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# ON THE LIMIT BEHAVIOUR OF SUMS OF A RANDOM NUMBER OF INDEPENDENT RANDOM VARIABLES

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**1. Introduction and notation.** This paper deals with the asymptotic distribution of the sums of a random number of independent random variables. For the first time the limit behaviour of sums with random indices was investigated by Robbins [4]. Some generalizations of his results and an estimate of the rapidity of the convergence of sums distribution function to the limit law may be found in [3], [5], [6], p. 154-162, and [7]. We shall give generalizations and extensions of the results of the above-mentioned papers.

Let $\{X_k, k \geqslant 1\}$ be a sequence of independent random variables, $F_k$ the distribution function of the $X_k$, and $S_n = \sum_{k=1}^{n} X_k$.

Let us put

$$a_k = \mathrm{E}X_k = \int_{-\infty}^{\infty} x dF_k(x), \quad a_0 = 0, \quad A_n = \sum_{k=0}^{n} a_k,$$

$$b_k^2 = \mathrm{E}X_k^2 = \int_{-\infty}^{\infty} x^2 dF_k(x), \quad b_0^2 = 0,$$

$$\sigma_k^2 = \sigma^2 X_k = b_k^2 - a_k^2, \quad \sigma_0^2 = 0, \quad s_n^2 = \sum_{k=0}^{n} \sigma_k^2,$$

$$\beta_k^{2+p} = \mathrm{E}(|X_k - \mathrm{E}X_k|^{2+p}), \quad \beta_0^{2+p} = 0, \quad \gamma_n^{2+p} = \sum_{k=0}^{n} \beta_k^{2+p}.$$

Let

$$(1) \qquad f_k(t) = \mathrm{E}\exp(itX_k) = \int_{-\infty}^{\infty} [\exp(itx)] dF_k(x), \quad f_0(t) \equiv 1.$$

By $N$ we denote a non-negative integer-valued random variable which is independent of the $X_k$, $k = 1, 2, \ldots$ We assume that the distribution function of $N$ depends on a parameter $\lambda$ and is determined by the values

$$p_n = \mathrm{P}[N = n],\ n = 0, 1, 2, \ldots, \qquad \sum_{n=0}^{\infty} p_n = 1,$$

where $p_n = p_n(\lambda)$.

Put

$$a = \mathrm{E}N = \sum_{n=0}^{\infty} np_n, \qquad \sigma^2 N = \sum_{n=0}^{\infty} (n-a)^2 p_n,$$

$$g(t) = \mathrm{E}\exp\left(it\frac{N-a}{\sigma N}\right) = \sum_{n=0}^{\infty} p_n \exp\left(it\frac{n-a}{\sigma N}\right).$$

Under these assumptions on $N$, the distribution function of $S_N = X_1 + X_2 + \ldots + X_N$ depends on the parameter $\lambda$, and

$$\mathrm{E}S_N = \sum_{n=0}^{\infty} A_n p_n = A, \tag{2}$$

$$\sigma^2 S_N = \sum_{n=0}^{\infty} s_n^2 p_n + \sum_{n=0}^{\infty} A_n^2 p_n - A^2 = \sigma^2,$$

$$\varphi(t) = \mathrm{E}\exp\left(it\frac{S_N - \mathrm{E}S_N}{\sigma S_N}\right) \tag{3}$$

$$= \sum_{n=0}^{\infty} p_n \exp\left(-\frac{itA}{\sigma}\right) \prod_{k=0}^{n} f_k\left(\frac{t}{\sigma}\right).$$

Now, let us observe that the sums $\sum_{k=0}^{N} a_k$, $\sum_{k=0}^{N} s_k^2$ and $\sum_{k=0}^{N} \beta_k^{2+p}$ define the new random variables $L$, $M$ and $R$, respectively. For these random variables we have

$$L = \sum_{k=0}^{N} a_k, \qquad \mathrm{P}[L = A_n] = p_n,$$

$$\mathrm{E}L = \sum_{n=0}^{\infty} A_n p_n = A, \qquad \sigma^2 L = \sum_{n=0}^{\infty} A_n^2 p_n - A^2 = \Delta^2, \tag{4}$$

$$h(t) = \mathrm{E}\exp\left(it\frac{L - \mathrm{E}L}{\Delta}\right) = \sum_{n=0}^{\infty} p_n \exp\left(it\frac{A_n - A}{\Delta}\right),$$

$$M = \sum_{k=0}^{N} \sigma_k^2, \qquad \mathrm{P}[M = s_k^2] = p_n,$$

$$\mathrm{E}M = \sum_{n=0}^{\infty} s_n^2 p_n = \varrho, \qquad \sigma^2 M = \sum_{n=0}^{\infty} s_n^4 p_n - \varrho^2 = u^2, \tag{5}$$

$$R = \sum_{k=0}^{N} \beta_k^{2+p}, \qquad \mathrm{P}[R = \gamma_n^{2+p}] = p_n, \qquad \mathrm{E}R = \sum_{n=0}^{\infty} \gamma_n^{2+p} p_n = w_{2+p}.$$

