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CONCERNING A CLOSED SUBSET OF A DENDROID CONTAINING 2-RAMIFICATION POINTS

BY

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In [7] the author* defined a 2-ramification point of a metric dendroid and presented results concerning sufficiency for the existence of such a point. He also described a subset $TR(M)$ of a dendroid M which was shown to be the smallest subset of a dendroid that contains either the ramification points, the terminal points or the 2-ramification points of M and which is necessarily closed. In the present paper the author presents results concerning the properties of $TR(M)$ and its complement in a dendroid M. It will also be shown that the decomposition of M into points of $M-TR(M)$ and components of $TR(M)$ is a monotone upper semi-continuous decomposition of M which is a dendrite.

In this paper all spaces are *metric*, a *continuum* is a compact, connected space and an *arc* is a non-degenerate continuum with no more than two non-separating points. A *dendroid* is a hereditarily unicoherent, arcwise connected continuum, and a *dendrite* is a locally connected dendroid. A point p is a *ramification point* of a dendroid M if there are three arcs in M such that the intersection of each two contains only p. A point p is a *terminal point* of a dendroid M if p is an endpoint of every arc in M which contains p. The point p is a *2-ramification point* of a dendroid M if p is not a terminal point of M and if there exists a sequence $p_1, p_2, p_3, \ldots$ of points of M which converges to p such that no two points of the sequence are contained in the same subarc of M with p. If M is a dendroid, then $TR(M)$ denotes the union of the set of terminal points, the set of ramification points and the set of 2-ramification points of M. Since, in a dendroid, there is only one arc between each two points, if p and q are points of a dendroid, $[p, q]$ will denote the arc from p to q. A set T is a *triod* if there exist three non-empty mutually exclusive open subsets H, J and K of T such that $N = T-(H \cup J \cup K)$ is connected.

A set M is *connected im kleinen* at a point p if, for every open set U containing p, there is an open set V containing p such that V is a subset

* This paper represents a part of the author's doctoral dissertation.

of U and such that if q is a point of V, some connected subset of U contains p and q. If G is any collection of sets, G^* denotes the union of all members in G. The statement that *the collection $g(M)$ of disjoint subsets of a space M is upper semi-continuous* means that if $G_1, G_2, G_3, \ldots$ is a sequence of elements of $g(M)$, each of $p_1, p_2, p_3, \ldots$ and $q_1, q_2, q_3, \ldots$ is a sequence of points such that, for each i, p_i and q_i are in G_i, G is an element of $g(M)$ and $p_1, p_2, p_3, \ldots$ converges to a point of G, then some subsequence of $q_1, q_2, q_3, \ldots$ converges to a point of G. An upper semi-continuous decomposition is *monotone* if each element in the collection is connected. If p is a point and ε is a positive number, then $S(p, \varepsilon)$ denotes the set of all points x such that the distance from p to x is less than ε. If M is a set, then $\bar{M}$ denotes the closure of M.

1. The set $TR(M)$ and its complement. It is interesting to note that if M is a dendroid, $TR(M)$ might be M, as illustrated in the following example:

Example 1.1. Let C denote the Cantor set and M the dendroid in the plane consisting of the union of line segments connecting the point $(0, 0)$ with each point in $1 \times C$ and line segments connecting the point $(1, 1)$ with each point in $0 \times C$. Each point of M is either a terminal point or a 2-ramification point. Hence $TR(M) = M$.

THEOREM 1.1. *If every non-terminal point of a dendroid M is a ramification point of M, then every non-terminal point of M is a 2-ramification point of M.*

Proof. Considering the non-trivial case that M is non-degenerate, let p denote a non-terminal point of M and let $[x, y]$ denote an arc of which p is a cutpoint. Let ε_1 denote a positive number such that $\varepsilon_1 \leqslant 1$ and $S(p, \varepsilon_1)$ contains neither x nor y, and let p_1 denote a cutpoint of $[p, y]$ that is in $S(p, \varepsilon_1)$. Then p_1 is a non-terminal point of M, thus a ramification point of M; and there is a point q_1 such that q_1 is not in $[x, y]$, q_1 is in $S(p, \varepsilon_1)$ and $[p_1, q_1] \cap [p, y]$ contains only p_1. Sequences $\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots, p_1, p_2, p_3, \ldots$ and $q_1, q_2, q_3, \ldots$ can be constructed so that, for each integer $i > 1$,

(1) $0 < \varepsilon_i < \varepsilon_{i-1}/2$ and $S(p, \varepsilon_i)$ does not intersect any of $[p_1, q_1]$, $[p_2, q_2], \ldots, [p_{i-1}, q_{i-1}]$,

(2) p_i is a cutpoint of $[p, p_{i-1}]$ which is also in $S(p, \varepsilon_i)$, and

(3) q_i is a point of $S(p, \varepsilon_i)$ such that $[q_i, p_i] \cap [x, y]$ contains only p_i.

