

TO PROFESSOR A. D. WALLACE  
ON HIS SIXTIETH BIRTHDAY

ON PARTIALLY ORDERED ALGEBRAS I

BY

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In the theory of universal algebras a great progress has been made recently, and universal algebras have been dealt with by several authors from different points of view. However, to the best of author's knowledge, universal algebras with partial order or lattice-order have not been investigated systematically. The aim of this paper\* is to call attention to partially ordered universal algebras and to point out some directions which — according to the author's opinion — deserve more attention.

The paper does not contain essentially new results. It merely gives some basic definitions and in few cases it shows that the known methods can be applied (with slight generalizations) to derive results which are universal algebraic in character.

The relevant discussion consists of two parts. The first one deals with the fundamental concepts and some properties of algebras<sup>(1)</sup> which admit full orders or are subdirect unions of such algebras. A brief treatment of order topology and completion processes is also given. The second part will show how every algebra gives rise to a lattice-ordered algebra, which is simply the algebra of all (or some only) its subalgebras. For these algebras the analogues to a number of theorems known for groups and rings can be obtained without any difficulty.

**1. Fundamental concepts.** Let  $(A; F)$  be an algebra, where  $A$  denotes the underlying set and  $F$  is the family of fundamental operations on  $A$ . All the operations  $f$  in  $F$  are supposed to be finitary. We assume tacitly that the operations in  $A$  satisfy some given set of axioms, though in most cases no explicit mention will be made of them.

Let further  $A$  also be a partially ordered set under a relation  $\leq$ .

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<sup>(1)</sup> We use "universal algebra" and "algebra" synonymously.

An  $n$ -ary operation  $f$  on  $A$  is said to satisfy a *monotony law* with the *monotony domain*  $C_f$  (where  $C_f$  is a subset of  $A$ ) if

1.  $x_1, \dots, x_n \in C_f$  implies  $f(x_1, \dots, x_n) \in C_f$ ,
2. for each  $i$  ( $i = 1, \dots, n$ ),  $f$  is either isotone or antitone or both in the variable  $x_i$ , that is,  $x_i \leq y_i$  and  $x_i, y_i \in A$  imply for all  $x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n \in C_i$  either

$$(1) \quad f(x_1, \dots, x_i, \dots, x_n) \leq f(x_1, \dots, y_i, \dots, x_n)$$

or (1) with  $\leq$  replaced by  $\geq$  or by  $=$ , respectively <sup>(2)</sup>.

According to that  $f$  is called of *monotony type*  $\uparrow, \downarrow$  or  $\updownarrow$  in the variable  $x_i$ . In case when  $x_i < y_i$  implies strict inequality in (1), we say that  $f$  obeys the *strict monotony law*.

By a *partially ordered algebra* we mean a set  $A$  such that

- (i)  $(A; F)$  is an algebra,
- (ii)  $(A, \leq)$  is a partially ordered set,
- (iii) with each  $n$ -ary operation  $f \in F$  there is associated a symbol  $\langle C_f; \gamma_1^f, \dots, \gamma_n^f \rangle$ , where  $C_f$  is the monotony domain of  $f$  and  $\gamma_i^f$  ( $= \uparrow, \downarrow$  or  $\updownarrow$ ) denotes the monotony type of  $f$  in  $x_i$  <sup>(3)</sup>.

We shall denote the just defined partially ordered algebra as  $(A; F, \leq)$  or more explicitly as

$$(2) \quad (A; F, \leq, \langle C_f; \gamma_1^f, \dots, \gamma_n^f \rangle_{f \in F}).$$

For instance <sup>(4)</sup>, if  $A$  is a group and  $F$  consists of a binary (the group operation), a unary (the inverse) and a nullary operation (the neutral element), then all the three operations have the whole of  $A$  as monotony domains, and they are of the monotony types  $(\uparrow, \uparrow), (\downarrow)$  and  $(\updownarrow)$ , respectively.

<sup>(2)</sup> The definition of monotony domain given above is not the most general one for which our following arguments are valid. In fact, condition 1 is not essential. But since this rather natural condition is satisfied in all cases of interest, we assume it. Note that in the traditional particular cases only the multiplication in rings is an operation for which the monotony domain is not the whole ring. It would be natural to suppose that  $C_f$  is never void; however, should we do so, the definition of subalgebras to be given had to be changed by adding a condition guaranteeing that the monotony domains are not void.

<sup>(3)</sup> This definition of partially ordered algebras is essentially the same as that given in [2], p. 5. A more general definition can allow an operation to be isotone in a subset of  $A$  and antitone in another one. This possibility for semigroups has been considered by Tamari [9] and Clifford [1].

<sup>(4)</sup> We shall frequently refer to partially ordered groups, rings etc. On them see e. g. [2].

In most cases of interest the monotony domains coincide with the set  $A$ , but sometimes they form a proper subsets of  $A$ . If  $C_f \neq A$ , then  $C_f$  is usually of the form

$$C_f = \{x \in A \mid x \geq e\},$$

where  $e \in A$  is some nullary operation. More generally, we shall say that the monotony domain  $C_f$  of  $f \in F$  is *natural* if it can be defined in the form

$$(3) \quad C_f = \{x \in A \mid \Gamma_f(x)\},$$

where  $\Gamma_f(x)$  is a meaningful sentence containing individual variables  $x, x_1, \dots, x_k$  in  $A$ , operations from  $F$ , the relation  $\leq$ , equality, negation, disjunction, conjunction, implication, and is such that quantifiers bind all of  $x_1, \dots, x_k$ . We may also allow  $k$  to vary on the non-negative integers such that a quantifier binds  $k$  too. Roughly speaking,  $C_f$  is natural if it can be defined in terms of operations in  $F$  and order relation.

For example, in a partially ordered ring  $R$  the monotony domain  $C_\mu = \{x \in R \mid x \geq 0\}$  of multiplication is natural, and it remains natural if it is defined as the set of sums of squares.