Moreover, according to (2), (4) and (5), we obtain $\mathrm{E}S_N = \mathrm{E}L = A$ and $\sigma^2 S_N = \varrho + \Delta = \sigma^2$.

It is easy to see that, for the independent random variables with $\mathrm{E}X_k = a_k = a$ for every $k = 1, 2, \ldots$, we have $\mathrm{E}L = aa$, $\mathrm{E}L^2 = a^2\mathrm{E}N^2$ and $\sigma^2 L = a^2\sigma^2 N$.

**2. The asymptotic distribution of sums of a random number of independent random variables.** In what follows we assume that random variables $X_k$, $k = 1, 2, \ldots$, satisfy Lindeberg's condition.

THEOREM 1. *If*

$$\sigma^2 \to \infty, \qquad (M - \mathrm{E}M)/\sigma^2 \xrightarrow{\mathrm{P}} 0 \ (\mathrm{P} = \textit{in probability}) \tag{6}$$

*with* $\lambda \to \infty$, *then*

$$\lim_{\lambda\to\infty} \varphi(t) = h(td)\exp\left[-\frac{t^2}{2}(1-d^2)\right], \tag{7}$$

*where*

$$d = \frac{\Delta}{\sigma} = \left(\frac{\sum_{n=0}^{\infty} A_n^2 p_n - A^2}{\sum_{n=0}^{\infty} s_n^2 p_n + \sum_{n=0}^{\infty} A_n^2 p_n - A^2}\right)^{1/2}, \qquad 0 \leqslant d \leqslant 1.$$

Proof. Let

$$\psi(t) = \sum_{n=0}^{\infty} p_n \exp\left(it\frac{A_n - A}{\sigma}\right)\exp\left(-\frac{\varrho t}{\sigma}\right).$$

By (3), we have

$$\varphi(t) = \sum_{n=0}^{\infty} p_n \exp\left(it\frac{A_n - A}{\sigma}\right)\prod_{k=0}^{n} f_k\left(\frac{t}{\sigma}\right)\exp\left(-\frac{ia_k t}{\sigma}\right).$$

Hence

$$|\varphi(t) - \psi(t)| \leqslant \sum_{n=0}^{\infty} p_n \left|\prod_{k=0}^{n} f_k\left(\frac{t}{\sigma}\right)\exp\left(-\frac{ia_k t}{\sigma}\right) - \exp\left(-\frac{\varrho t^2}{2\sigma^2}\right)\right|. \tag{8}$$

Choosing an arbitrary $\varepsilon > 0$ and using (8), we have

$$(9)\quad |\varphi(t)-\psi(t)| \leqslant 2\mathrm{P}\left[\left|\frac{M-\mathrm{E}M}{\sigma^2}\right| \geqslant \varepsilon\right] + \\ +\max\left|\prod_{k=0}^{n} f_k\left(\frac{t}{\sigma}\right)\exp\left(-\frac{ia_k t}{\sigma}\right) - \exp\left(-\frac{\varrho t^2}{2\sigma^2}\right)\right|,$$

where the maximum is taken over all $n$ such that $|s_n^2-\varrho| < \varepsilon\sigma^2$.

In view of (1), we have, as $t\to 0$,

$$f_k(t) = 1+ia_k t - \frac{b_k^2 t^2}{2} + o(t^2);$$

hence, as $\sigma^2\to\infty$,

$$\exp\left(-\frac{ia_k t}{\sigma}\right) f_k\left(\frac{t}{\sigma}\right) = 1 - \frac{\sigma_k^2 t^2}{2\sigma^2} + o\left(\frac{1}{\sigma^2}\right).$$

Thus

$$(10)\quad \prod_{k=0}^{n} \exp\left(-\frac{ia_k t}{\sigma}\right) f_k\left(\frac{t}{\sigma}\right) = \exp\left(-\frac{t^2}{2\sigma^2}\sum_{k=0}^{n}\sigma_k^2\right) + o(1).$$

