COLLOQUIUM MATHEMATICUM

XIV WARSZAWA—WROCŁAW 1966

*A THEORY OF P-HOMOMORPHISMS**

BY

JÓZEF SŁOMIŃSKI (TORUŃ)

The notion of homomorphisms between two algebras or quasi-algebras is usually defined in the case they are similar. In this paper we introduce a more general notion of a P-homomorphism which does not require the mentioned similarity but still has many properties of an ordinary homomorphism. In particular, we obtain the existence theorems on free $\mathfrak{B}$-P-homomorphisms, free $\mathfrak{B}$-P-bilinears, $\mathfrak{B}$-P-direct sums and $\mathfrak{B}$-P-tensor products of quasi-algebras, where $\mathfrak{B}$ is an arbitrary quasi-primitive class of quasi-algebras. We also consider the notion of independence with respect to P-homomorphisms, i. e. a P-independence, and we obtain some results similar to that of Marczewski [5] and Schmidt [6]. The results of § 3, section E, are generalizations of my paper [10]. The paragraphs 1 and 2 may be considered as an introduction to the theory of quasi-algebras. The notion of a P-homomorphism is related to that of a P-mapping due to Fujiwara [3].

§ 1. Partial operations. Let A be an arbitrary set and let k be an ordinal number. A *k-ary partial operation* (or a partial operation of the type k) in the set A is any mapping $f: D \subseteq A^k \to A$, where the domain D is a set of some sequences $(a_\xi, \xi < k)$ in A of the type k. The value of partial operation f for a sequence $(a_\xi, \xi < k)$ — if it exists — will be denoted by $f(a_\xi, \xi < k)$ or briefly by $f(a_\xi)$. A k-ary partial operation f in the set A defined over the whole set A^k of all sequences $(a_\xi, \xi < k)$ with $a_\xi \in A$ for $\xi < k$ is called a *k-ary operation in the set A*. The partial operations of the same type are called *similar*. A subset $B \subseteq A$ is called *closed with respect to a k-ary partial operation f in the set A* provided that for all sequences $(b_\xi, \xi < k) \in B^k$, if f is defined for $(b_\xi, \xi < k)$, then the value $f(b_\xi, \xi < k)$ belongs to B.

A. Ordinary homomorphisms of similar partial operations. A mapping h of a set A into (onto) a set B is said to be a *homomorphism of a k-ary*

* Presented to the Conference on General Algebra, held in Warsaw, September 7-11 1964.

partial operation f in the set A into (onto) a k-ary partial operation f' in the set B provided that for all sequences $(a_\xi, \xi < k)$ belonging to A^k, if f is defined for $(a_\xi, \xi < k)$, then f' is defined for $\big(h(a_\xi), \xi < k\big)$ and, moreover, that

$$h\big(f(a_\xi, \xi < k)\big) = f'\big(h(a_\xi), \xi < k\big). \tag{1}$$

A homomorphism h of a partial operation f in a set A into a partial operation f' in a set B is called *strong* provided that for all sequences $(a_\xi, \xi < k) \in A^k$, if f' is defined for $\big(h(a_\xi), \xi < k\big)$, then there exist elements $a'_\xi \in A$, $\xi < k$ such that $h(a'_\xi) = h(a_\xi)$ for $\xi < k$ and f is defined for $(a'_\xi, \xi < k)$.

The homomorphisms between operations are always strong. The one-to-one homomorphisms are isomorphisms. An isomorphism of partial operation f onto a partial operation f' is strong if and only if i^{-1} is an isomorphism of f' onto f.

B. Direct products of partial operations. Let T be any set and let f_t be a k-ary partial operation in a set A_t for $t \in T$. Let us denote by $A = \mathop{\mathrm{P}}_{t \in T} A_t$ the *cartesian product* of all sets A_t, $t \in T$, i. e. the set of all mappings $\varphi : T \to \bigcup_{t \in T} A_t$ with $\varphi(t) \in A_t$ for $t \in T$. The *direct product of partial operations* f_t, $t \in T$, is a k-ary partial operation f in the set A such that, for all sequences $(\varphi_\xi, \xi < k) \in A^k$, f is defined for $(\varphi_\xi, \xi < k)$ if and only if f_t is defined for $(\varphi_\xi(t), \xi < k)$ for all $t \in T$ and, moreover, that we have

$$f(\varphi_\xi, \xi < k) = \varphi, \quad \text{where} \quad \varphi(t) = f_t\big(\varphi_\xi(t), \xi < k\big) \quad \text{for all } t \in T. \tag{2}$$

Hence we obtain for all $t \in T$:

$$p_t\big(f(\varphi, \xi < k)\big) = f_t\big(p_t(\varphi_\xi), \xi < k\big) \quad \text{for all} \quad t \in T, \tag{3}$$

where p_t is the *natural projection* of A onto A_t, i. e. $p_t(\varphi) = \varphi(t)$ for all $\varphi \in A$. By (3) the projection p_t is a homomorphism of f onto f_t. The direct product of operations is an operation.

