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ON ALMOST POLYNOMIAL FUNCTIONS

BY

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1. Let f be a real-valued function defined on the set R of all real numbers. We say that f is *almost additive* iff the relation

$$f(x+y) = f(x)+f(y)$$

holds for almost all $(x, y) \in R\times R$ (in the sense of Lebesgue's plane measure).

N. G. de Bruijn [1] and, independently, W. B. Jurkat [4] gave an affirmative answer to the following question raised by P. Erdös [2]: If f: $R\to R$ is an almost additive function, does there exist a function g: $R\to R$ such that

$$g(x+y) = g(x)+g(y)$$

for all $(x, y)\in R\times R$ and such that $f(x) = g(x)$ almost everywhere (in the sense of Lebesgue's linear measure)?

In [1] we can find also some natural generalizations of this problem.

The analogous problem for convex functions has been recently solved in affirmative by M. Kuczma [5].

A similar problem appears in the case of polynomial functions.

Definition 1. We say that f: $R\to R$ is a *polynomial function* (*almost polynomial function*) *of the n-th order* iff the relation

$$\Delta_y^{n+1}f(x) = 0 \tag{1}$$

holds for all $(x, y)\in R\times R$ (for almost all $(x, y)\in R\times R$), where $\Delta_y^p f(x)$ denotes the finite difference of p-th order of f with increment y (¹).

Thus we can ask if for a given almost polynomial function f of n-th order there exists a polynomial function g of the same order and such that $f(x) = g(x)$ almost everywhere in R.

(¹) In the literature we meet usually the symbol $\Delta_h^p f(x)$ with $h>0$, but an elementary calculation shows that the relations $\Delta_h^p f(x) = 0$ for $h>0$ and $\Delta_y^p f(x) = 0$ for $y\in R$ are equivalent.

The answer is positive. It results as a particular case of theorem 1.

2. It follows by induction that we can write (1) in the form

$$\sum_{j=0}^{n+1} (-1)^{n+1-j} \binom{n+1}{j} f(x+jy) = 0 \tag{2}$$

or, equivalently, in the form

$$f(x) = \sum_{i=1}^{n+1} (-1)^{i-1} \binom{n+1}{i} f(x+iy). \tag{3}$$

In the sequel we shall use both these forms.

Deviating slightly from the terminology adopted by de Bruijn let us introduce the following notions:

Definition 2. A non-empty family $\mathscr{I}^k$ of subsets of k-dimensional euclidean space R^k is called a *linearly invariant proper ideal* (²) iff

(i) $A, B \in \mathscr{I}^k$ implies $A \cup B \in \mathscr{I}^k$,

(ii) $A \in \mathscr{I}^k$, $B \subset A$ implies $B \in \mathscr{I}^k$,

(iii) $R^k \notin \mathscr{I}^k$,

(iv) for every real number $\alpha \neq 0$, $\beta \in R^k$, and $A \in \mathscr{I}^k$ we have $\alpha A + \beta \in \mathscr{I}^k$ (³).

It is easily seen that the family $\mathscr{L}_0^k$ of all subsets of the space R^k with the Lebesgue measure zero is a linearly invariant proper ideal. Similarly, families $\mathscr{F}^k$ and $\mathscr{L}_f^k$ of all sets of the first category in R^k and of all sets with the finite outer Lebesgue measure in R^k, are also linearly invariant proper ideals.

Definition 3. We say that two linearly invariant proper ideals $\mathscr{I}^2$ and $\mathscr{I}^1$ are *conjugate* iff for every $M \in \mathscr{I}^2$ there exists a set $U \in \mathscr{I}^1$ such that for every $x \notin U$ the set

$$V_x = \{y\colon (x, y) \in M\} \tag{4}$$

belongs to $\mathscr{I}^1$.

Remark 1. In view of Fubini's theorem the ideals $\mathscr{L}_0^2, \mathscr{F}^2, \mathscr{L}_f^2$ and $\mathscr{L}_0^1, \mathscr{F}^1, \mathscr{L}_f^1$ are, respectively, conjugate.

Let $\mathscr{I}^1$ be a given ideal on the real line. It generates a certain ideal on the plane in a natural way. Namely, let $\pi(\mathscr{I}^1)$ be the family consisting of all subsets of R^2 which are of the form $(S \times R) \cup (R \times S)$ with $S \in \mathscr{I}^1$, and of all their subsets.