By (6), for every $\delta > 0$, there is $\lambda_0$ such that

$$(11)\quad \mathrm{P}\left[\left|\frac{M-\mathrm{E}M}{\sigma}\right| \geqslant \varepsilon\right] < \frac{\delta}{5} \quad \text{and} \quad |o(1)| < \frac{\delta}{5} \quad \text{for } \lambda > \lambda_0.$$

By virtue of (9), (10) and (11), we have

$$(12)\quad |\varphi(t)-\psi(t)| \leqslant \frac{3\delta}{5} + \max\left|\exp\left(-\frac{t^2 s_n^2}{2\sigma^2}\right) - \exp\left(-\frac{\varrho t^2}{2\sigma^2}\right)\right| \\ \leqslant \frac{3\delta}{5} + \max\left|\exp\left[-\frac{t^2}{2\sigma^2}(s_n^2-\varrho)\right] - 1\right| \leqslant \frac{3\delta}{5} + \frac{\varepsilon t^2}{2} + o(1),$$

where the maximum is taken over all $n$ such that $|s_n^2-\varrho| < \varepsilon\sigma^2$.

Now, fix $t$ and $\delta > 0$. Taking $\varepsilon > 0$ (until now arbitrary) such that

$$(13)\quad \varepsilon t^2/2 < \delta/5 \quad \text{and} \quad |o(1)| < \delta/5 \quad \text{for } \lambda > \lambda_1 > \lambda_0,$$

we have, according to (12) and (13), $|\varphi(t)-\psi(t)| < \delta$ for $\lambda > \lambda_1$. Since $\delta$ was chosen arbitrary, $\varphi(t) = \psi(t)+o(1)$. In view of

$$\psi(t) = h(td)\exp\left[-\frac{t^2}{2}(1-d^2)\right],$$

the proof of the theorem is complete.

Remark. It is easy to see that the assumption $u = o(\sigma^2)$ implies

$$(M - \mathrm{E}M)/\sigma^2 \xrightarrow{\mathrm{P}} 0 \quad \text{with } \lambda \to \infty.$$

Theorem 1 and Remark yield

COROLLARY 1. *If* $\sigma^2 \to \infty$ *and* $u = o(\sigma^2)$ *with* $\lambda \to \infty$, *then* (7) *holds.*

An extension of Robbins' theorem [4] gives the following

COROLLARY 2. *If* $\{X_n, n \geqslant 1\}$ *is a sequence of independent random variables identically distributed, and*

$$\sigma^2 \to \infty, \quad (N - a)\sigma^2 \xrightarrow{\mathrm{P}} 0 \text{ with } \lambda \to \infty,$$

*then*

$$\varphi(t) = g\left(\frac{a\sigma N}{\sigma S_N} t\right) \exp\left[-\frac{t^2}{2}\left(1 - \frac{a^2 \sigma^2 N}{\sigma^2 S_N}\right)\right],$$

*where* $\mathrm{E}X_n = a$, $n = 1, 2, \ldots$

Proof. Since in this case we have

$$(M - \mathrm{E}M)/\sigma^2 = \theta^2 (N - a)\sigma^2 \xrightarrow{\mathrm{P}} 0,$$

where $\theta^2 = \sigma^2 X_k$, $k = 1, 2, \ldots$, so (7) is satisfied. And since

$$d^2 \doteq \Delta^2/\sigma^2 = a^2\sigma^2 N/\sigma^2 \quad \text{and} \quad h(td) = g\left(\frac{a\sigma N}{\sigma} t\right),$$

the proof of the corollary is complete.

From Theorem 1 one can also deduce

COROLLARY 3. *If* $\mathrm{E}X_k = a_k = 0$, $k = 1, 2, \ldots$, *and if*

$$\sigma^2 \to \infty, \quad M/\mathrm{E}M \xrightarrow{\mathrm{P}} 1 \text{ with } \lambda \to \infty,$$

*then*

$$\lim_{\lambda \to \infty} \varphi(t) = \exp(-t^2/2). \tag{14}$$

Proof. In this case $\sigma^2 = \varrho$ and $\psi(t) = \exp(-t^2/2)$, whence (14) holds by Theorem 1.

COROLLARY 4. *If* (6) *is satisfied, and if* $\Delta^2 = o(\sigma^2)$ *with* $\lambda \to \infty$, *then* $S_N$ *obeys* (14), *i.e.* $S_N$ *is asymptotically normal with parameters* $A$ *and* $\sigma$.