C. Direct sums of partial operations. Let T be any set and let f_t be a k-ary partial operation in a set A_t for $t \in T$. Let $A = \mathop{\mathrm{S}}_{t \in T} A_t$ be the *direct sum of sets* A_t, $t \in T$, i. e. $A = \{(t, a) : t \in T, a \in A_t\}$ is the set of all pairs (t, a) with $t \in T$ and $a \in A_t$. The *direct sum of partial operations* f_t, $t \in T$, is a k-ary partial operation f in the set A such that for all sequences (t_ξ, a_ξ), $\xi < k$, f is defined for this sequence if and only if there exists an element $t_0 \in T$ such that $t_\xi = t_0$ for all $\xi < k$ and f_{t_0} is defined for $(a_\xi, \xi < k)$, and, moreover, that

$$f\big((t_\xi, a_\xi), \xi < k\big) = \big(t_0, f_{t_0}(a_\xi, \xi < k)\big).$$

The mapping $i_t: A_t \to A$ with $i_t(a) = (t, a)$, which is said to be the *natural injection* of A_t into A, has the following property:

$$i_t\big(f_t(a_\xi, \xi < k)\big) = f\big(i_t(a_\xi), \xi < k\big). \tag{4}$$

By (4) the injection i_t is an isomorphism of f_t into f; i_t is a strong isomorphism of f_t onto $f|i_t(A_t)$, where $f|i_t(A_t)$ is the restriction of f to the closed subset $i_t(A_t)$.

The direct sum of operation is not an operation but it is a partial operation.

D. Quasi-tensor products of partial operations. Let f' and f'' be any k-ary partial operations in a set A and in a set B, respectively. The *quasi-tensor product of partial operations* f' and f'' is a k-ary partial operation f in the cartesian product $A \times B$ of sets A and B such that for all sequences $(\langle a_\xi, b_\xi\rangle, \xi < k) \in A \times B^k$, f is defined for this sequence if and only if either 1) there exists an element $a \in A$ such that $a_\xi = a$ for all $\xi < k$ and f'' is defined for $(b_\xi, \xi < k)$ or 2) there exists an element $b \in B$ such that $b_\xi = b$ for all $\xi < k$ and f' is defined for $(a_\xi, \xi < k)$; in the first case we put $f(\langle a_\xi, b_\xi\rangle, \xi < k) = \langle a, f''(b_\xi, \xi < k)\rangle$, in the second case we put $f(\langle a_\xi, b_\xi\rangle, \xi < k) = \langle f'(a_\xi, \xi < k), b\rangle$. Moreover, by definition, f is not defined for any constant sequence $(\langle a_\xi, b_\xi\rangle, \xi < k)$ with $\langle a_\xi, b_\xi\rangle = \langle a, b\rangle$ not for all $\xi < k$ (since in the opposite case, by 1), $f(\langle a, b\rangle, \langle a, b\rangle, \ldots) = \langle a, f''(b, b, \ldots)\rangle$ and, by 2), $f(\langle a, b\rangle, \langle a, b\rangle, \ldots) = \langle f'(a, a, \ldots), b\rangle$ but these values may be different).

§ 2. Quasi-algebras. Let $G = \{g, \ldots\}$ be an arbitrary set of *operator symbols*. We denote by $n(g)$ the *rank of the operator symbol* g, i. e. the ordinal number n for which g is n-ary. The *rank of* G is the least initial number ϱ such that $n(g) \leqslant \varrho$ for all $g \in G$. The *dimension* of G is the least initial number γ such that, for all $g \in G$, γ is not cofinal with any ordinal number $a \leqslant n(g)$, and $\gamma > n(g)$.

By a *quasi-algebra of the type* G we understand any sequence

$$\boldsymbol{A} = \langle A, (g_{\boldsymbol{A}}, g \in G)\rangle,$$

where A is a set and $g_{\boldsymbol{A}}$ is a $n(g)$-ary partial operation in the set A for all $g \in G$ [1]. If moreover for $g \in G$, $g_{\boldsymbol{A}}$ is an operation in the set A, then the sequence $\boldsymbol{A}$ is said to be an *algebra of the type* G. The *rank* and the *dimension of a quasi-algebra* $\boldsymbol{A}$ of the type G is the rank and the dimension of G. Let T be any set and let $\boldsymbol{A}_t = \langle A_t, (g_{\boldsymbol{A}_t}, g \in G\rangle)$ be a quasi-algebra of the type G for $t \in T$. Moreover let $A = \mathop{\mathrm{P}}_{t \in T} A_t$ be the cartesian product of sets A_t, $t \in T$, and let $A' = \mathop{\mathrm{S}}_{t \in T} A_t$ be the direct sum of sets A_t, $t \in T$.

(1) In the sequel the quasi-algebras will be denoted by $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \ldots$ and their sets by $A, B, C, \ldots$

The quasi-algebras $\boldsymbol{A} = \langle A, (g_A, g \in G)\rangle$ and $\boldsymbol{A}' = \langle A', (g_{A'}, g \in G)\rangle$ of the type G, where g_A and $g_{A'}$ are for all $g \in G$ the direct product and direct sum of partial operations $g_{A_t}, t \in T$, are called the *direct product* and *direct sum of quasi-algebras* $\boldsymbol{A}_t, t \in T$, respectively (see § 1, sections B and C). The direct product and direct sum of quasi-algebras $\boldsymbol{A}_t, t \in T$, will be denoted by $\mathop{\mathrm{P}}_{t \in T} \boldsymbol{A}_t$ and $\mathop{\mathrm{S}}_{t \in T} \boldsymbol{A}_t$, respectively. The quasi-algebra $\boldsymbol{C} = \langle C, (g_C, g \in G)\rangle$ of the type G is said to be the *quasi-tensor product* of quasi-algebras $\boldsymbol{A} = \langle A, (g_A, g \in G)\rangle$ and $\boldsymbol{B} = \langle B, (g_B, g \in G)\rangle$ of the type G if $C = A \times B$ is the cartesian product of sets A and B and g_C is the quasi-tensor product of partial operations g_A and g_B for all $g \in G$ (see § 1, sec. D).