If all the monotony domains  $C_f$  ( $f \in F$ ) of a partially ordered algebra  $(A; F, \leq)$  are natural, then the algebra will be called a *natural partially ordered algebra*. The usual definitions of partially ordered groups, rings, fields and semigroups indicate that they all are natural partially ordered algebras. Natural partially ordered algebras may be denoted by the symbol <sup>(5)</sup>

$$(A; F, \leq, \langle \Gamma_f; \gamma_1^f, \dots, \gamma_n^f \rangle_{f \in F})$$

where  $\Gamma_f$  stands for the sentence in (3) defining  $C_f$ .

If  $\leq$  defines a full (i. e. linear) order on  $A$ , the partially ordered algebra  $(A; F, \leq)$  is called a *fully* (or *linearly*) *ordered algebra*. If  $\leq$  defines a lattice-order, then it is a *lattice-ordered algebra*. In this case the monotony laws may be written in either one of the equivalent forms:

$$(4) \quad \begin{aligned} & f(x_1, \dots, x_i, \dots, x_n) \vee f(x_1, \dots, y_i, \dots, x_n) \leq f(x_1, \dots, x_i \vee y_i, \dots, x_n), \\ & f(x_1, \dots, x_i, \dots, x_n) \wedge f(x_1, \dots, y_i, \dots, x_n) \geq f(x_1, \dots, x_i \wedge y_i, \dots, x_n). \end{aligned}$$

for all  $x_i, y_i \in A$  and all  $x_j \in C_f$  ( $j \neq i$ ) provided that  $f$  is of monotony type  $\uparrow$  in the  $i$ -th variable. For type  $\downarrow$  corresponding inequalities hold.

Partial order can be expressed in terms of a partial operation, call it  $\top$ , by defining  $a \top b$  ( $a, b \in A$ ) if and only if  $a \leq b$  and putting, in that case,  $a \top b = a$ . The axioms for  $\top$  are reformulations of axioms for partial order.

<sup>(5)</sup> Our notation is not consequent in one respect. Namely, we do not mention the axioms satisfied by the operations in  $F$ , but we do write out explicitly the axioms of monotony.

Partially ordered algebras can be also considered from another point of view which proved to be fruitful in the case of groups (see [3]). It consists in introducing not everywhere defined binary operations, the intersection  $\wedge$  and the union  $\vee$ , by the following axioms:

1° for each  $a \in A$ ,  $a \wedge a$  and  $a \vee a$  are defined and satisfy

$$a \wedge a = a, \quad a \vee a = a;$$

2° for all  $a, b \in A$ , if  $a \wedge b$  exists, then so does  $b \wedge a$ , and

$$b \wedge a = a \wedge b,$$

and dually for the union  $\vee$ ;

3° for all  $a, b, c \in A$  we have

$$(a \wedge b) \wedge c = a \wedge (b \wedge c),$$

whenever all intersections occurring here exist, and dually for the union  $\vee$ ;

4° for all  $a, b \in A$ , if  $a \vee b$  exists, then so does  $a \wedge (a \vee b)$ , and

$$a \wedge (a \vee b) = a,$$

and dually.

The transition from  $\leq$  to the partial operations  $\wedge, \vee$  is wholly obvious. In the opposite direction one defines  $a \leq b$  if and only if  $a \wedge b = a$ . This is equivalent to  $a \vee b = b$ , for  $a \wedge b = a$  implies, by the dual part of 4°, that  $b \vee (a \wedge b) = b \vee a$  exists and is equal to  $b$ ; and conversely.

In terms of  $\top$  or  $\wedge, \vee$  partially ordered algebras are regarded as partial algebras (i. e. algebras where operations need not be defined for all  $n$ -tuples). In spite of the very close connection between the two points of view, they are not equivalent if the corresponding categories are considered. In fact, in the first case the morphisms are those homomorphisms of an algebra  $(A; F)$  into another one  $(B; F)$  which preserve order relations, while in the second case it is also required that these homomorphisms preserve intersections and unions whenever they exist — which is a more restrictive condition<sup>(6)</sup>. In what follows we shall confine our attention to the first category, but it is to be kept in mind that only the second point of view makes it possible to obtain results yielding those on lattice-ordered algebras as special cases.

In accordance with our confinement we are going to define the concepts of subalgebras and homomorphisms.

Let  $(A; F, \leq, \langle C_f : \gamma_1^f, \dots, \gamma_n^f \rangle_{f \in F})$  be a partially ordered algebra. If  $(B; F)$  is a subalgebra of  $(A; F)$  (in the pure algebraic sense), then

<sup>(6)</sup> In fact, intersection  $\wedge$  can be regarded as an extension of the operation  $\top$  considered above.

the restriction of  $\leq$  to  $B$  turns  $(B; F)$  into a partially ordered algebra with  $C_f \cap B$  as the monotony domain of  $f \in F$ ; the monotony type  $(\gamma_1^f, \dots, \gamma_n^f)$  of  $f$  in  $C_f \cap B$  is the same as in  $C_f$ . The algebra

$$(B; F, \leq, \langle C_f \cap B; \gamma_1^f, \dots, \gamma_n^f \rangle_{f \in F})$$

is called a *subalgebra* of  $(A; F, \leq)$  <sup>(7)</sup>.

In natural partially ordered algebras there is another way of defining the notion of subalgebra. If  $(A; F, \leq, \langle \Gamma_f; \gamma_1^f, \dots, \gamma_n^f \rangle_{f \in F})$  is a natural partially ordered algebra, then the monotony domain of an operation  $f \in F$  in  $(B; F, \leq)$  can be defined by the same formula, with the variables restricted to  $B$  and preserving the monotony types of  $f$ . We shall call

$$\langle B; F, \leq, \langle \Gamma_f; \gamma_1^f, \dots, \gamma_n^f \rangle_{f \in F} \rangle$$

a *natural subalgebra* of the given partially ordered algebra, provided it is a partially ordered algebra. If in a natural partially ordered algebra the subalgebras coincide with the natural subalgebras, then we shall call it *half-smooth*. It is evident that if every monotony domain is the whole  $A$  or if the formula  $\Gamma_f(x)$  contains no individual variables except  $x$ , then the partially ordered algebra is half-smooth. Thus partially ordered groups and semigroups are half-smooth, and so are partially ordered rings.