Proof. It follows from the equality $\Delta^2 = o(\sigma^2)$ that $d = o(1)$ with $\lambda \to \infty$. Now, putting $L_1 = (L - \mathrm{E}L)d/\Delta$, we have $L_1 \xrightarrow{\mathrm{P}} 0$ as $\mathrm{E}L_1 = 0$, and $\mathrm{E}L_1^2 = d^2 \to 0$ with $\lambda \to \infty$. Hence

$$\mathrm{E}\exp(itL_1) = \sum_{n=0}^{\infty} p_n \exp\left[it\left(\frac{A_n - A}{\Delta}\right)d\right] = h(td) \to 1 \quad \text{with } \lambda \to \infty.$$

But also

$$\exp\left[-\frac{t^2}{2}(1-d^2)\right] \to \exp\left(-\frac{t^2}{2}\right) \quad \text{with } \lambda\to\infty,$$

so according to (7) $\lim_{\lambda\to\infty}\varphi(t) = \exp(-t^2/2)$.

COROLLARY 5. *If* (6) *holds, and if $L$ is asymptotically normal* $(A, \Delta)$, *then also $S_N$ is asymptotically normal* $(A, \sigma)$.

Proof. Under the assumptions of Corollary 5, we have

$$\lim_{\lambda\to\infty} h(\tau) = \exp(-\tau^2/2)$$

uniformly for $0 \leqslant \tau \leqslant t$. But $0 \leqslant d \leqslant 1$, so

$$h(td) = \exp(-t^2d^2/2)+o(1) \quad \text{with } \lambda\to\infty.$$

Hence, according to (7), $\lim_{\lambda\to\infty}\varphi(t) = \exp(-t^2/2)$.

COROLLARY 6. *If* (6) *is satisfied, and if* $(L-A)/\Delta$ *has a non-normal limiting distribution function* $G_1$ *such that*

$$\lim_{\lambda\to\infty} h(t) = h_1(t) = \int_{-\infty}^{\infty}[\exp(itx)]\,dG(x) \not\equiv \exp(-t^2/2),$$

*and if the limit*

$$\lim_{\lambda\to\infty}(\varrho/\Delta^2) = s \qquad (0\leqslant s<\infty)$$

*does exist, then*

$$\lim_{\lambda\to\infty}\varphi(t) = h_1\left(\frac{t}{\sqrt{1+s}}\right)\exp\left[-\frac{t^2}{2}\left(\frac{s}{1+s}\right)\right] \not\equiv \exp\left(-\frac{t^2}{2}\right).$$

Proof. In this case

$$\lim_{\lambda\to\infty}(\Delta/\sigma) = 1/\sqrt{1+s}.$$

Hence

$$\lim_{\lambda\to\infty} h(td) = h_1(t/\sqrt{1+s}),$$

and so we also have

$$\lim_{\lambda\to\infty}\exp\left(-\frac{\varrho t^2}{2\sigma^2}\right) = \exp\left[-\frac{t^2}{2}\left(\frac{s}{1+s}\right)\right],$$

the two equalities giving the corollary.

Remarks. If $s = 0$ (it holds if $\varrho = o(\Delta^2)$), then $\lim_{\lambda\to\infty} \varphi(t) = h_1(t)$.

If $s = \infty$ (it holds if $\Delta^2 = o(\varrho^2)$), then $\lim_{\lambda\to\infty} \varphi(t) = \exp(-t^2/2)$ (see Corollary 4).

We need the following

LEMMA. *If* $\sigma_k^2 \leqslant c < \infty$, $k = 1, 2, \ldots$, *where* $c$ *is a positive constant, and if* $\sigma^2 \to \infty$, $\sigma^2 N = o(\sigma^2)$ *with* $\lambda \to \infty$, *and either* $a = o(\sigma^2)$ *or* $a = O(\sigma^2)$ *with* $\lambda \to \infty$, *then*

$$(M - \mathrm{E}M)/\sigma^2 \xrightarrow{\mathrm{P}} 0 \quad \textit{with } \lambda \to \infty.$$

Proof. Let us choose $\varepsilon > 0$. By Chebyschev's inequality, we have

$$\mathrm{P}[|M - \mathrm{E}M| \geqslant \varepsilon\sigma^2] \leqslant c^2\sigma^2 N/\varepsilon^2\sigma^4 + c^2 a^2/\varepsilon^2\sigma^4 \to 0 \quad \text{with } \lambda \to \infty,$$

when $a = o(\sigma^2)$ with $\lambda \to \infty$.