A. Subquasi-algebras and sets of generators. Let $\boldsymbol{A} = \langle A, (g_A, g \in G)\rangle$ be a quasi-algebra of the type G and let B be an arbitrary subset of A closed with respect to all partial operations $g_A, g \in G$. Then the set B and the sequence $\boldsymbol{B} = \langle B, (g_B, g \in G)\rangle$, where $g_B = g_A | B$ is the restriction of g_A to the set B for all $g \in G$, are called *a subquasi-algebra* of $\boldsymbol{A}$. Any intersection of subquasi-algebras of $\boldsymbol{A}$ is also a subquasi-algebra of $\boldsymbol{A}$. Hence it follows that for an arbitrary subset M of A there exists the least subquasi-algebra $\overline{M}$ of $\boldsymbol{A}$ which contains the set M. It is called *generated* by M. If $\overline{M} = \boldsymbol{A}$, then M is said to be a *set of generators for* $\boldsymbol{A}$. Now we determine the form of the subquasi-algebra $\overline{M}$ of a quasi-algebra $\boldsymbol{A}$ of the type G. Let γ be the dimension of G. We define by induction the sequence $M = M_0, M_1, \ldots, M_\sigma, \ldots$ $(\sigma < \gamma)$ such that

$$M_\sigma = \bigcup_{\xi<\sigma} M_\xi \cup \bigcup_{g \in G} g_A\Big(\bigcup_{\xi<\sigma} M_\xi\Big), \tag{5}$$

where for $X \subseteq A$, $g_A(X)$ denotes the set of all elements of A which are the values of partial operation g_A for elements belonging to X. In an analogical way as in the proof of my theorem (2.3) in [9] we obtain the following theorem:

(2.1) $\overline{M} = \bigcup_{\sigma<\gamma} M_\sigma$.

Theorem (2.1) determines the form of the subquasi-algebra of $\boldsymbol{A}$ generated by a set M. The sets $M_\sigma, \sigma < \gamma$, are called the *Borel-classes of the set M* in the quasi-algebra $\boldsymbol{A}$.

The cardinal number of a set X will be denoted by $|X|$ and that of an ordinal number ϱ by $\bar{\varrho}$. We obtain from (2.1) in a similar way to that of my paper [9] (theorem (4.1)) that

(2.2) $|\overline{M}| \leqslant (|M| \cdot |G| \cdot \bar{\varrho})^{\bar{\varrho}}$, *where ϱ is the rank of G.*

From (2.2) results

(2.3) *For every cardinal number* $\mathfrak{m}$ *there exists a number* $\bar{\bar{\mathfrak{m}}}$ ($\mathfrak{m} = \cdot |G| \cdot \bar{\varrho})^{\bar{\varrho}}$) *such that for all quasi-algebras* $\boldsymbol{A}$ *of the type* G (*of rank* ϱ) *and all subsets* M *of* A *with* $|M| \leqslant \mathfrak{m}$ *we have* $|\bar{M}| \leqslant \bar{\bar{\mathfrak{m}}}$.

B. Ordinary homomorphisms of quasi-algebras. Let $\boldsymbol{A} = \langle A, (g_{\boldsymbol{A}}, g \in G) \rangle$ and $\boldsymbol{B} = \langle B, (g_{\boldsymbol{B}}, g \in G) \rangle$ be any two quasi-algebras of the type G. A mapping h of A into B is said to be *a* (*strong*) *homomorphism* of $\boldsymbol{A}$ *into* $\boldsymbol{B}$ if h is a (strong) homomorphism of the partial operation $g_{\boldsymbol{A}}$ *into* the partial operation $g_{\boldsymbol{B}}$ (see § 1 section A) for all $g \in G$.

The homomorphisms between algebras are always strong. One-to-one homomorphisms are isomorphisms. If h is a homomorphism of a quasi-algebra $\boldsymbol{A}$ of the type G into a quasi-algebra $\boldsymbol{B}$ of the type G, then h considered as a subset of the cartesian product $A \times B$ of sets A and B is a subquasi-algebra of the direct product $\boldsymbol{A} \times \boldsymbol{B}$ of $\boldsymbol{A}$ and $\boldsymbol{B}$ such that

(6) for all $a \in A$ there exists only one element $b \in B$ with $\langle a, b \rangle \in h$.

The converse is not always true. Any subquasi-algebra h of the direct product $\boldsymbol{A} \times \boldsymbol{B}$ of quasi algebras $\boldsymbol{A}$ and $\boldsymbol{B}$ which has the property (6) is called *a full-homomorphism of* $\boldsymbol{A}$ *into* $\boldsymbol{B}$. Every homomorphism of $\boldsymbol{A}$ into $\boldsymbol{B}$ is a full-homomorphism of $\boldsymbol{A}$ into $\boldsymbol{B}$, but the full-homomorphism of $\boldsymbol{A}$ into $\boldsymbol{B}$ is not always a homomorphism of $\boldsymbol{A}$ into $\boldsymbol{B}$.