Note that the property of being half-smooth is hereditary for subalgebras.

Next let  $(A; F, \leq, \langle C_f; \gamma_1^f, \dots, \gamma_n^f \rangle_{f \in F})$  and  $(B; F, \leq, \langle D_f; \gamma_1^f, \dots, \gamma_n^f \rangle_{f \in F})$  be partially ordered algebras of the same type indicated by using the same  $F$  to denote the families of fundamental operations and the same  $\gamma_1^f, \dots, \gamma_n^f$  for the monotony types of  $f$  both in  $A$  and in  $B$ . Then a map

$$(5) \quad \varphi : A \rightarrow B$$

is called an *o-homomorphism* (*o* for order), if

- 1)  $\varphi$  is an algebraic homomorphism of the algebra  $(A; F)$  into  $(B; F)$ ,
- 2)  $\varphi$  is order-preserving, i. e.  $a_1 \leq a_2$  in  $A$  implies  $\varphi(a_1) \leq \varphi(a_2)$  in  $B$ ,
- 3)  $\varphi$  maps  $C_f$  into  $D_f$  for each  $f \in F$ .

The third condition is automatically satisfied if the monotony domains exhaust  $A$  and  $B$ .

The map (5) is called an *o-monomorphism* if it is simultaneously a monomorphism of the algebra  $(A; F)$  into the algebra  $(B; F)$  and an *o-homomorphism*. An *o-epimorphism*  $\varphi$  is an *o-homomorphism* such that

<sup>(7)</sup> If we would have assumed that the monotony domains could never be empty, then in the definition of subalgebras we ought to have assume that  $B \cap C_f$  is not void.

- a) it is an epimorphism (i. e. it is surjective),
- b)  $b_1 \leq b_2$  in  $B$  implies that some preimages  $a_j$  of  $b_j$  satisfy  $a_1 \leq a_2$  in  $A$ ,
- c)  $\varphi$  maps  $C_f$  onto  $D_f$  for each  $f \in F$ .

Finally, an *o-isomorphism* is a one-to-one mapping which is an *o*-homomorphism in both directions. Observe that an *o*-monomorphism need not induce an *o*-isomorphism with the image.

The definition of a congruence relation  $\theta$  of a partially ordered algebra (2) can be given if the congruence classes mod  $\theta$  again form a partially ordered algebra under the definition

- (\*)  $\theta(a) \leq \theta(b)$  if and only if  $a_0 \leq b_0$  holds for some  $a_0 \in \theta(a)$  and some  $b_0 \in \theta(b)$ ,

where  $\theta(a)$  denotes the class mod  $\theta$  that contains  $a$ .

In general, the classes mod  $\theta$ , where  $\theta$  is an algebraic congruence relation of  $(A; F)$ , do not form a partially ordered factor algebra  $(A; F, \leq)/\theta$ , not even if the classes mod  $\theta$  are convex subsets of  $A$ . In fact, under  $\theta$ , neither antisymmetry nor transitivity is necessarily satisfied. Nevertheless, it is possible to show that under some special hypotheses on  $\theta$  or on  $(A; F: \leq)$ ,  $(A; F: \leq)/\theta$  will be again a partially ordered algebra. In such a case the monotony domain of  $f$  in  $(A; F, \leq)/\theta$  is defined to consist of the classes containing at least one element from the monotony domain  $C_f$  of  $f$  in  $A$ , and the monotony type of  $f$  in the factor algebra is the same as previously. The arising algebra of classes mod  $\theta$  is called the *factor algebra* of (2) mod  $\theta$ .

In the case of natural partially ordered algebras  $(A; F, \leq, \langle \Gamma_f; \gamma_1^f, \dots, \gamma_n^f \rangle_{f \in F})$  there is yet another way of introducing the monotony domains in the factor algebra, the one using the same sentence  $\Gamma_f$  but with variables ranging over the elements of the factor. If we prefer to do so, then we get the definition of a *natural factor algebra*, provided it is actually a partially ordered algebra under the induced ordering.

Let us call a natural partially ordered algebra *smooth* if it is half-smooth and if its factor algebras are the same as its natural factor algebras. Clearly, a partially ordered algebra  $A$  is necessarily smooth if  $C_f$  coincides with  $A$  for every operation  $f$ ; and partially ordered rings are smooth too. It would be of interest to find conditions under which natural partially ordered algebras are smooth (**P 516**).

We have the *o*-epimorphism theorem:

**THEOREM 1.** *If  $\varphi$  is an *o*-epimorphism of the partially ordered algebra  $(A; F, \leq, \langle C_f; \gamma_1^f, \dots, \gamma_n^f \rangle_{f \in F})$  onto the partially ordered algebra  $(B; F, \leq, \langle D_f; \gamma_1^f, \dots, \gamma_n^f \rangle_{f \in F})$ , then the relation  $\theta$  defined by*

$$a_1 \equiv a_2(\theta) \text{ if and only if } \varphi(a_1) = \varphi(a_2)$$



is a congruence relation of  $(A; F, \leq)$ , and

$$\theta(a) \rightarrow \varphi(a), \quad a \in A,$$

is an  $\theta$ -isomorphism of  $(A; F, \leq)/\theta$  with  $(B; F, \leq)$ .

The proof is straightforward and may be left to the reader.

**2. Products of algebras.** We are turning now our attention to different kinds of direct products of partially ordered algebras.