The sequence $q_1, q_2, q_3, \ldots$ converges to p and no two points in the sequence are in the same arc with p. Hence, p is a 2-ramification point of M.

The following corollary, which is more general than Theorem 1.1 of [7], gives a sufficient condition that a dendroid contain a 2-ramification point:

COROLLARY 1.1. *If a dendroid M is non-degenerate and $M = TR(M)$, then M contains a 2-ramification point.*

This follows immediately from Theorem 1.1 and the fact that M is non-degenerate.

THEOREM 1.2. *If M is a dendroid, $M-TR(M)$ is non-empty if and only if $TR(M)$ is not connected.*

Proof. Suppose $M-TR(M)$ is non-empty. Any subcontinuum of a dendroid is a dendroid [1]. Then, if $TR(M)$ is connected, $TR(M)$ is a proper subcontinuum of M (cf. [7], Theorem 2.1) which contains all the terminal points of M. This is a contradiction to Miller's Theorem (cf. [3], Theorem 3.5). Thus, if $M \neq TR(M)$, then $TR(M)$ is not connected.

Assume that $TR(M)$ is not connected. Then $M-TR(M)$ must be non-empty since M is connected.

THEOREM 1.3. *If M is a dendroid and C is a component of $M-TR(M)$, then C is arcwise connected and strongly connected.*

Proof. $M-TR(M)$ is locally connected and open (cf. [7], Theorems 1.3 and 2.1). Then C is an open (cf. [4], Theorem 2, p. 86), connected, connected im kleinen, inner limiting set and, consequently, C is arcwise connected (cf. [4], Theorem 13, p. 91).

THEOREM 1.4. *If M is a dendroid and C is a component of $M-TR(M)$, then $\bar{C}$ is an arc.*

Proof. The procedure is to show first that $\bar{C}$ is not a triod. Suppose that $\bar{C}$ is a triod. Then there exist mutually exclusive subsets H, J and K of $\bar{C}$, each open in $\bar{C}$, such that $N = \bar{C}-(H \cup J \cup K)$ is connected. Since each of H, J and K is open in $\bar{C}$, there is a point h in $H \cap C$, a point j in $J \cap C$ and a point k in $K \cap C$; and since C is connected, there is a point x in $N \cap C$. By Theorem 1.3, C is arcwise connected, so each of the arcs $[h, x]$, $[j, x]$, and $[k, x]$ is in C. Since M is hereditarily unicoherent, each of h, j and k is in only one of $[h, x]$, $[j, x]$ and $[k, x]$. Thus $[h, x] \cup [j, x] \cup \cup [k, x]$ contains a simple triod and C contains a ramification point of M. This is a contradiction to the definition of $TR(M)$. Hence, $\bar{C}$ is not a triod. Then $\bar{C}$ is a compact, non-degenerate, unicoherent continuum which is not a triod; and, by Sorgenfrey's Theorem (cf. [5], Theorem 3.2), $\bar{C}$ is irreducible between two points p and q. Since the arc $[p, q]$ is in $\bar{C}$, $\bar{C} = [p, q]$.

THEOREM 1.5. *If M is a non-degenerate dendroid which is not an arc and C is a component of $M-TR(M)$, then $\bar{C}$ contains either a ramification point or a 2-ramification point.*

Proof. Since M is not an arc, M contains a ramification point. Then, if $\bar{C}$ is the arc $[x, y]$, where x and y are both terminal points of M, the set composed of x and y separates M into C and $M-\bar{C}$, two non-empty mutually separated sets. This contradicts Miller's Theorem (cf. [3], Theorem 3.6) which asserts that no subset of terminal points of a non-degenerate hereditarily decomposable continuum separates the continuum. Thus $\bar{C}$ must contain either a ramification point or a 2-ramification point.