In the case $a = O(\sigma^2)$, we have

$$\mathrm{P}[|M - \mathrm{E}M| \geqslant \varepsilon\sigma^2] \leqslant \Big[\sum_{n=0}^{\infty} s_n^4 p_n - s_{[a]}^4\Big]/\varepsilon^2\sigma^4 + \Big[s_{[a]}^4 - \Big(\sum_{n=0}^{\infty} s_n^2 p_n\Big)^2\Big]/\varepsilon^2\sigma^4,$$

where here and in what follows $[x]$ denotes the integer part of the real number $x$.

First, we are going to estimate the second term of the last inequality. We have for it

$$\Big|s_{[a]}^4 - \Big(\sum_{n=0}^{\infty} s_n^2 p_n\Big)^2\Big|/\varepsilon^2\sigma^4 \leqslant \frac{2c}{\varepsilon^2}\Big(\frac{a}{\sigma^2}\Big)\Big|s_{[a]}^2 - \sum_{n=0}^{\infty} s_n^2 p_n\Big|/\sigma^2 = o(1)$$

as $a = O(\sigma^2)$ (by the assumption), and

$$\Big|s_{[a]}^2 - \sum_{n=0}^{\infty} s_n^2 p_n\Big|/\sigma^2 = o(1),$$

what was proved in [6], p. 154-162.

Now, we infer, taking into account the assumption $\sigma N = o(\sigma^2)$, that

$$(N - a)/\sigma^2 \xrightarrow{\mathrm{P}} 0 \quad \text{with } \lambda \to \infty.$$

Let $\delta > 0$ be arbitrary. For the first term of the considered inequality we have

$$\Big|\sum_{n=0}^{\infty} s_n^4 p_n - s_{[a]}^4\Big|/\varepsilon^2\sigma^4 \leqslant \sum_{n\in B} s_n^4 p_n/\varepsilon^2\sigma^4 + \Big|\sum_{n\in B} s_n^4 p_n - s_{[a]}^4\Big|/\varepsilon^2\sigma^4,$$

where $B = \{n\colon |n-a| \geqslant \delta\sigma^2\}$.

Further,

$$\sum_{n\in B} s_n^4 p_n/\varepsilon^2\sigma^4 \leqslant c^2 \sum_{n\in B} n^2 p_n/\varepsilon^2\sigma^4 = c^2\Big(\mathrm{E}N^2 - \sum_{n\in B} n^2 p_n\Big)/\varepsilon^2\sigma^4$$
$$= c^2\{\sigma^2 N + 2a\delta\sigma^2 - \delta^2\sigma^4 + (a-\delta\sigma^2)^2 \mathrm{P}[|N-a| \geqslant \delta\sigma^2]\}/\varepsilon^2\sigma^4 = o(1).$$

Now, if

$$\sum_{n\in B} s_n^4 p_n - s_{[a]}^4 \geqslant 0,$$

then we have

$$\Big|\sum_{n\in B} s_n^4 p_n - s_{[a]}^4\Big|/\varepsilon^2\sigma^4 \leqslant \{s_{[a+\delta\sigma^2]}^4 \mathrm{P}[|N-a| < \delta\sigma^2] - s_{[a]}^4\}/\varepsilon^2\sigma^4$$
$$\leqslant \Big\{\Big(s_{[a]}^2 + \sum_{k=[a]}^{[a+\delta\sigma^2]} \sigma_k^2\Big)^2 (1-\mathrm{P}[|N-a| \geqslant \delta\sigma^2]) - s_{[a]}^4\Big\}/\varepsilon^2\sigma^4 = o(1)$$

with $\lambda\to\infty$.

If

$$\sum_{n\in B} s_n^4 p_n - s_{[a]}^4 < 0,$$

then we have

$$\Big|\sum_{n\in B} s_n^4 p_n - s_{[a]}^4\Big|/\varepsilon^2\sigma^4 \leqslant \{s_{[a]}^4 - s_{[a-\delta\sigma^2]}^4 \mathrm{P}[|N-a| < \delta\sigma^2]\}/\varepsilon^2\sigma^4 = o(1)$$

with $\lambda\to\infty$, which completes the proof of the lemma.