Given a set of partially ordered algebras of the same type,

$$(A_\lambda; F, \leq, \langle C_f^\lambda; \gamma_1^f, \dots, \gamma_n^f \rangle_{f \in F}), \quad \lambda \in A,$$

we can form their *cartesian product*  $\Pi(A_\lambda; F, \leq)$  in the following rather obvious way. It is an algebra  $(A; F, \leq)$ , where  $A$  is the cartesian product of the underlying sets  $A_\lambda$  ( $\lambda \in A$ ), the operations  $f$  are to be performed componentwise and for any two elements of  $A$  one puts

$$\langle \dots, a_\lambda, \dots \rangle \leq \langle \dots, b_\lambda, \dots \rangle$$

if and only if, for each  $\lambda \in A$ ,  $a_\lambda \leq b_\lambda$  holds in  $A_\lambda$ . The cartesian product of the monotony domains  $C_f^\lambda$  of  $f$  in the  $A_\lambda$  is taken as the monotony domain  $C_f$  of  $f$  in  $A$ , while the monotony type of  $f$  in  $A$  is its common monotony type in the components. It is easy to check that  $(A; F, \leq, \langle C_f; \gamma_1^f, \dots, \gamma_n^f \rangle_{f \in F})$ , defined in this way, is again a partially ordered algebra. Moreover, it is natural if so are the components, where the sentence  $\Gamma_f$  is the same for every  $A_\lambda$  and for  $A$ , and  $\Gamma_f$  contains no disjunction.

The projection of  $(A; F, \leq)$  onto  $(A_\lambda; F, \leq)$  is evidently an  $\theta$ -epimorphism for every  $\lambda$  provided none of  $C_f^\lambda$  is empty.

Manifestly, if each  $(A_\lambda; F, \leq)$  is lattice-ordered, then so is their cartesian product.

Having introduced the concept of the cartesian product and subalgebra, it is evident what should be meant by *subdirect products*.

An important special case is formed by algebras possessing a nullary operation  $e$  (built up in terms of the fundamental operations  $f \in F$ ) such that  $e$  is a *subalgebra* (and hence every subalgebra contains  $e$ ). It is possible then to introduce the *discrete direct product* of algebras by requiring that all but a finite number of components should be equal to  $e$ . Using the same definitions for ordering and monotony domains it is obvious that the *discrete direct product of partially ordered algebras is a subalgebra of their cartesian product*. The components  $(A_\lambda; F, \leq, \langle C_f^\lambda; \gamma_1^f, \dots, \gamma_n^f \rangle_{f \in F})$  will be — under the natural correspondence —  $\theta$ -isomorphic to subalgebras of the discrete direct product whenever  $e$  belongs to  $C_f^\lambda$  for every  $f \in F$ .

We can also speak of inner direct products if we make the following definition.

A partially ordered algebra  $(A; F, \leq, \langle C_f; \gamma_1^f, \dots, \gamma_n^f \rangle_{f \in F})$  is an *inner direct product* of its subalgebras  $(A_\lambda; F, \leq, \langle C_f \cap A_\lambda; \gamma_1^f, \dots, \gamma_n^f \rangle_{f \in F})$  if there is an  $\phi$ -isomorphism  $\varphi$  of  $(A; F, \leq)$  with the discrete direct product of the  $(A_\lambda; F, \leq)$  such that, for each  $\lambda$ ,  $\varphi$  maps  $a_\lambda$  ( $\in A_\lambda$ ) on the vector with  $a_\lambda$  as  $\lambda$ -th component and  $e$  elsewhere. It is easy to check that this definition implies that either the element  $e$  must be contained in each  $C_f^\lambda$  or all  $C_f^\lambda$  (for some  $f$ ) are void. Also, the  $A_\lambda$  are necessarily convex subalgebras<sup>(8)</sup>.

The definition of inner direct product has the disadvantage that it needs not be unique and it does not tell the precise way how to obtain the vector with the  $\lambda$ -th component  $a_\lambda$  and with the  $\mu$ -th component  $a_\mu$  from the vectors with the  $\lambda$ -th component  $a_\lambda$  and with the  $\mu$ -th component  $a_\mu$ , respectively, where their remaining components are equal to  $e$ . If we wish to have an operation doing that, then we have to assume the existence of an operation  $g$  on  $A$  for which  $e$  is the neutral element; and in order that the usual properties of direct products should be preserved, we also have to assume that  $g$  is a binary associative operation. Accordingly, we introduce the following definition.

A partially ordered algebra  $(A; F, \leq)$  is called the *direct union* of its subalgebras  $(A_\lambda; F, \leq)$ ,  $\lambda \in \Lambda$ , if

(i) in terms of the operations in  $F$ , a nullary operation  $e$  and a binary operation  $g$  can be defined in such a way that  $A$  is a partially ordered semigroup under  $g$  with identity  $e$ ;

(ii)  $A$ , as a partially ordered semigroup, is the discrete direct product of its partially ordered subsemigroups  $A_\lambda$  with  $e$ ;

(iii)  $e$  is a subalgebra of  $A$ ;

(iv) if  $A_1, A_2$  are different and if  $a'_i \in A_1, a''_i \in A_2$ , then

$$f(\dots, g(a'_i, a''_i), \dots) = g(f(\dots, a'_i, \dots), f(\dots, a''_i, \dots))$$

for every  $n$ -ary operation  $f \in F$ ;

(v)  $g(a_1, \dots, a_k)$  belongs to the monotony domain  $C_f$ , where  $a_i \in A$  and  $A_1, \dots, A_k$  are different if and only if each  $a_i \in C_f$ .

Since the operation  $g$  has a distinguished role in the definition, it is better to call, what we have defined under (i)-(v), a  *$g$ -direct union*. Note that in view of (ii) the  $A_\lambda$  are convex subalgebras of  $A$ .

Using theorems due to Šimbireva [8] and Šik [7], it is easy to conclude:

**THEOREM 2.** *Let  $(A; F, \leq)$  be a partially ordered algebra and  $g$  a binary operation in  $A$  under which  $A$  is a partially ordered group. If  $A$  is directed, then*

<sup>(8)</sup> It is clear that inner cartesian products can also be defined.