It can be easily checked that if $\mathscr{I}^1$ is a linearly invariant proper ideal, then so is also $\pi(\mathscr{I}^1)$, and that $\pi(\mathscr{I}^1)$ and $\mathscr{I}^1$ are conjugate.

(²) A non-empty family $\mathscr{I}^k$ fulfilling (i) and (ii) is called an *ideal* (cf., e.g., [6]). The word "proper" refers to (iii) and "linearly invariant" to (iv).

(³) $\alpha A + \beta \overset{\mathrm{df}}{=} \{x\colon x = \alpha a + \beta, a \in A\}$.

Definition 4. Let $\mathscr{I}^2$ be a linearly invariant proper ideal. We say that $f\colon R \to R$ is an *almost polynomial function of n-th order with respect to* $\mathscr{I}^2$ iff there exists a set $M \in \mathscr{I}^2$ such that relation (1) holds for all $(x, y) \in R^2 \setminus M$.

Remark 2. For $\mathscr{I}^2 = \mathscr{L}_0^2$ this notion reduces to that of an almost polynomial function of n-th order as defined in Definition 1.

3. Now, we shall prove the following:

THEOREM 1. *Let $\mathscr{I}^2$ and $\mathscr{I}^1$ be conjugate linearly invariant proper ideals. If $f\colon R \to R$ is an almost polynomial function of n-th order with respect to $\mathscr{I}^2$, then there exists exactly one polynomial function g of n-th order and a set $U \in \mathscr{I}^1$ such that $f(x) = g(x)$ for all $x \notin U$.*

Proof. There exists a set $M \in \mathscr{I}^2$ such that relation (1) holds for all $(x, y) \notin M$. Let us put

$$U \overset{\mathrm{df}}{=} \{x\colon V_x \notin \mathscr{I}^1\},$$

where V_x is given by (4). Since $\mathscr{I}^2$ and $\mathscr{I}^1$ are conjugate, we have $U \in \mathscr{I}^1$.

To every $x \in R$ we assign a set $A_x \in \mathscr{I}^1$ as follows:

$$A_x = \bigcup_{i=1}^{n+1} \frac{1}{i}(U - x).$$

Let $\varphi\colon R \to R$ be an arbitrary function such that

(a) $\varphi(x) = 0$ for $x \notin U$,

(b) $\varphi(x) \notin A_x$ for $x \in U$.

Condition (b) implies that for $x \in U$ we have

$$x + i\varphi(x) \notin U \quad \text{for } i = 1, 2, \ldots, n+1.$$

In the sequel the function $\varphi(x)$ is regarded as fixed.

We put

$$g(x) \overset{\mathrm{df}}{=} \sum_{i=1}^{n+1} (-1)^{i-1} \binom{n+1}{i} f\big(x + i\varphi(x)\big). \tag{5}$$

It is obvious that $f(x) = g(x)$ for $x \notin U$.

In order to show that g is a polynomial function of n-th order we shall first prove that

$$g(x) = \sum_{k=1}^{n+1} (-1)^{k-1} \binom{n+1}{k} f(x + ky) \quad \text{for } y \notin A_x. \tag{6}$$

In fact, let us fix arbitrarily an $x \in R$ and a $y \notin A_x$. Since $\varphi(x), y \notin A_x$, the sets $V_{x+i\varphi(x)}$ and V_{x+ky} are elements of $\mathscr{I}^1$ for $i, k = 1, 2, \ldots, n+1$.

Thus there exists at least one $\psi(y)$ such that

$$\psi(y) \notin \bigcup_{k=1}^{n+1} \frac{1}{k}\big(V_{x+ky}-\varphi(x)\big) \cup \bigcup_{i=1}^{n+1} \frac{1}{i}\left(V_{x+i\varphi(x)}-y\right) \in \mathscr{I}^1.$$

Then we have

$$x+ky \notin U, \quad \varphi(x)+k\psi(y) \notin V_{x+ky}, \quad y+i\psi(y) \notin V_{x+i\varphi(x)} \tag{7}$$

and so, in particular,

$$\big(x+ky, \varphi(x)+k\psi(y)\big) \notin M \quad \text{for } k = 1, 2, \ldots, n+1. \tag{8}$$