From Theorem 1 and the lemma, we get the following extension of the results given in [6]:

THEOREM 2. *If $\sigma_k^2 \leqslant c < \infty$, $k = 1, 2, \ldots$, if $\sigma^2\to\infty$, $\sigma N = o(\sigma^2)$ with $\lambda\to\infty$, and either $a = o(\sigma^2)$ or $a = O(\sigma^2)$ with $\lambda\to\infty$, then* (7) *holds.*

From Theorem 2 one can obtain, in a simple way,

COROLLARY 7. *If $\mathrm{E}X_k = a$, $\sigma^2 X_k \leqslant c < \infty$, $k = 1, 2, \ldots$, if $\sigma^2\to\infty$ with $\lambda\to\infty$, and either $a = o(\sigma^2)$ or $a = O(\sigma^2)$ with $\lambda\to\infty$, then*

$$\lim_{\lambda\to\infty} \varphi(t) = g\Big(t\frac{a\sigma N}{\sigma}\Big)\exp\Big\{-\frac{t^2}{2}\Big(1-\frac{a^2\sigma^2 N}{\sigma^2}\Big)\Big\}.$$

**3. An estimation of the deviation of the distribution of the sum of a random number of independent random variables from its limit distribution function.** Let $F$ and $G$ be the distribution functions of the random variables $(S_N - A)/\sigma$ and $(L-\mathrm{E}L)/\Delta$, respectively.

THEOREM 3. *If $w_{2+p} < \infty$ $(0 < p \leqslant 1)$ and, for every $n$, $\gamma_n^{2+p}/s_n^2 \leqslant K$, where $K$ is a constant, and if* (6) *holds, then*

$$\Big|\sup_x F(x) - G\Big(\frac{x}{d}\Big) * \Phi\Big(\frac{x}{\sqrt{1-d^2}}\Big)\Big| \leqslant c\Big(\frac{u}{\varrho} + \frac{u^2}{\varrho^2} + \frac{w_{2+p}}{\varrho^{1+p/2}} + \frac{\sigma}{\varrho}\Big), \tag{15}$$

*where $c$ is a positive constant, $\Phi$ is a normal distribution function, and $*$ denotes the convolution operation.*

Proof. Let us consider the function

$$\varphi_1(t) = \sum_{n\in C} p_n \exp\left(it\frac{A_n-A}{\sigma}\right)\prod_{j=0}^{n}\left[f_k\left(\frac{t}{\sigma}\right)\exp\left(-\frac{ita_k}{\sigma}\right)\right],$$

where $C = \{n: s_n^2 \geqslant \varrho/2\}$.

It can be observed that

$$\varphi_1(t) = \sum_{n\in C} p_n \exp\left(it\frac{A_n-A}{\sigma}\right)\left[\prod_{k=0}^{n}\tilde{f}_k\left(\frac{t}{\sigma}\right)-\exp\left(-\frac{s_n^2t^2}{2\sigma^2}\right)\right]+$$
$$+\sum_{n\in C} p_n \exp\left(it\frac{A_n-A}{\sigma}\right)\left[\exp\left(-\frac{s_n^2t^2}{2\sigma^2}\right)-\exp\left(-\frac{\varrho t^2}{2\sigma^2}\right)\right]+$$
$$+\sum_{n\in C} p_n \exp\left(it\frac{A_n-A}{\sigma}\right)\exp\left(-\frac{\varrho t^2}{2\sigma^2}\right),$$

where

$$\tilde{f}_k\left(\frac{t}{\sigma}\right) = f_k\left(\frac{t}{\sigma}\right)\exp\left(-\frac{ita_k}{\sigma}\right).$$

Now, putting

$$h_1(td) = \sum_{n\in C} p_n \exp\left(-it\frac{A_n-A}{\sigma}\right),$$

we obtain

$$(16)\quad \varphi_1(t)-h_1(td)\exp\left(-\frac{\varrho t^2}{2\sigma^2}\right)$$
$$= \sum_{n\in C} p_n \exp\left(it\frac{A_n-A}{\sigma}\right)\left[\prod_{k=0}^{n}\tilde{f}_k\left(\frac{t}{\sigma}\right)-\exp\left(-\frac{s_n^2t^2}{2\sigma^2}\right)\right]+$$
$$+\sum_{n\in C} p_n \exp\left(it\frac{A_n-A}{\sigma}\right)\left[\exp\left(-\frac{s_n^2t^2}{2\sigma^2}\right)-\exp\left(-\frac{\varrho t^2}{2\sigma^2}\right)\right].$$

It is obvious that

$$(17)\quad \left|\exp\left(-\frac{s_n^2t^2}{2\sigma^2}\right)-\exp\left(-\frac{\varrho t^2}{2\sigma^2}\right)\right|$$
$$\leqslant \frac{|s_n^2-\varrho|t^2}{2\sigma^2}\exp\left\{-\min(s_n^2,\varrho)\frac{t^2}{2\sigma^2}\right\}.$$