1. any two  $g$ -direct union decompositions of  $(A; F, \leq)$  have a common refinement;

2. the  $g$ -direct factors of  $(A; F, \leq)$  form a Boolean algebra.

The proof is the same as for groups, only the fulfilment of conditions (iv) and (v) must be verified, but this is straightforward.

Assume again that the algebras to be considered have a nullary operation  $e$  such that  $e$  is a subalgebra. Then lexicographic products over fully ordered index sets also make sense. Let  $A$  be a fully ordered set and  $(A_\lambda; F, \leq, \langle C_f^\lambda; \gamma_1^f, \dots, \gamma_n^f \rangle_{f \in F})$ ,  $\lambda \in A$ , a set of partially ordered algebras of the same type. Let  $\Gamma A_\lambda$  denote the subset of the cartesian product of the sets  $A_\lambda$  which consists of all vectors  $\langle \dots, a_\lambda, \dots \rangle$  whose support  $\{\lambda \in A \mid a_\lambda \neq e\}$  is well-ordered in the ordering of  $A$ , and define the *lexicographic product*  $\Gamma(A_\lambda; F, \leq)$  of the  $(A_\lambda; F, \leq)$  with  $\lambda \in A$  as the algebra  $(\Gamma A_\lambda; F, \leq, \langle C_f; \gamma_1^f, \dots, \gamma_n^f \rangle_{f \in F})$  with componentwise operations and by putting  $\langle \dots, a_\lambda, \dots \rangle < \langle \dots, b_\lambda, \dots \rangle$  if and only if  $a_\lambda < b_\lambda$  for the first  $\lambda$  for which  $a_\lambda \neq b_\lambda$ . We let the monotony domain  $C_f$  of  $f \in F$  consist of all  $\langle \dots, a_\lambda, \dots \rangle$  for which every  $a_\lambda$ , with  $\lambda \in A$ , belongs to the monotony domain of  $f$  in  $A_\lambda$ . It is then a matter of routine to check: if the operations  $f \in F$  obey the strict monotony law, then  $\Gamma(A_\lambda; F, \leq)$  will be a partially ordered algebra containing  $(A_\lambda; F, \leq)$  as subalgebras<sup>(9)</sup>.

Note that the hypothesis on the strictness of the monotony laws is essential as shown by the rings. Also, if we have natural partially ordered algebras to start with, then we don't get in general a natural partially ordered algebra as lexicographic product, but we do if all the monotony domains coincide with the whole underlying set.

Obviously, the lexicographic product is fully ordered if all the components are fully ordered. Under additional assumptions it is possible to establish a refinement theorem on directed algebras (cf. [2], p. 26).

**3.  $O$ -algebras.** Given an algebra  $(A; F)$ , in several cases it is desirable to introduce in  $A$  an order relation  $\leq$  under which it is a fully ordered algebra. If nothing is presupposed about the monotony domains of the operations in  $F$ , then the problem of introducing  $\leq$  in  $A$  does not make much sense in view of the fact that every set can be fully ordered. Therefore, in order to have a meaningful problem (reducing to the corresponding problems on groups and rings as special cases), we have to restrict ourselves to algebras with a natural definition of monotony domains. More explicitly, we assume that together with  $(A; F)$  we are also given

<sup>(9)</sup> If we start with a well-ordered set  $A$ , then we can get rid of the hypothesis on the existence of  $e$  in  $A$ . In this case monotony domains must be defined as for cartesian products.

1. the defining formulas  $I_f$  of the monotony domains  $C_f$  for each  $f \in F$ , expressed in terms of operations in  $F$ , order relation (to be defined), element variables in  $A$ , and logical operations in the manner described in section 1;

2. the monotony types of  $f$  in  $C_f$ .

Thus we are given  $(A; F, \langle I_f; \gamma_1^f, \dots, \gamma_n^f \rangle_{f \in F})$  and we proceed to consider partial or linear orders in  $(A; F)$  such that  $(A; F, \leq, \langle I_f; \gamma_1^f, \dots, \gamma_n^f \rangle_{f \in F})$  is a partially or linearly ordered algebra. E. g. in the case of rings  $A$  the monotony domains for all the operations are the whole  $A$  except for multiplication whose monotony domain is defined by the formula  $(x \geq 0)$ ; the monotony types of the operations are enumerated in the axioms of partially ordered rings.

An algebra  $(A; F)$  with given  $\langle I_f; \gamma_1^f, \dots, \gamma_n^f \rangle$  for all  $f \in F$  is called an *O-algebra*, if it admits a linear order  $\leq$  such that  $(A; F, \leq, \langle I_f; \gamma_1^f, \dots, \gamma_n^f \rangle_{f \in F})$  is a fully ordered algebra. It is called an *O\*-algebra* if every partial order on  $A$  satisfying 1 and 2 above can be extended to a linear order again with 1 and 2.

In the class of semigroups, groups or rings, the property of being an *O-algebra* depends on the finitely generated subalgebras. Which other classes of algebras have this property? For obvious reasons, it seems necessary to restrict ourselves to natural algebras  $(A; F, \langle I_f; \gamma_1^f, \dots, \gamma_n^f \rangle_{f \in F})$  which are half-smooth in every partial order. Under this hypothesis we have:

**THEOREM 3.** *Let  $(A; F, \langle I_f; \gamma_1^f, \dots, \gamma_n^f \rangle_{f \in F})$  be an open sentence algebra <sup>(10)</sup> half-smooth in every partial order. Then it is an *O-algebra* if and only if all of its finitely generated subalgebras are *O-algebras*.*

Recall that an open sentence algebra is an algebra in which the postulates on the operations are all open sentences in the sense of McKinsey [5]; i. e. they consist of individual variables in  $A$ , operations in  $F$ , equality, negation, disjunction, conjunction, and implication, without any quantifiers, but interpreted as if they were preceded by universal quantifiers binding all individual variables <sup>(11)</sup>. Then the statement of Theorem 3 follows at once from a more general result by B. H. Neumann [6].