THEOREM 1.6. *Let t denote a terminal point of a dendroid M and $D(t)$ the component of $TR(M)$ which contains t. If there is a component C of $M-TR(M)$ such that $\bar{C}$ contains t, then $D(t)$ is degenerate.*

Proof. If M is an arc, the theorem is true. Suppose that M is not an arc and that $D(t)$ is non-degenerate. Let s denote a point of $D(t)$ distinct from t. Since $TR(M)$ is closed (cf. [7], Theorem 2.1), $D(t)$ is a subcontinuum of M, and $D(t)$ contains the arc $[t, s]$. By Theorem 1.5, $\bar{C}$ contains a point r which is either a ramification point or a 2-ramification point of M. Of course, $t \neq r$, and $\bar{C} = [t, r]$. Since t is a terminal point of M, $[t, r] \cap [t, s]$ is non-degenerate and must contain a point q such that $q \neq t$, $q \neq r$, $q \neq s$ and q is also in $C \cap D(t)$. However, since $D(t)$ is a subset of $TR(M)$, q in $C \cap D(t)$ is a contradiction to the fact that C and $D(t)$ are mutually exclusive. Thus $D(t)$ contains only the point t.

Example 1.2. Let M denote the dendroid in the plane which is the union of the lines $[0, 1] \times 0$ and $t \times [0, t]$ for $t = 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$ The point $(0, 0)$ is a terminal point of M, is a degenerate component of $TR(M)$, yet $(0, 0)$ is not in $\bar{C}$ for any component C of $M-TR(M)$. Hence, the converse of Theorem 1.6 is not true.

THEOREM 1.7. *If t is a point of a dendroid M which is contained in a degenerate component of $TR(M)$, then M is locally connected at t. Moreover, if t is a terminal point of M and a point of $\bar{C}$ for some component C of $M-TR(M)$, then $TR(M)$ is also locally connected at t.*

Proof. Based on Theorems 1.1 and 1.3 of [7], any point at which M is not locally connected will be contained in an arc each point of which is in $TR(M)$. Then, if $D(t)$ denotes the component of $TR(M)$ containing t, it is degenerate, and M is locally connected at t.

Suppose t is a terminal point of M and t is in $\bar{C}$ for some component C of $M-TR(M)$. Let r denote a point of M such that $\bar{C} = [t, r]$. By Theorem 1.6, t is in a degenerate component of $TR(M)$, and, by the first part of this theorem, M is locally connected at t. There is a connected open set U containing t such that $\bar{U}$ does not contain r. Suppose y is a point of U not in $[t, r]$. The arc $[y, t]$ is a subset of U and does not contain r. Since t is a terminal point of M, $[t, r] \cap [y, t]$ is non-degenerate and $[t, r] \cup [y, t]$ contains a ramification point q of M which is in $[t, r]$ and such that $q \neq r$. But, since $[t, r] = \bar{C}$ and C is a component of $M-TR(M)$, the only possible ramification point in $[t, r]$ is r. Thus $U \cap M = U \cap [t, r] = U \cap TR(M)$, and $TR(M)$ is locally connected at t.

THEOREM 1.8. *Suppose M is a dendroid, t is a point of M which is in* *a degenerate component of $TR(M)$ and H is the collection to which D belongs if and only if there is a component C of $M - TR(M)$ such that $\bar{C}$ contains t and $D = \bar{C}$. Then H^* is a dendrite.*

Proof. The set H^* is connected; so, if H^* can be shown to be closed, it is a dendroid and all remaining to be shown is local connectedness. Suppose that H^* is not closed and that p is a point of $\bar{H}^* - H^*$. Then there is a sequence $A_1, A_2, A_3, \ldots$ of distinct elements of H and a sequence of points $p_1, p_2, p_3, \ldots$ such that, for each i, p_i is a point of A_i distinct from t, and p is the sequential limit point of $p_1, p_2, p_3, \ldots$ Some subsequence $[p_{n_1}, t]$, $[p_{n_2}, t]$, $[p_{n_3}, t], \ldots$ of the sequence of arcs $[p_1, t]$, $[p_2, t]$, $[p_3, t], \ldots$ has a sequential limiting set L which is closed and connected [4] and contains p and t. Since M is hereditarily unicoherent, L contains $[p, t]$. At most one of the arcs $[p_{n_i}, p]$ can fail to contain t. Then, if q is a cutpoint of $[p, t]$, there is a sequence of points of M converging to q no two of which are in the same arc with q. Thus, q is a 2-ramification point of M and $[p, t]$ is a subset of $TR(M)$. This contradicts the fact that t is in a degenerate component of $TR(M)$. Hence H^* is closed and H^* is a dendroid.