It is easy to verify that

(2.4) *For algebras the notions of homomorphisms, strong homomorphisms, and full-homomorphisms are identical.*

The quasi-algebras h of the direct product $\boldsymbol{A} \times \boldsymbol{B}$ of quasi-algebras $\boldsymbol{A}$ and $\boldsymbol{B}$ of the type G such that

(7) for all $a \in A$ there exists at most one element $b \in B$ with $\langle a, b \rangle \in h$

are called *partial-homomorphisms of* $\boldsymbol{A}$ *into* $\boldsymbol{B}$. The partial-homomorphisms of $\boldsymbol{A}$ into $\boldsymbol{B}$ are those subquasi-algebras of $\boldsymbol{A} \times \boldsymbol{B}$ which are the partial mappings of A into B.

Let us observe that

(2.5) *If* $\boldsymbol{A}$ *and* $\boldsymbol{B}$ *are any algebras of the type* G *and* h *is a partial-homomorphism of* $\boldsymbol{A}$ *into* $\boldsymbol{B}$ *such that the set* $p_1(h)$, *where* p_1 *is the natural projection of* $A \times B$ *onto* A, *contains a set of generators for* $\boldsymbol{A}$, *then* h *is a full-homomorphism* (*and therefore, by* (2.4), *a homomorphism*) *of* $\boldsymbol{A}$ *into* $\boldsymbol{B}$.

Indeed, $p_1(h)$ is a subalgebra of $\boldsymbol{A}$ containing a set of generators for $\boldsymbol{A}$ and hence $p_1(h) = A$. Thus (2.5) is proved.

C. Peano-algebra. Let X be a set. An algebra $\boldsymbol{W} = \langle W, (g_{\boldsymbol{W}}, g \in G) \rangle$ of the type G is called a *Peano-algebra of the type G generated by X* if it has the following properties:

(2.a) *the elements of X are not values of the operations $g_{\boldsymbol{W}}$, $g \in G$, for elements of W;*

(2.b) *for all $g, g' \in G$ and all sequences $(w_\xi, \xi < n(g)) \in W^{n(g)}$ and $(w'_\xi, \xi < n(g')) \in W^{n(g')}$, if $g_{\boldsymbol{W}}(w_\xi, \xi < n(g)) = g'_{\boldsymbol{W}}(w'_\xi, \xi < n(g'))$, then $g = g'$ and $w_\xi = w'_\xi$ for $\xi < n(g) = n(g')$,*

(2.c) *the set X generates the algebra $\boldsymbol{W}$.*

Now we prove that

(2.6) *For every set X there exists the Peano-algebra $\boldsymbol{W}$ of the type G generated by the set X.*

Proof. For $g \in G$ and any set B we denote by $g[B]$ the set of all pairs $\langle g, (b_\xi < n(g))\rangle$, where $(b_\xi, \xi < n(g))$ is a sequence of the type $n(g)$ the elements of which belong to B. Let γ be the dimension of G. We define by induction a sequence of sets $X = X_0, X_1, \ldots, X_\sigma, \ldots, \sigma < \gamma$, such that

$$X_\sigma = \bigcup_{\xi<\sigma} X_\xi \cup \bigcup_{g \in G} g[\bigcup_{\xi<\sigma} X_\xi].$$

Let $W = \bigcup_{\sigma<\gamma} X_\sigma$. Putting

$$g_{\boldsymbol{W}}(w_\xi, \xi < n(g)) = \langle g, (w_\xi, \xi < n(g))\rangle$$

for all $g \in G$ and all sequences $(w_\xi, \xi < n(g))$ with $w_\xi \in W$, we obtain an $n(g)$-ary operation in the set W. It is easy to verify that the sequence $\boldsymbol{W} = \langle W, (g_{\boldsymbol{W}}, g \in G)\rangle$ is a Peano-algebra of the type G generated by X.

(2.7) *Let $\boldsymbol{W} = \langle W, (g_{\boldsymbol{W}}, g \in G)\rangle$ be a Peano-algebra of the type G generated by a set X and let $\boldsymbol{A} = \langle A, (g_{\boldsymbol{A}}, g \in G)\rangle$ be an arbitrary quasi-algebra of the type G. Moreover, let $\varphi : X \to A$ be any mapping of X into A. Then the subquasi-algebra $\overline{\varphi}$ of $\boldsymbol{W} \times \boldsymbol{A}$ generated by φ is a partial-homomorphism of $\boldsymbol{W}$ into $\boldsymbol{A}$.*

Proof. At first let us observe that

(i) no pair $\langle x, a\rangle$, where $x \in X$ and $a \neq \varphi(x)$, is in the set $\overline{\varphi}$.

Indeed, the set $U = \overline{\varphi} - \{\langle x, a\rangle\}$, where $a \neq \varphi(x)$, is a subquasi-algebra of direct product $\boldsymbol{W} \times \boldsymbol{A}$ of $\boldsymbol{W}$ and $\boldsymbol{A}$ containing φ, because

1° $\varphi \subset U$ by virtue of $\langle x, a\rangle \notin \varphi$,

2° if the pairs $\langle w_\xi, a_\xi\rangle$, $\xi < n(g)$, belong to U, then the pair $\langle g_{\boldsymbol{W}}(w_\xi), g_{\boldsymbol{A}}(a_\xi)\rangle = g_{\boldsymbol{W}\times\boldsymbol{A}}(\langle w_\xi, a_\xi\rangle)$ — provided it exists — is by (2.a) different from $\langle x, a\rangle$, and therefore U is closed with respect to partial-operations $g_{\boldsymbol{W}\times\boldsymbol{A}}$ for all $g \in G$.