It seems to be a rather hard problem to describe those classes of algebras for which the *O-algebras* are necessarily *O\*-algebras*. (Remember that the abelian groups form such a class.)

Let  $(A; F)$  be an algebra with given  $\langle I_f; \gamma_1^f, \dots, \gamma_n^f \rangle$  for all  $f \in F$ . If it is an *O-algebra*, then — under the assumption of half-smoothness — so are all its subalgebras, but the analogous statement for homomorphic images does not hold in general, as shown e. g. by groups. The

<sup>(10)</sup> Thus we also assume that  $I_f(x)$  is an open sentence.

<sup>(11)</sup> The order relation is then interpreted as a partial operation.

cartesian product of  $O$ -algebras of the same type need not be an  $O$ -algebra (cf. rings), but we can prove the following general result:

**THEOREM 4.** *Let  $(A_\lambda; F)$  be a set of  $O$ -algebras with the same monotony types of operations in  $F$  such that*

(i) *for each  $f \in F$ , the monotony domain of  $f$  coincides with  $A_\lambda$  for every  $\lambda$ ;*

(ii) *every  $n$ -ary operation ( $n \geq 1$ ) in  $F$  obeys the strict monotony law.*

*Then the cartesian product of the  $(A_\lambda; F)$  is again an  $O$ -algebra, and so are their subdirect products.*

By virtue of (i) we need only to show the statement for the cartesian product. Take arbitrary but fixed full orders on the  $A_\lambda$  and well-order the index set  $\Lambda$  in some way. Then the lexicographic product of the  $(A_\lambda; F, \leq)$  will be fully ordered and the monotony laws hold with the whole of  $\prod A_\lambda$  as monotony domain.

Much more difficult problem arises if we consider a class  $\mathcal{A}$  of algebras of the same type and ask under what circumstances a certain subclass contains only  $O$ -algebras. For instance, assume that  $\mathcal{A}$  contains enough free algebras or  $\mathcal{A}$  is a class in which free products exist. It would be of interest to know conditions guaranteeing that free algebras in  $\mathcal{A}$  are  $O$ -algebras or that free products of  $O$ -algebras are again  $O$ -algebras (both are true for the class of groups and for the class of abelian groups). Of course we suppose that 1 and 2 are prescribed. **(P 517)**

As to  $O^*$ -algebras let us observe that even for the class of  $O^*$ -groups it is not known whether or not this class is closed under passage to subgroups. **(P 518)**

**4. Varieties.** Next we wish to consider varieties of partially ordered algebras. We may evidently restrict ourselves *ab initio* to the case when the class of underlying algebras is a variety. Also, since partial order must be expressed in terms of everywhere defined operations (which are evidently  $\vee$  and  $\wedge$ ), the varieties have to be defined in terms of certain operations  $f \in F$  together with  $\vee$  and (or)  $\wedge$ . But also the monotony laws ought to be expressed by identically valid equations. Hence the monotony domains are natural and, moreover, they can be defined by a formula like (3) in section 1, where  $\Gamma_f(x)$  contains individual variables  $x, x_1, \dots, x_k$  in  $A$ , operations from  $F$ , the operations  $\vee$  and  $\wedge$ , equality and conjunction, and if they are substituted in the monotony laws, then the formula itself is preceded by universal quantifiers binding all  $x_1, \dots, x_k$ . For instance, in lattice-ordered rings the monotony domain for multiplication is defined by the formula  $(x \vee 0)$ . Furthermore, even the monotony laws ought to be written in form of identities. In so-called function rings e. g. the monotony laws for multiplication are:  $\forall x, y, z$

$$(y \vee z)(x \vee 0) = y(x \vee 0) \vee z(x \vee 0), \quad (x \vee 0)(y \vee z) = (x \vee 0)y \vee (x \vee 0)z.$$

In general, the monotony laws are of the form (4) in section 1 with  $\leq$  replaced by  $=$ , where the condition that certain elements should belong to  $C_f$  is replaced by substituting these elements by their defining formulas  $\Gamma_f$ . For the sake of brevity we shall call monotony laws of this kind *identical monotony laws*.

We are particularly interested in the variety generated by the fully ordered algebras in a certain variety  $\mathcal{V}$  with a common family  $F$  of fundamental operations. We add  $\vee$  and  $\wedge$  to  $F$  and express monotony laws as identical monotony laws in the subclass of fully ordered algebras in  $\mathcal{V}$ .

**THEOREM 5.** *Let  $\mathcal{V}$  be a variety of algebras with fundamental operations  $f \in F$  and with prescribed identical monotony laws in terms of  $f \in F$  and  $\vee, \wedge$ . Let  $f\mathcal{V}$  denote the class of the fully ordered algebras in  $\mathcal{V}$  with the given monotony laws. If the algebras in  $f\mathcal{V}$  are smooth, then the smallest variety containing  $f\mathcal{V}$  with the operations in  $F$  and  $\vee, \wedge$  consists of subdirect products of algebras in  $f\mathcal{V}$ .*

In order to prove the assertion of the theorem, let us begin with observing that by a well-known result of G. Birkhoff and a remark of A. Tarski, the variety in question consists of all homomorphic images of subalgebras of cartesian products of algebras in  $f\mathcal{V}$ . Note that all the algebras in the arising variety are lattices, and therefore their lattices of congruence relations (as sublattices of the distributive lattices of all congruence relations of lattices) are necessarily distributive. A recent result of Jónsson [4] states that if all the algebras of the arising variety have distributive lattices of congruence relations, then the variety consists of subalgebras of cartesian products of homomorphic images of ultraproducts (in the sense of J. Łoś) of algebras in the generating class. In our present case the ultraproducts and homomorphic images of algebras in  $f\mathcal{V}$  are again in  $f\mathcal{V}$ , hence by making use of Jónsson's theorem we conclude that the variety in question consists of subalgebras of cartesian products of algebras in  $f\mathcal{V}$ . Moreover,  $f\mathcal{V}$  being closed with respect to taking subalgebras, subalgebras of cartesian products are nothing else than subdirect products. This completes the proof.