Now, let us note that for an arbitrary i, $1 \leqslant i \leqslant n+1$, and for every $y_i \notin V_{x+i\varphi(x)}$ there is (see (3))

$$f\big(x+i\varphi(x)\big) = \sum_{k=1}^{n+1} (-1)^{k-1} \binom{n+1}{k} f\big(x+i\varphi(x)+ky_i\big).$$

In particular, in virtue of (7) we may take $y_i = y+i\psi(y)$ and thus we can write

$$\begin{aligned} f\big(x+i\varphi(x)\big) &= \sum_{k=1}^{n+1} (-1)^{k-1} \binom{n+1}{k} f\Big(x+i\varphi(x)+k\big(y+i\psi(y)\big)\Big) \\ &= \sum_{k=1}^{n+1} (-1)^{k-1} \binom{n+1}{k} f\Big(x+ky+i\big(\varphi(x)+k\psi(y)\big)\Big). \end{aligned} \tag{9}$$

Finally, (5) and (9) give

$$\begin{aligned} g(x) &= \sum_{i=1}^{n+1} (-1)^{i-1} \binom{n+1}{i} \sum_{k=1}^{n+1} (-1)^{k-1} \binom{n+1}{k} f\Big(x+ky+i\big(\varphi(x)+k\psi(y)\big)\Big) \\ &= \sum_{k=1}^{n+1} (-1)^{k-1} \binom{n+1}{k} \sum_{i=1}^{n+1} (-1)^{i-1} \binom{n+1}{i} f\Big(x+ky+i\big(\varphi(x)+k\psi(y)\big)\Big), \end{aligned}$$

whence (6) results in virtue of (3) and (8).

Now, let us fix arbitrary $u, v \in R$. By (6) we can write

$$g(u+jv) = \sum_{k=1}^{n+1} (-1)^{k-1} \binom{n+1}{k} f(u+jv+ky_j) \tag{10}$$

for $y_j \notin A_{u+jv}$, $j = 0, 1, \ldots, n+1$.

Let us take a $y \notin A_u = \bigcup_{k=1}^{n+1} \frac{1}{k}(U-u) \in \mathscr{I}^1$, and choose $\zeta(y)$ such that

$$\begin{aligned} &y+j\zeta(y) \notin A_{u+jv} \quad \text{for } j = 0, 1, \ldots, n+1, \\ &v+k\zeta(y) \notin V_{u+kv} \quad \text{for } k = 1, 2, \ldots, n+1. \end{aligned} \tag{11}$$

Such a choice is always possible, because the set

$$\bigcup_{j=1}^{n+1} \frac{1}{j}(A_{u+jv}-y) \cup \bigcup_{k=1}^{n+1} \frac{1}{k}(V_{u+ky}-v)$$

belongs to $\mathscr{I}^1$.

Since $y+j\zeta(y) \notin A_{u+jv}$, we obtain from (10) that

$$g(u+jv) = \sum_{k=1}^{n+1} (-1)^{k-1}\binom{n+1}{k} f\big(u+jv+k(y+j\zeta(y))\big) \tag{12}$$

$$= \sum_{k=1}^{n+1} (-1)^{k-1}\binom{n+1}{k} f\big(u+ky+j(v+k\zeta(y))\big)$$

for $j = 0, 1, \ldots, n+1$.

Finally, from (2) and (12) we have

$$\Delta_v^{n+1} g(u) = \sum_{j=0}^{n+1} (-1)^{n+1-j}\binom{n+1}{j} g(u+jv)$$

$$= \sum_{j=0}^{n+1} (-1)^{n+1-j}\binom{n+1}{j} \sum_{k=1}^{n+1} (-1)^{k-1}\binom{n+1}{k} f\big(u+ky+j(v+k\zeta(y))\big)$$

$$= \sum_{k=1}^{n+1} (-1)^{k-1}\binom{n+1}{k} \sum_{j=0}^{n+1} (-1)^{n+1-j}\binom{n+1}{j} f\big(u+ky+j(v+k\zeta(y))\big) = 0$$

in view of the fact that $\big(u+ky, v+k\zeta(y)\big) \notin M$ for $k = 1, 2, \ldots, n+1$ (see (11)).

Thus $g(x)$ is a polynomial function of n-th order.