Hence it follows that $U = \overline{\varphi}$ and thus we obtain (i). Now we prove that for all $w \in W$

(ii) there exists at most one element $a \in A$ with $\langle w, a\rangle \in \overline{\varphi}$.

For let us denote by W' the set of all elements $w \in W$ which have the property (ii). By (i), $X \subset W'$. Let $w_\xi \in W'$ for $\xi < n(g)$. Then there exists at most one element $a_\xi \in A$ with $\langle w_\xi, a_\xi \rangle \in \bar{\varphi}$ for $\xi < n(g)$. Let us observe that

(iii) No pair $\langle g_{\boldsymbol{W}}(w_\xi, \xi < n(g)), a \rangle$ with $a \neq g_{\boldsymbol{A}}(a_\xi, \xi < n(g))$ is in the set $\bar{\varphi}$.

Indeed, the set $U = \bar{\varphi} - \{\langle g_{\boldsymbol{W}}(w_\xi, \xi < n(g)), a \rangle\}$, where $a \neq g_{\boldsymbol{A}}(a_\xi, \xi < n(g))$, is a subquasi-algebra of $\boldsymbol{W} \times \boldsymbol{A}$ containing the set φ, because

1' $\varphi \subset U$ by (2.a),

2' if $\langle w'_\xi, a'_\xi \rangle \in U$ for $\xi < n(g')$, then, by (2.b), the pair (if it exists)

$$g'_{\boldsymbol{W}\times\boldsymbol{A}}(\langle w'_\xi, a'_\xi \rangle, \xi < n(g')) = \langle g'_{\boldsymbol{W}}(w'_\xi, \xi < n(g')), g'_{\boldsymbol{A}}(a'_\xi, \xi < n(g')) \rangle$$

is different from $\langle g_{\boldsymbol{W}}(w_\xi, \xi < n(g)), a \rangle$, and hence U is closed with respect to the partial operation $g'_{\boldsymbol{W}\times\boldsymbol{A}}$ for all $g' \in G$. Thus $U = \bar{\varphi}$, and therefore we obtain (iii). From (iii) it follows that for the element $w = g_{\boldsymbol{W}}(w_\xi, \xi < n(g))$ there exists at most one element $a \in A$ with $\langle w, a \rangle \in \bar{\varphi}$ $(a = g_{\boldsymbol{A}}(a_\xi, \xi < n(g)))$ if the element a exists. Hence the element $w = g_{\boldsymbol{W}}(w_\xi, \xi < n(g))$ belongs to W' if $\omega_\xi \in W'$ for all $g \in G$ and thus W' is a subalgebra of $\boldsymbol{W}$ containing the set X of generators for $\boldsymbol{W}$; hence $W' = W$, and thus lemma (ii) is true for all $w \in W$. Therefore $\bar{\varphi}$ is a partial-homomorphism of $\boldsymbol{W}$ into $\boldsymbol{A}$ and theorem (2.7) is proved.

(2.8) *The Peano-algebra $\boldsymbol{W}$ of the type G generated by a set X is the absolutely free algebra of the type G freely generated by X, i. e. for every algebra $\boldsymbol{A}$ of the type G and each mapping $\varphi : X \to A$, the mapping φ can be extended to the unique homomorphism $\bar{\varphi}$ of $\boldsymbol{W}$ into $\boldsymbol{A}$, where $\boldsymbol{W}$ is a Peano-algebra of the type G generated by X.*

Proof. By (2.7) and (2.5) the subalgebra $\bar{\varphi}$ of $\boldsymbol{W} \times \boldsymbol{A}$ generated by φ is a full-homomorphism of $\boldsymbol{W}$ into $\boldsymbol{A}$, and thus, by (2.4), it is a homomorphism of $\boldsymbol{W}$ into $\boldsymbol{A}$ which is the unique extension of φ. Theorem (2.8) is proved.

In this way we have obtained a correct proof of theorem (1.3) of my paper [9, Chap. III] (2).

From (2.8) it follows that a Peano-algebra of the type G generated by a set X is uniquely determined up to an isomorphism by the cardinal number of set X. Let $\mathfrak{B}$ be an arbitrary class of quasi-algebras of the type G. An algebra $\boldsymbol{B}$ is said to be a *$\mathfrak{B}$-partial-free algebra freely generated by a set X* if it has the following properties:

(2) From the Henkin's considerations in [4] it follows that the proof of theorem (1.3) in my paper [9] (chap. III) is false. This fact has been observed also by Schmidt [7].

1'' $\boldsymbol{B} \in \mathfrak{B}$,

2'' X is a set of generators for $\boldsymbol{B}$,

3'' for each quasi-algebra $\boldsymbol{A} \in \mathfrak{B}$ and for each mapping $\varphi: X \to A$, the subquasi-algebra $\bar{\varphi}$ of $\boldsymbol{B} \times \boldsymbol{A}$ generated by φ is a partial-homomorphism of $\boldsymbol{B}$ into $\boldsymbol{A}$ (if $\boldsymbol{A}$ is an algebra, then by (2,5) the condition "$\bar{\varphi}$ is a partial-homomorphism of $\boldsymbol{B}$ into $\boldsymbol{A}$" is equivalent to the condition "$\bar{\varphi}$ is a homomorphism of $\boldsymbol{B}$ into $\boldsymbol{A}$").