Note that the variety generated by  $f\mathcal{V}$  can be defined by equations identically valid in  $\mathcal{V}$ , identical monotony laws and possibly additional equations valid identically in each member of  $f\mathcal{V}$ . For instance, in the case of groups we have the single additional equation

$$(x \vee e) \wedge y^{-1}(x^{-1} \vee e)y = e,$$

while in the case of rings we have

$$(x \vee 0) \wedge (y \vee 0)(-x \vee 0) = 0 \quad \text{and} \quad (x \vee 0) \vee (-x \vee 0)(y \vee 0) = 0$$

for all  $x, y$ . In general algebras the author has been unable to find identically valid equations giving a necessary and sufficient condition for an algebra to be representable as a subdirect product of fully ordered algebras. (P 519)

**5. Completion of partially ordered algebras.** There are several methods of completing partially ordered algebras. First we consider the Dedekind-MacNeille completion process.

We start with a partially ordered algebra  $A$ . Consider its non-vacuous  $u$ -bounded subsets  $X$  <sup>(12)</sup>. The correspondence

$$(6) \quad X \rightarrow X^\# = L(U(X))$$

is a closure operation. Assuming  $A$  directed, the set  $A^\#$  of all closed sets  $C = C^\#$  is a conditionally complete lattice under set inclusion. Here intersection is the same as intersection of sets, while union means the closure of set union. The correspondence

$$(7) \quad a \rightarrow a^\# = L(a)$$

is an isotone embedding of the partially ordered set  $A$  in  $A^\#$ , preserving  $\vee$  and  $\wedge$ , whenever they exist.

Let  $f$  be an  $n$ -ary operation whose monotony domain is the whole  $A$ . If  $f$  is isotone in each of its variables, then for  $X_1, \dots, X_n \in A^\#$  we define

$$(8) \quad f^\#(X_1, \dots, X_n) = \{\bigcup f(x_1, \dots, x_n) \text{ for all } x_i \in X_i\}^\#,$$

and we replace  $X_i$  by  $U(X_i)$  in the right-hand member of (8) if  $f$  happens to be antitone in the variable  $x_i$ . (Note that there is a Galois correspondence between  $A^\#$  and the set of all  $U(X)$  with  $X \in A^\#$ .) In this way, the operations  $f$  with the whole of  $A$  as monotony domain can be extended to  $A^\#$ ; in fact, the correspondence (7) preserves these  $f$ 's.

However, the identities between operations of  $A$  carry over only exceptionally to  $A^\#$ . In order to obtain a sufficient condition for an identity to carry over to  $A^\#$ , let us extend the definition of  $f^\#$  for all subsets  $X_i$  of  $A$  bounded from above by the formula (8). In general we have the obvious inclusion

$$f^\#(X_1, \dots, X_n) \subseteq f^\#(X_1^\#, \dots, X_n^\#).$$

If here equality holds true (and if  $f$  is isotone in each variable), we shall call  $f$ , for the sake of brevity, a *coherent* operation. We then have

**THEOREM 6.** *Let  $(A; F, \leq)$  be a partially ordered algebra and  $A^\#$  its Dedekind-MacNeille completion. If*

$$\varphi(x_1, \dots, x_k) = \psi(x_1, \dots, x_k) \quad (x_i \in A)$$

<sup>(12)</sup>  $u$ -bounded means bounded from above.  $U(X)$  and  $L(X)$  are the sets of all upper and lower bounds for  $X$  in  $A$ .

is an identity in  $A$  containing only coherent operations, then the corresponding identity in  $A^\#$

$$\varphi^\#(X_1, \dots, X_k) = \psi^\#(X_1, \dots, X_k) \quad (X_i \in A^\#)$$

is also valid.

The proof is straightforward by writing out  $\varphi$  and  $\psi$  explicitly.

In the theory of partially ordered groups it is shown that multiplication is coherent in the above sense. Hence, by the theorem, associativity is preserved in  $A^\#$ . But the inverse is not coherent in groups, and indeed a new condition must be imposed on a partially ordered group  $G$  to guarantee  $G^\#$  to be a group again. That is a rather restrictive condition.

Let us turn now our attention to completions with respect to order topology. The order topology is introduced into  $A$  in the usual fashion. Let  $A$  be a  $u$ -directed index set of fixed type, without greatest element. The set  $\{u_\alpha\}_{\alpha \in \Lambda}$  is isotone or antitone according to whether  $\alpha \leq \beta$  implies  $u_\alpha \leq u_\beta$  or  $u_\alpha \geq u_\beta$ . If  $\{u_\alpha\}$  is isotone and  $\{v_\alpha\}$  is antitone such that

$$U(\dots, u_\alpha, \dots) = U(a), \quad L(\dots, v_\alpha, \dots) = L(b)$$

for some  $a, b \in A$ , then we write  $u_\alpha \uparrow a$  and  $v_\alpha \downarrow b$ , and call  $a$  and  $b$  the  $o$ -limits of  $\{u_\alpha\}$  and  $\{v_\alpha\}$ , respectively. If to  $\{x_\alpha\}_{\alpha \in \Lambda}$  there exist an isotone set  $\{u_\alpha\}$  and an antitone set  $\{v_\alpha\}$  such that

$$u_\alpha \leq x_\alpha \leq v_\alpha \quad \text{with} \quad u_\alpha \uparrow a \quad \text{and} \quad v_\alpha \downarrow a,$$

then we write  $x_\alpha \rightarrow a$  and call  $a$  the  $o$ -limit of  $\{x_\alpha\}$ .  $\{x_\alpha\}$  is also said to  $o$ -converge to  $a$ . It is obvious that a constant set has an  $o$ -limit equal to its members, the  $o$ -limits — if exist — are uniquely determined, and cofinal subsets have the same  $o$ -limits.