In order to prove the uniqueness it is sufficient to observe that two polynomial functions g_1, g_2 of n-th order which coincide besides a set $S \in \mathscr{I}^1$ are identical.

In fact, let $x_0 \in R$. We have, by (3), for every y,

$$g_1(x_0) = \sum_{i=1}^{n+1} (-1)^{i-1}\binom{n+1}{i} g_1(x_0+iy),$$

$$g_2(x_0) = \sum_{i=1}^{n+1} (-1)^{i-1}\binom{n+1}{i} g_2(x_0+iy).$$

Now, it is sufficient to take a y such that $x_0+iy \notin S$ for $i = 1, 2, \ldots, n+1$, which is possible, since

$$\bigcup_{i=1}^{n+1} \frac{1}{i}(S-x_0) \in \mathscr{I}^1.$$

Then $g_1(x_0+iy) = g_2(x_0+iy)$ for $i = 1, 2, \ldots, n+1$, and consequently $g_1(x_0) = g_2(x_0)$, which completes the proof.

4. In the case $\mathscr{I}^2 = \mathscr{L}_f^2$, $\mathscr{I}^1 = \mathscr{L}_f^1$ one can prove a little more. Namely, the following theorem is true:

THEOREM 2. *If* (1) *holds except for* $(x, y) \in M \subset R^2$, $m_e(M) < \infty$ (4), *then f is equal to a polynomial function g of n-th order almost everywhere in R (in particular, f is almost polynomial of n-th order).*

Proof. Let us fix positive real numbers a and β such that $m_e(M) < \beta$ and let

$$U \overset{\mathrm{df}}{=} \left\{x\colon\ m_e(V_x) > \frac{\beta}{a}\right\},$$

where V_x is given by (4). By Fubini's theorem, $m_e(U) < a$.

Now, by the same argument as in the proof of theorem 1, we obtain that there exists a polynomial function g of n-th order such that $f(x) = g(x)$ for $x \notin U$ (note that in view of (6) g does not depend on a). Thus, letting a tend to zero we obtain our assertion.

Similarly, following [1] we can obtain a stronger result in the case where the ideal $\mathscr{I}^2$ consists of the sets of the form $S_1 \times S_2$, where $S_1, S_2 \in \mathscr{I}^1$ (cf. also [3]). Namely, we have the following

THEOREM 3. *Let $\mathscr{I}^1$ be a proper linearly invariant ideal on the real line. If $\Delta_y^{n+1} f(x) = 0$ for all $x, y \notin S \in \mathscr{I}^1$, then f is a polynomial function of n-th order.*

Proof. Theorem 1 with $\mathscr{I}^2 = \pi(\mathscr{I}^1)$ implies the existence of a polynomial function g of n-th order such that

$$T = \{x\colon f(x) \neq g(x)\} \in \mathscr{I}^1.$$

Observe that we can write (1) in an equivalent form

$$f(x+y) = \frac{1}{n+1}\left\{f(x) + \sum_{j=2}^{n+1} (-1)^j \binom{n+1}{j} f(x+jy)\right\}. \tag{13}$$

According to our hypothesis, (13) holds for all $x, y \notin S$.

Let us take an arbitrary $x_0 \in R$. Since the set

$$(T \cup S) \cup (x_0 - S) \cup \bigcup_{j=2}^{n+1} \frac{1}{1-j}(T - jx_0)$$

belongs to $\mathscr{I}^1$, we can find an x such that $x \notin T \cup S$, $x_0 - x \notin S$, $x + j(x_0 - x) \notin T$ for $j = 2, 3, \ldots, n+1$. Thus, for $y = x_0 - x$, we have

(4) $m_e(M)$ denotes here the outer Lebesgue measure of M.

$$f(x_0) = f(x+y) = \frac{1}{n+1}\Big\{f(x)+\sum_{j=2}^{n+1}(-1)^j\binom{n+1}{j}f(x+jy)\Big\}$$

$$= \frac{1}{n+1}\Big\{g(x)+\sum_{j=2}^{n+1}(-1)^j\binom{n+1}{j}g(x+jy)\Big\} = g(x+y) = g(x_0),$$

which ends the proof.

Remark 3. The results of the present paper extend almost without any change to the case where the variables lie in a vector space over rationals, or even in an abelian (additive) group in which the division by integers can be always uniquely performed.

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