The $\mathfrak{B}$-partial-free algebra freely generated by a set X is uniquely determined (if it exists) up to isomorphisms by the cardinal number of the set X. By (2.7) the Peano-algebra of the type G generated by a set X is the $\mathfrak{B}^*$-partial-free algebra freely generated by X, where $\mathfrak{B}^*$ is the class of all quasi-algebras of the type G. If $\mathfrak{B}$ is a class of algebras, then the $\mathfrak{B}$-partial-free algebra freely generated by a set X is identical with *the* $\mathfrak{B}$*-free algebra freely generated by* X (see remark in the parenthesis of condition 3''). Let us observe that

(2.9) *If $\mathfrak{B}$ is an equationally definable class of quasi-algebras of the type G, then for any set X there exists the $\mathfrak{B}$-partial-free algebra freely generated by the set X and it is identical with the $\mathfrak{B}'$-free algebra freely generated by X, where $\mathfrak{B}' \subset \mathfrak{B}$ is the class of all algebras defined by the same equations as the class $\mathfrak{B}$.*

For a definition of equationally definable class of quasi-algebras see next section D.

D. The partial operations defined in quasi-algebras by terms. Let $\boldsymbol{W} = \langle W, (g_{\boldsymbol{W}}, g \in G)\rangle$ be a fixed Peano-algebra of the type G generated by a set $X = (x_0, x_1, \ldots, x_\sigma, \ldots, \sigma < \varrho)$, where all x_σ are pairwise different elements. The elements of W are called *G-terms*, and the elements of X are considered as *variables*. For every G-term $\tau \in W$ there exists the least subset $X' \subset X$ with $\tau \in \bar{X}'$, which is called the support of τ. The support of τ will be denoted by X_τ. Let $X_\tau = (x_{\beta_0}, x_{\beta_1}, \ldots, x_{\beta_\xi}, \ldots, \xi < a)$, where $\beta_0 < \beta_1 < \beta_2 < \ldots$ is the support of G-term τ. Then the ordinal number $a = n(\tau)$ is said to be the *rang* of τ and the term τ will be also denoted by $\tau(x_{\beta_\xi}, \xi < n(\tau))$. Let $\boldsymbol{A}$ be an arbitrary quasi-algebra of the type G and let $\tau = \tau(x_{\beta_\xi}, \xi < n(\tau))$ be any G-term in W. Let k be an arbitrary ordinal number such that $k \geqslant n(\tau)$. The term τ defines in the set A of quasi-algebra $\boldsymbol{A}$ a k-ary partial operation $\tau_{\boldsymbol{A}}$. We define $\tau_{\boldsymbol{A}}$ as follows. Let $(a_\xi, \xi < k)$ be any sequence of the type k in A. Let us denote by φ any mapping of X into A such that $\varphi(x_{\beta_\xi}) = a_\xi$ for all $\xi < n(\tau)$. By theorem (2.7)' the subquasi-algebra $\bar{\varphi}$ of the direct product $\boldsymbol{W} \times \boldsymbol{A}$ of $\boldsymbol{W}$ and $\boldsymbol{A}$ generated by φ is a partial-homomorphism of $\boldsymbol{W}$ into $\boldsymbol{A}$. The partial operation $\tau_{\boldsymbol{A}}$ is defined for $(a_\xi, \xi < k)$ if and only if the term τ belongs to the domain of $\bar{\varphi}$, i. e. to the set $p_1(\bar{\varphi})$, where p_1 is the natural projection of $W \times A$ onto W, and, moreover, we put $\tau_{\boldsymbol{A}}(a_\xi, \xi < k) = \bar{\varphi}(\tau)$.

The partial-operation τ_A depends only on the first $n(\tau)$ arguments. If A is an algebra, then τ_A is also an operation. The partial operation τ_A is said *to be defined in the quasi-algebra* $\boldsymbol{A}$ *by the G-term* τ (³). It is easy to verify that

(2.10) *If h is a homomorphism of a quasi-algebra $\boldsymbol{A}$ of the type G into a quasi-algebra $\boldsymbol{B}$ of the type G, then, for every G-term τ, h is a homomorphism of τ_A into τ_B, where τ_A and τ_B are the partial operations defined by G-term τ in the quasi-algebras $\boldsymbol{A}$ and $\boldsymbol{B}$, respectively.*

Now we give a definition:

(2.11) *For any term $\tau = \tau(x_{\beta_\xi}, \xi < n(\tau))$ we denote by τ' the term obtained from τ by substituting x_{β_ξ} by x_ξ for $\xi < n(\tau)$, i. e. $\tau' = h(\tau)$, where h is a homomorphism of $\boldsymbol{W}$ into $\boldsymbol{W}$ such that $h(x_{\beta_\xi}) = x_\xi$ for $\xi < n(\tau)$.*

We observe that

(2.12) *For any term τ and any quasi-algebra $\boldsymbol{A}$ the partial operations τ_A and τ'_A are identical.*

(2.13) *The subquasi-algebra $\overline{M}$ of a quasi-algebra $\boldsymbol{A}$ of the type G generated by a subset M has the form*

$$\overline{M} = \bigcup_{\tau\in W} \tau_A(M) = \bigcup_{\tau\in W} \tau'_A(M),$$

where $\tau_A(M)$ and $\tau'_A(M)$ are the sets of all elements of $\boldsymbol{A}$ which are the values of partial operations τ_A and τ'_A (we assume for (2.13) that $|X| \geqslant \bar{\varrho}$, where is the rank of G).