If y is a terminal point of H^*, then y is an endpoint of an element of H and y is distinct from t except for the trivial case where H contains only one element. Thus y is in $\bar{C}$ for some component C of $M - TR(M)$ and y is not in C. Then y is in $TR(M)$. Also, if y is either a ramification point or a 2-ramification point of H^*, y is a ramification point or a 2-ramification point of M. Hence, $TR(H^*)$ is a subset of $TR(M)$, and $TR(H^*)$ consists of the points of H^* which are endpoints of arcs in H. Since t is in a degenerate component of $TR(M)$, t is in a degenerate component of $TR(H^*)$, and, by Theorem 1.7, H^* is locally connected at t. Each point x of $TR(H^*)$, such that $x \neq t$, is a terminal point of H^* which is in $\bar{K}$ for some component K of $H^* - TR(H^*)$, and, again by Theorem 1.7, H^* is locally connected at x. Thus H^* is a locally connected dendroid and H^* is a dendrite.

2. Upper semi-continuous decompositions of dendroids. In this section an upper semi-continuous decomposition of a dendroid will be described and will be shown to be a dendrite.

THEOREM 2.1. *Suppose M is a dendroid and $g(M)$ is the collection of subsets of M whose only non-degenerate elements are the non-degenerate components of $TR(M)$. Then*

(1) *if $TR(M) = M$, $g(M)$ is degenerate,*

(2) *$g(M)$ is a monotone upper semi-continuous decomposition of M, and*

(3) *the decomposition space $g(M)$ is a dendrite.*

Proof. Statement (1) is obvious. Suppose $G_1, G_2, G_3, \ldots$ and $p_1, p_2, p_3, \ldots$ are sequences such that, for each i, G_i is an element of $g(M)$ and p_i

is a point of G_i. Suppose further that $p_1, p_2, p_3, \ldots$ has a sequential limit point p in some element G of $g(M)$ and that $q_1, q_2, q_3, \ldots$ is a sequence such that, for each i, q_i is a point of G_i. If there exists a subsequence $G_{n_1}, G_{n_2}, G_{n_3}, \ldots$ of $G_1, G_2, G_3, \ldots$ such that, for each i and j, $G_{n_i} = G_{n_j}$, then each of $q_{n_1}, q_{n_2}, q_{n_3}, \ldots$ is in G, and, since G is compact, some subsequence of $q_{n_1}, q_{n_2}, q_{n_3}, \ldots$ has a sequential limit point in G. Also, if there is a subsequence $G_{m_1}, G_{m_2}, G_{m_3}, \ldots$ of $G_1, G_2, G_3, \ldots$ such that each G_{m_i} is degenerate, then, for each i, $p_{m_i} = q_{m_i}$ and p is the sequential limit point of $q_{m_1}, q_{m_2}, q_{m_3}, \ldots$

Now suppose that all terms of $G_1, G_2, G_3, \ldots$ are distinct and that each is non-degenerate. Then, the set consisting of the points of the sequence $q_1, q_2, q_3, \ldots$ is an infinite subset of the compact set M and has a limit point, and some subsequence $q_{n_1}, q_{n_2}, q_{n_3}, \ldots$ of $q_1, q_2, q_3, \ldots$ has a sequential limit point. It then follows [4] that some subsequence $H_1, H_2, H_3, \ldots$ of $G_{n_1}, G_{n_2}, G_{n_3}, \ldots$ has a sequential limiting set A which is a continuum containing p and q. Since each H_i is non-degenerate, each H_i is a subset of $TR(M)$ and $H_1 \cup H_2 \cup H_3 \cup \ldots$ is a subset of $TR(M)$; and, since each point of A is a limit point of $H_1 \cup H_2 \cup H_3 \cup \ldots$ and $TR(M)$ is closed [7], A is a subset of $TR(M)$. Since A is connected and contains p, A is a subset of the component G of $TR(M)$ which contains p. Thus q is in G and $g(M)$ is upper semi-continuous; and, since each element of $g(M)$ is connected and $g(M)^* = M$, $g(M)$ is a monotone upper semi-continuous decomposition of M.