Moreover, we have

$$\int_0^{c\varrho^{1/2}} t\exp\left(-\frac{\varrho t^2}{2\sigma^2}\right)dt = \frac{\sigma^2}{\varrho}\int_0^{c\varrho/\sigma} z\exp\left(-\frac{z^2}{2}\right)dz \leqslant \frac{\sigma^2}{\varrho} \tag{18}$$

and

$$\int_0^{c\varrho^{1/2}} t\exp\left(-\frac{s_n^2t^2}{2\sigma^2}\right)dt = \frac{\sigma^2}{s_n^2}\int_0^{c\sqrt{\varrho s_n^2}/\sigma} z\exp\left(-\frac{z^2}{2}\right)dz \leqslant \frac{\sigma^2}{s_n^2}, \tag{19}$$

where

$$c = \left(\frac{s_n^2}{24\sum_{k=0}^{n}\beta_k^{2+p}}\right)^{1/p}.$$

On the basis of (17), (18) and (19), we have

$$\int_{|t|<c\varrho^{1/2}}\left|\exp\left(-\frac{s_n^2t^2}{2\sigma^2}\right)-\exp\left(-\frac{\varrho t^2}{2\sigma^2}\right)\right|\frac{dt}{|t|} \leqslant \begin{cases}(s_n^2-\varrho)/\varrho & \text{if } s_n^2 \geqslant \varrho,\\ (\varrho-s_n^2)/s_n^2 & \text{if } s_n^2 < \varrho.\end{cases}$$

And, finally,

$$\int_{|t|<c\varrho^{1/2}}\sum_n p_n\left|\left[\exp\left(-\frac{s_n^2t^2}{2\sigma^2}\right)-\exp\left(-\frac{\varrho t^2}{2\sigma^2}\right)\right]\exp\left(it\frac{A_n-A}{\sigma}\right)\right|\frac{dt}{|t|} \tag{20}$$

$$\leqslant \sum_{n\in D}\frac{|s_n^2-\varrho|}{s_n^2}p_n + \sum_{n\in E}\frac{|s_n^2-\varrho|}{\varrho}p_n \leqslant \frac{2}{\varrho}\sum_{n\in D}|s_n^2-\varrho|\,p_n \leqslant \frac{2u}{\varrho}$$

for $D = \{n: \varrho/2 \leqslant s_n^2 \leqslant \varrho\}$, $E = \{n: s_n^2 > \varrho\}$ an$\delta$ $\sum_{n=o}^{\infty} p_n|s_n^2-\varrho| \leqslant \sigma M = u.$

Now, by Lemma 1 of [1], we have

$$\left|\prod_{k=0}^{n}\tilde{f}_k\left(\frac{t}{\sigma}\right)-\exp\left(-\frac{s_n^2t^2}{2\sigma^2}\right)\right| \leqslant c(p)\frac{\sum_{k=0}^{n}\beta_j^{2+p}|t|^{2+p}}{\sigma^{2+p}}\exp\left(-\frac{s_n^2t^2}{4\sigma^2}\right) \tag{21}$$

for

$$|t| < \frac{\sigma(s_n^2)^{1/p}}{(24\sum_{k=0}^{n}\beta_j^{2+p})^{1/p}} = c\sigma,$$

where a positive constant $c(p)$ depends only on $p$.

Hence

$$\int_0^{c\varrho^{1/2}} t^{1+p}\exp\left(-\frac{s_n^2 t^2}{4\sigma^2}\right)dt = \int_0^{c\sqrt{\varrho s_n^2/2}/\sigma} \sigma^{2+p}\sqrt{\frac{1}{2}s_n^2}\left(\frac{2}{s_n^2}\right)^{(1+p)/2} z^{1+p}\exp\left(-\frac{z^2}{2}\right)dz$$

$$\leqslant \frac{\sigma^{2+p}2^{1+p/2}}{(s_n^2)^{1+p/2}} = \sigma^{2+p}\left(\frac{2}{s_n^2}\right)^{1+p/2}.$$

Thus

$$\int_0^{c\varrho^{1/2}} t^{1+p}\exp\left(-\frac{s_n^2 t^2}{4\sigma^2}\right)dt \leqslant \sigma^{2+p}\left(\frac{2}{s_n^2}\right)^{1+p/2}. \tag{22}$$

Taking into account (21), (22) and the evident inequality $\sigma > \varrho^{1/2}$, we obtain

$$\int_{|t|<c\varrho^{1/2}} \sum_{n\in C} p_n\left|\prod_{k=0}^{n}\tilde{f}_k\left(\frac{t}{\sigma}\right) - \exp\left(-\frac{s_n^2t^2}{2\sigma^2}\right)\right|\frac{dt}{|t|} \tag{23}$$

$$\leqslant c(p)2^{2+p/2}\sum_{n\in C}p_n\sum_{k=0}^{n}\beta_j^{2+p}(s_n^2)^{-1-p/2} \leqslant c_1\mathrm{E}\left(\sum_{k=0}^{N}\beta_j^{2+p}\right)/\varrho^{1+p/2},$$

where $c_1$ is a positive constant.