The term τ defines in quasi-algebra $\boldsymbol{A}$ other partial operation ${}_A\tau$ as follows: ${}_A\tau$ is defined for a mapping $\varphi: X \to A$ if and only if the term τ belongs to the domain of $\bar{\varphi}$ and ${}_A\tau(\varphi) = \bar{\varphi}(\tau)$.

${}_A\tau$ is called the *partial operation of the type X* defined by term τ in quasi-algebra $\boldsymbol{A}$. From the definition results

(2.14) *If $\tau = \tau(x_{\beta_\xi}, \xi < n(\tau))$ and $\vartheta = \vartheta(x_{\beta_\xi}, \xi < n(\vartheta))$, where $n(\tau) \leqslant n(\vartheta)$, then the identity $\tau_A = \vartheta_A$ is equivalent to the identity ${}_A\tau = {}_A\vartheta$.*

From (2.14), (2.11) and (2.12) we obtain

(2.15) *For any terms τ and ϑ and any quasi-algebra $\boldsymbol{A}$ the identity ${}_A\tau' = {}_A\vartheta'$ is equivalent to the identity $\tau'_A = \vartheta'_A$* (resp. $\tau_A = \vartheta_A$).

The pairs $\langle\tau, \vartheta\rangle$, where $\tau, \vartheta \in W$ are G-terms, will be called *G-equations*. A G-equation $\langle\tau, \vartheta\rangle$ will be also denoted by $\ulcorner\tau = \vartheta\urcorner$. A G-equation $\ulcorner\tau = \vartheta\urcorner$ is called *valid* in a quasi-algebra $\boldsymbol{A}$ of the type G if ${}_A\tau = {}_A\vartheta$ (i. e. $\bar{\varphi}(\tau) = \bar{\varphi}(\vartheta)$ for all $\varphi \in D$, where D is the domain of ${}_A\tau$ and ${}_A\vartheta$. Hence the equation $\ulcorner\tau = \vartheta\urcorner$ is valid in $\boldsymbol{A}$ if and only if the partial opera-

(³) The partial operations τ_A defined in a quasi-algebra $\boldsymbol{A}$ by terms τ are called also algebraic partial operations in $\boldsymbol{A}$.

tions ${}_A\tau$ and ${}_A\vartheta$ are identical. By (2.15) the equation $\ulcorner\tau' = \vartheta'\urcorner$ is valid in $\boldsymbol{A}$ if and only if the partial operations $\tau'_{\boldsymbol{A}}$ and $\vartheta'_{\boldsymbol{A}}$ (resp. $\tau_{\boldsymbol{A}}$ and $\vartheta_{\boldsymbol{A}}$) are identical.

The set of all G-equations which are valid in $\boldsymbol{A}$ will be denoted by $E(\boldsymbol{A})$. If E_0 is a set of G-equations, then $G(E_0)$ denotes the class of all quasi-algebras $\boldsymbol{A}$ of the type G such that $E_0 \subset E(\boldsymbol{A})$. The classes of the form $G(E_0)$ are called *equationally definable*.

§ 3. P-homomorphisms. Let $F = \{f, \ldots\}$ and $G = \{g, \ldots\}$ be two sets of operator symbols. Let $\boldsymbol{A} = \langle A, (f_{\boldsymbol{A}}, f \in F)\rangle$ and $\boldsymbol{B} = \langle B, (g_{\boldsymbol{B}}, g \in G)\rangle$ be two quasi-algebras, the first of type F and the second of type G. In order to define the notion of a homomorphism for this general case, let us consider two Peano-algebras $\boldsymbol{F}^* = \mathrm{Free}(F, X)$ and $\boldsymbol{G}^* = \mathrm{Free}(G, X)$, the first of type F and the second of type G, both generated by the same set $X = (x_0, x_1, \ldots, x_\sigma, \ldots, \sigma < \varrho)$. The elements of $\boldsymbol{F}^*$ and of $\boldsymbol{G}^*$ will be considered as F-terms and G-terms, respectively.

A mapping $P: F \to 2^{G^*}$, where G^* is the set of all G-terms, is said to be a *$P_{F,G}$-mapping* if for all $f \in F$ and all $\tau \in G^*$ the relation $\tau \in P(f)$ implies $n(\tau) \leqslant n(f)$, where $n(\tau)$ and $n(f)$ are the ranks of the G-term τ and the operator symbol f, respectively. A $P_{F,G}$-mapping P is called *proper* if $P(f)$ is a one-element set for all $f \in F$. Let P be an arbitrary $P_{F,G}$-mapping. A mapping h of A into B is called a *P-homomorphism of a quasi-algebra $\boldsymbol{A} = \langle A, (f_{\boldsymbol{A}}, f \in F)\rangle$ of the type F into a quasi-algebra $\boldsymbol{B} = \langle B, (g_{\boldsymbol{B}}, g \in G)\rangle$* of the type G if, for all $f \in F$, all the sequences $(a_\xi, \xi < n(f)) \in A^{n(f)}$, and all $\tau \in P(f)$, if $f_{\boldsymbol{A}}$ is defined for $(a_\xi, \xi < n(f))$, then $\tau_{\boldsymbol{B}}$ is defined for $(h(a_\xi), \xi < n(f))$ and $h(f_{\boldsymbol{A}}(a_\xi, \xi < n(f))) = \tau_{\boldsymbol{B}}(h(a_\xi), \xi < n(f))$.

