for all $x \in [a,b]$.

Proof. We shall prove this by induction on n. This result is true for $n = 2$ [Theorem 1]. Assume that it is true for n and suppose

$$\sum_{k=0}^{n+1}(-1)^k \binom{n+1}{k} f^{n-k+1}(x) = 0 \qquad \text{for all } x \in [a,b]. \tag{10}$$

This can be rewritten as

$$\binom{n}{0} f^{n+1}(x) + \sum_{k=1}^{n}(-1)^k \left[\binom{n}{k} + \binom{n}{k-1}\right] f^{n-k+1}(x) - (-1)^n \binom{n}{n} x = 0$$

for all $x \in [a,b]$,

or

$$\sum_{k=0}^{n}(-1)^k \binom{n}{k} f^{n-k+1}(x) = \sum_{k=0}^{n}(-1)^k \binom{n}{k} f^{n-k}(x) \tag{11}$$

for all $x \in [a,b]$.

In (11), if we replace x successively by x, $f(x)$, $f^2(x)$, ..., $f^{m-1}(x)$, we obtain (using (11) again) the following system of equations:

(12)

$$
\begin{array}{llll}
\binom{n}{0}f^{n+1}(x) & -\binom{n}{1}f^{n}(x) & +\ldots+(-1)^n f(x) & =\sum_{k=0}^{n}(-1)^k\binom{n}{k}f^{n-k}(x),\\
\binom{n}{0}f^{n+2}(x) & -\binom{n}{1}f^{n+1}(x) & +\ldots+(-1)^n f^2(x) & =\sum_{k=0}^{n}(-1)^k\binom{n}{k}f^{n-k}(x),\\
\vdots & & & \\
\binom{n}{0}f^{n+n+1}(x) & -\binom{n}{1}f^{n+n}(x) & +\ldots+(-1)^n f^{n+1}(x) & =\sum_{k=0}^{n}(-1)^k\binom{n}{k}f^{n-k}(x),\\
\binom{n}{0}f^{n+n+2}(x) & -\binom{n}{1}f^{n+n+1}(x) & +\ldots+(-1)^n f^{n+2}(x) & =\sum_{k=0}^{n}(-1)^k\binom{n}{k}f^{n-k}(x),\\
\vdots & & & \\
\binom{n}{0}f^{n+m}(x) & -\binom{n}{1}f^{n+m-1}(x) & +\ldots+(-1)^n f^{m}(x) & =\sum_{k=0}^{n}(-1)^k\binom{n}{k}f^{n-k}(x).
\end{array}
$$

Here we let $m > n$. Note that along the dotted diagonals the sum of terms is equal to 0. Thus by adding (12) and noting that $|f^j(x)| \le \max\{|a|,|b|\}$ for all j, we obtain, on the left-hand side, a sum of terms whose absolute value is bounded by a function of n only. That is,

$$m\left|\sum_{k=0}^{n}(-1)^k\binom{n}{k}f^{n-k}(x)\right| \le g(n)(b-a)$$

or

$$\left|\sum_{k=0}^{n}(-1)^k\binom{n}{k}f^{n-k}(x)\right| \le \frac{g(n)(b-a)}{m}, \tag{13}$$

where $g(n)$ is a constant which is independent of m.

Letting $m \to \infty$ in (13), we conclude that

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}f^{n-k}(x) = 0 \qquad \text{for all } x \in [a,b] \tag{14}$$

and by the induction hypothesis, we have $f(x) = x$ for all $x \in [a,b]$.

Returning to equation (3), $f^2(x) - 2f(x) + x = 0$, we note that the sum of its coefficients is zero. The question arises: "If instead of integral coefficients we have other constants with zero sum does the conclusion of Theorem 1 remain true?" The answer is given in the following:

THEOREM 3. *Let $f : [a,b] \to [a,b]$ be bijective, let α be any real number satisfying $0 < \alpha < 1$, and suppose that*

$$\alpha f(x) + (1-\alpha)f^{-1}(x) = x \qquad \textit{for all } x \in [a,b]. \tag{15}$$

Then $f(x) = x$ for all $x \in [a, b]$.

Proof. The case where $\alpha = \frac{1}{2}$ is Theorem 1, therefore in the sequel we assume that $0 < \alpha < \frac{1}{2}$. (The case where $\frac{1}{2} < \alpha < 1$ can be reduced to the former by interchanging the roles of f and f^{-1}.)

We replace x by $f(x)$ in (15) to obtain the equivalent equation

$$f^2(x) - f(x) = \frac{1-\alpha}{\alpha}[f(x) - x] \qquad \text{for all } x \in [a, b], \tag{16}$$

from which we get, for any positive integer k,

$$f^{k+2}(x) - f^{k+1}(x) = \left(\frac{1-\alpha}{\alpha}\right)^{k+1}[f(x) - x] \qquad \text{for all } x \in [a, b]. \tag{17}$$

From (17) it follows that

$$|f(x) - x| \leq \left(\frac{\alpha}{1-\alpha}\right)^{k+1}(b - a) \qquad \text{for all } x \in [a, b]. \tag{18}$$

Since $0 < \alpha < \frac{1}{2}$, we have $0 < \frac{\alpha}{1-\alpha} < 1$. Thus, letting $k \to \infty$ in (18), we obtain the assertion.

The following theorem is motivated by Theorem 2 where f now represents a linear transformation.

THEOREM 4. *Let V be a vector space and let $A : V \to V$ be a linear transformation. Then A is invertible and*

$$\sum_{k=0}^{n}(-1)^k \binom{n}{k} A^{n-k}(x) = 0 \qquad \textit{for all } x \in V \tag{19}$$

if and only if there exists a linear transformation $T : V \to V$ such that

$$T^n = 0, \tag{20}$$

$$A = I + T + T^2 + \ldots + T^{n-1}. \tag{21}$$

Proof. (i) Suppose there is a T fulfilling (20) and (21). Applying T to both sides of (21) we obtain according to (20)

$$TA = AT = T + T^2 + \ldots + T^{n-1}. \tag{22}$$

Subtracting (22) from (21) we get

$$A(I - T) = I = (I - T)A.$$

Thus, A is invertible.

Next, equations (21) and (20) yield

$$A - I = T(I + T + T^2 + \ldots + T^{n-1}) = TA$$

and hence

$$(A - I)^n = (TA)^n = T^n A^n = 0 \tag{23}$$

since by (22), T and A commute and (20) holds. Equation (19) is a consequence of (23).

(ii) Now assume that A is invertible and satisfies (19). Replacing x by $A^{-n}(x)$ we get

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}A^{-k}(x)=0 \qquad \text{for all } x \in V,$$

that is,

$$(I-A^{-1})^n=0. \tag{24}$$

Setting $T = I - A^{-1}$ in (24) we obtain $T^n = 0$, hence (20) and consequently $A = (I-T)^{-1}$ fulfils (21).

3. A remark. Theorems 1 through 3 remain valid for open (or half-open) bounded intervals.

Acknowledgements. The authors are grateful to the referee for thoughtful comments.

REFERENCE

[1] Problem E 2893, Amer. Math. Monthly 88 (6) (1981), 444.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF SOUTH FLORIDA
TAMPA, FLORIDA 33620-5700, U.S.A.

Reçu par la Rédaction le 13.4.1990;
en version modifiée le 27.6.1990