The closed subsets  $B$  of  $A$  are defined by the property of containing the  $o$ -limits of  $o$ -convergent subsets whose elements belong to  $B$ . Then  $A$  becomes a  $T_1$ -space relatively to this topology.

In order to discover the continuity properties of the operations  $f \in F$ , we verify the multiple limit property:

LEMMA 1. Let  $(A; F, \leq)$  be a partially ordered algebra and  $f \in F$  an  $n$ -ary operation such that for each  $i$ ,

$$(9) \quad \begin{aligned} &\text{if } x_i^a \rightarrow x_i, \text{ then for every fixed } x_j \ (j \neq i), \\ &f(x_1, \dots, x_i^a, \dots, x_n) \rightarrow f(x_1, \dots, x_i, \dots, x_n). \end{aligned}$$

Then  $x_i^a \rightarrow x_i$  for each  $i$  implies

$$f(x_1^a, \dots, x_n^a) \rightarrow f(x_1, \dots, x_n).$$



Assume that  $n > 1$ . We prove by induction on  $k$  that  $x_i^a \rightarrow x_i$  ( $i = 1, \dots, k$ ) implies  $f(x_1^a, \dots, x_k^a, x_{k+1}, \dots, x_n) \rightarrow f(x_1, \dots, x_k, \dots, x_n)$  for all  $x_{k+1}, \dots, x_n$ . If, for instance,  $f$  is isotone in its variables, and if  $u_i^a$  is an isotone set  $\rightarrow x_i$  such that  $u_i^a \leq x_i$ , then

$$\begin{aligned} & U_a(f(u_1^a, \dots, u_{k-1}^a, u_k^a, x_{k+1}, \dots, x_n)) \\ &= U_{a,\beta}(f(u_1^a, \dots, u_{k-1}^a, u_k^\beta, x_{k+1}, \dots, x_n)) \\ &= \bigcap_{\beta} U_a(f(u_1^a, \dots, u_{k-1}^a, u_k^\beta, x_{k+1}, \dots, x_n)) \\ &= \bigcap_{\beta} U(f(x_1, \dots, x_{k-1}, u_k^\beta, x_{k+1}, \dots, x_n)) \\ &= U(f(x_1, \dots, x_k, \dots, x_n)), \end{aligned}$$

where the index at  $U$  denotes the varying index over which the upper bounds are to be taken. In the other cases similar inference may be applied.

To simplify our considerations, let us restrict ourselves to ordinary sequences  $\{x_n\}$  of type  $\omega$ , and assume that to each  $a \in A$  there exist isotone and antitone sequences  $\{u_n\}$  and  $\{v_n\}$  with  $o$ -limit  $a$  such that  $u_n \neq a \neq v_n$ . In the general case the arguments are a bit more complicated, but no essentially new ideas are necessary.

We have to assume the following conditions:

(a) if every infinite subsequence of a sequence contains a subsequence  $o$ -converging to  $a$ , then the sequence  $o$ -converges to  $a$ ;

(b) if  $x^{n,m}$  is a double sequence such that  $x^{n,m} \rightarrow x^n$  for varying  $m$ , and if  $x^n \rightarrow x$ , then there exists a subsequence of  $x^{n,m}$  which  $o$ -converges to  $x$ .

Then we have the pure topological conclusion<sup>(13)</sup>:

LEMMA 2. Under (a) and (b), the  $o$ -convergence is equivalent to the convergence in the sense of order topology.

Now we have

THEOREM 7. If  $(A; F, \leq)$  is a partially ordered algebra in which order convergence satisfies (a) and (b), and if the operations  $f \in F$  satisfy condition (9) of Lemma 1, then  $(A; F)$  is a topological algebra in the order topology.

We have to verify the continuity of the operations  $f \in F$ . Let  $f(x_1, \dots, x_n) = x$  and let  $\mathcal{V}_m(x_i)$  ( $m = 1, 2, \dots$ ) be systems of neighbourhoods around  $x_i$ , and, say,  $\mathcal{V}_m(x_i) \supseteq \mathcal{V}_{m+1}(x_i)$ . If, given a neighbourhood  $\mathcal{V}$  of  $x$ , no  $m$  satisfies  $f(\mathcal{V}_m(x_1), \dots, \mathcal{V}_m(x_n)) \subset \mathcal{V}$ , then there exist elements  $x_i^m \in \mathcal{V}_m(x_i)$  such that  $f(x_1^m, \dots, x_n^m) \notin \mathcal{V}$ . By Lemma 2,

<sup>(13)</sup> Cf. [2], p. 31. It follows from our hypothesis on  $A$  that no element in  $A$  is isolated and that every element has a countable system of neighbourhoods.

$x_i^m \rightarrow x_i$ , and by Lemma 1,  $f(x_1^m, \dots, x_n^m) \rightarrow f(x_1, \dots, x_n)$  as  $m \rightarrow \infty$ . Since  $A \setminus \mathcal{V}$  is closed,  $f(x_1, \dots, x_n) \in A \setminus \mathcal{V}$ , a contradiction. This completes the proof.

Note that condition (9) of Lemma 1 can be weakened by stipulating the same condition only for isotone or for antitone sequences. Then the operations  $f$  will not prove to be continuous in general, but the analogue of Lemma 1 will still hold.

Having turned our partially ordered algebra  $(A; F, \leq)$  into a topological algebra, we can try to define a uniform structure on it. This is not always possible, but we may restrict ourselves to the case when there is a binary operation  $g$  on  $A$  which can be used to introduce uniformity (like subtraction is used in abelian groups). Then we are confronted with the hard problem of finding conditions under which the completion becomes a partially ordered algebra of the same type satisfying the same postulates as  $(A; F, \leq)$  (**P 520**). Here we do not intend to enter the discussion of this problem.

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