There is a natural continuous function f (see, for example, [6]) from M onto $g(M)$ which maps each point p of M onto the element of $g(M)$ which contains p. Then $g(M)$ is the image of a dendroid under a continuous monotone function, and $g(M)$ is a dendroid [2]. So, in order to complete the argument to statement (3), it is only necessary to show that $g(M)$ is locally connected.

Suppose p is a point of $M - TR(M)$. Then p is contained in an open set U such that $\overline{U} = [x, y]$ for some points x and y and such that $\overline{U}$ does not intersect $TR(M)$. Since p is a cutpoint of $[x, y]$, $f(p)$ is a cutpoint of the arc $[f(x), f(y)]$ and $f(p)$ is not a terminal point of $g(M)$. Each point of $\overline{U}$ is in a degenerate element of $g(M)$. Then $f(U)$ is open in $g(M)$, $f(U)$ contains $f(p)$ but $f(U)$ contains no point of $g(M)$ except points of the arc $[f(x), f(y)]$. Thus $f(p)$ is neither a ramification point nor a 2-ramification point of $g(M)$ and p is not in $TR(g(M))$.

Suppose C is a component of $TR(g(M))$. Since C is a connected subset of $f(M)$ and f is monotone, the set $f^{-1}(C)$ is connected (cf. [6], p. 138). Also, according to the preceding section, $f^{-1}(C)$ is a subset of $TR(M)$. Then $f^{-1}(C)$ is contained in a component of $TR(M)$, and, by definition of $g(M)$, $f(f^{-1}(C)) = C$. Thus every component of $TR(g(M))$ is degenerate.

Finally, $g(M)$ is locally connected at each point of $g(M) - TR(g(M))$

(cf. [7], Theorem 1.4). Each component of $TR(g(M))$ is degenerate, so $g(M)$ is locally connected at each point of $TR(g(M))$ by Theorem 1.7. Hence, $g(M)$ is locally connected and it is a dendrite.

The decomposition described in Theorem 2.1 is not necessarily minimal with respect to being a monotone upper semi-continuous decomposition of a dendroid to a dendrite since examples are known of dendrites with dense sets of ramification points.

An obvious consequence to Theorem 2.1 is the following theorem which gives a necessary and sufficient condition that $M-TR(M)$ be non-empty:

THEOREM 2.2. *Let M denote a dendroid and $g(M)$ the monotone upper semi-continuous decomposition of M described in Theorem* 2.1. *Then $M \neq TR(M)$ if and only if $g(M)$ is non-degenerate.*

THEOREM 2.3. *Suppose M is a dendroid, $g(M)$ is a monotone upper semi-continuous decomposition of M to a dendrite, f is the continuous function from M onto $g(M)$ which maps each point of M onto the element of $g(M)$ which contains it, and N is a subcontinuum of M at each point of which M is not locally connected. Then, for each point p of N, $f(p)$ is a non-degenerate element of $g(M)$.*

Proof. Suppose that there is a point p of N such that $f(p)$ is degen erate. There is an open set U in M containing p such that if U' is any open set in M containing p which is a subset of U, then U' is not connected. There is a subcollection V of $g(M)$ such that p is a point of V^*, V^* is a subset of U and V^* is an open set in M. Then V is an open set in $g(M)$ containing $f(p)$. Since $g(M)$ is locally connected, there is a connected open set J in $g(M)$ such that J contains $f(p)$ and such that J is a subset of V. By [6], p. 138, $f^{-1}(J)$ is a connected open set in M which contains p and which is a subset of U. This is a contradiction. Thus $f(p)$ is non-degenerate.

The author considered the possibility that any monotone upper semi-continuous decomposition of a dendroid to a dendrite would have a degenerate element. For example, it is well known that an arc cannot be decomposed non-trivially into mutually exclusive subarcs.

Example 2.1. Charatonik in [1] described a function h from the Cantor set C in $[0,1]$ onto the interval $[0,1]$ in such a way that h is continuous and non-decreasing. He then considered the dendroid M in the plane consisting of the union of line segments connecting the point $(t,0)$ to $(h(t),1)$ for each t in C. Let f be the function from M onto the arc A from $(0,1)$ to $(1,1)$ such that if x is in M and x is in the segment from $(t,0)$ to $(h(t),1)$ for some t in C, then $f(x) = (h(t),1)$. The function f is continuous and monotone, and the collection $f^{-1}(y)$ for y in A is a monotone upper semi-continuous decomposition of the dendroid M to an arc such that each element of the decomposition is non-degenerate.

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