According to (16), (20) and (23), we get

$$\int_{|t|<c\varrho^{1/2}}\left|\varphi_1(t) - h_1(td)\exp\left[-\frac{t^2}{2}(1-d^2)\right]\right|\frac{dt}{|t|} \tag{24}$$

$$\leqslant \frac{2u}{\varrho} + c\mathrm{E}\left(\sum_{k=0}^{N}\beta_j^{2+p}\right)/\varrho^{1+p/2}.$$

Here we also observe that $\Phi'(x/\sqrt{1-d^2}) \leqslant \sigma/\varrho^{1/2}$. Let now $F_1$ and $G_1$ be distribution functions corresponding to the characteristic functions $\varphi_1$ and $h_1$, respectively. On the basis of (23), (24) and the well-known Esseen Theorem [1], it follows that

$$\sup_x |F_1(x) - G_1(x/d)*\Phi(x/\sqrt{1-d^2})| \leqslant 2u/\varrho + c_2 w_{2+p}/\varrho^{1+p/2} + c_3\sigma/\varrho, \tag{25}$$

where $c_2$ and $c_3$ are positive constants.

Further, we have

$$F(x) - F_1(x) \leqslant \sum_{n\in Y} p_n \leqslant 4u^2/\varrho^2 \tag{26}$$

and

(27) $$G(x)-G_1(x) \leqslant \sum_{n\in Y} p_n \leqslant 4u^2/\varrho^2,$$

where $Y = \{n: s_n^2 < \varrho/2\}$.

Taking into account (25), (26) and (27), we obtain (15).

COROLLARY 8. *If the assumptions of Theorem* 3 *are fulfilled and* $\mathrm{E}X_k = a$ *for* $k = 1, 2, \ldots,$ *then*

$$\sup_x |F(x)-H(x/d)*\Phi(x/\sqrt{1-d^2})| \leqslant c(u/\varrho+u^2/\varrho^2+w_{2+p}/\varrho^{1+p/2}+1/\varrho^{1/2}+a\sigma N/\varrho),$$

*where* $H$ *is the distribution function of the random variable* $(N-\mathrm{E}N)/\sigma N$, *and* $c$ *is a positive constant.*

Proof. In this case $\sigma/\varrho \leqslant 1/\varrho^{1/2}+a\sigma N/\varrho$. This inequality and Theorem 3 give the estimation of Corollary 8. Of course, in this case $d = a\sigma N/\sigma$.

The following corollaries extend the results given in [5]:

COROLLARY 9. *If in Corollary* 8 $\mathrm{E}X_k = 0$ *for* $k = 1, 2, \ldots,$ *then*

$$\sup_x |F(x)-\Phi(x)| \leqslant c(u/\varrho+u^2/\varrho^2+w_{2+p}/\varrho^{1/2}+1/\varrho^{1/2}),$$

*where* $c$ *is a positive constant.*

COROLLARY 10. *If the assumptions of Corollary* 2 *are satisfied and* $\beta^{2+p} = \mathrm{E}|X_k-a|^{2+p} < \infty$ *for* $k = 1, 2, \ldots,$ *then*

$$\sup_x |F(x)-H(x/d)*\Phi(x/\sqrt{1-d^2})| \leqslant c(\sigma N/a+\sigma^2 N/a^2+\beta^{2+p}/\theta^{2+p}a^{p/2}+\sigma/\theta^2 a).$$

*In this case* $d = a\sigma N/\sigma$.

COROLLARY 11. *If the assumptions of Corollary* 2 *are satisfied and* $a = 0$, $\theta^2 = 1$, *and* $\beta^3 < \infty$, *then*

$$|F(x)-\Phi(x)| < c\frac{\beta^3}{1+|x|^3}\left(\frac{1}{\sqrt{a}}+\frac{\sigma^2 N}{a^2}+\frac{\sigma N}{a}\right).$$

The proof of Corollary 11 follows by Corollary 2 and by the estimations given in [2] and [5].

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