If $F \subset G$ and P is a proper $P_{F,G}$-mapping such that $P(f) = f$ for all $f \in F$, then P-homomorphisms are ordinary homomorphisms.

It is easy to verify that

(3.1) *A mapping h of a quasi-algebra $\boldsymbol{A}$ of the type F into a quasi-algebra $\boldsymbol{B}$ of the type G is a P-homomorphism of $\boldsymbol{A}$ into $\boldsymbol{B}$, where P is a proper $P_{F,G}$-mapping, if and only if h is an ordinary homomorphism of $\boldsymbol{A}$ into a quasi-algebra $P(\boldsymbol{B}) = \langle B, P(f)_{\boldsymbol{B}}, f \in F\rangle$ of the type F, called P-quasi-algebra over $\boldsymbol{B}$, where $P(f)_{\boldsymbol{B}}$ is the $n(f)$-ary partial operation in $\boldsymbol{B}$ defined by the F-term $P(f)$.*

(3.2) *If h is a P-homomorphism, where P is an arbitrary $P_{F,G}$-mapping, of a quasi-algebra $\boldsymbol{A}$ of the type F into a quasi-algebra $\boldsymbol{B}$ of the type G, and q is a homomorphism of $\boldsymbol{B}$ into a quasi-algebra $\boldsymbol{C}$ of the type G, then the mapping qh is a P-homomorphism of $\boldsymbol{A}$ into $\boldsymbol{C}$.*

(3.3) *If q is a homomorphism of a quasi-algebra $\boldsymbol{A}$ of the type F into a quasi-algebra $\boldsymbol{B}$ of the type F, and h is a P-homomorphism, where P is* *any $P_{F,G}$-mapping, of $\boldsymbol{B}$ into a quasi-algebra $\boldsymbol{C}$ of the type G, then hq is a P-homomorphism of $\boldsymbol{A}$ into $\boldsymbol{C}$.*

A. The direct product of P-homomorphisms. Let P be an arbitrary $P_{F,G}$-mapping and let $\boldsymbol{A} = \langle A, (f_{\boldsymbol{A}}, f \in F)\rangle$ be any quasi-algebra of the type F. Let T be any set and let $h_t, t \in T$, be a P-homomorphism of $\boldsymbol{A}$ into a quasi-algebra $\boldsymbol{B}_t = \langle B_t, (g_{\boldsymbol{B}_t}, g \in G)\rangle$ of the type G. Let us consider the quasi-algebra $\boldsymbol{B} = \mathop{\mathrm{P}}_{t\in T} \boldsymbol{B}_t = \langle B, g_{\boldsymbol{B}}, g \in G\rangle$ of the type G, which is the direct product of all quasi-algebras $\boldsymbol{B}_t, t \in T$. The mapping $h: A \to B$, where $B = \mathop{\mathrm{P}}_{t\in T} B_t$ is the cartesian product of sets B_t, such that for all $a \in A$, $h(a) = \varphi$ with $\varphi(t) = h_t(a)$ for all $t \in T$, is called the *direct product of mappings $h_t, t \in T$*. Let p_t be the natural projection of $\boldsymbol{B} = \mathop{\mathrm{P}}_{t\in T} B_t$ onto B_t. It is easy to verify that

(3.4) *The direct product h of P-homomorphisms $h_t, t \in T$, is the unique P-homomorphism of $\boldsymbol{A}$ into the direct product $\boldsymbol{B}$ of quasi-algebras $\boldsymbol{B}_t, t \in T$, such that $h_t = p_t h$ for all $t \in T$.*

(3.5) *The direct product h of P-homomorphisms $h_t, t \in T$, is one-to-one if and only if the homomorphisms $h_t, t \in T$, separate the quasi-algebra $\boldsymbol{A}$, i. e. if and only if for any pair of different elements $a, b \in A$ there exists a $t_0 \in T$ with $h_{t_0}(a) \neq h_{t_0}(b)$. Thus h is one-to-one if and only if*

$$\bigcap_{t\in T} R_t \subset \mathrm{id}_A,$$

where R_t is the equivalence relation in A induced by $h_t\big(\langle a, b\rangle \in R^t \Leftrightarrow h_t(a) = h_t(b)\big)$ and id_A is the identity relation in A ($\mathrm{id}_A = \{\langle a, a\rangle : a \in A\}$).

Theorems (3.4) and (3.5) may be considered as some generalizations of the product theorems of Birkhoff [2].

B. The direct sum of P-homomorphisms. Let P be any $P_{F,G}$-mapping. Let T be an arbitrary set and let $\boldsymbol{A}_t, t \in T$, be a family of quasi-algebras of the type F and let $\boldsymbol{B}$ be any quasi-algebra of the type G. Let us consider an arbitrary family of P-homomorphisms h_t of $\boldsymbol{A}_t$ into $\boldsymbol{B}$ for $t \in T$. Let $\boldsymbol{A}$ be the direct sum of quasi-algebras $\boldsymbol{A}_t, t \in T$. A mapping $h: A \to B$, where $A = \mathop{\mathrm{S}}_{t\in T} A_t$ is the direct sum of sets A_t of all elements of quasi-algebras $\boldsymbol{A}_t$ and B is the set of all elements of quasi-algebra $\boldsymbol{B}$ such that $h\big((t, a)\big) = h_t(a)$ for all $t \in T$ and all $a \in A_t$ is called the *direct sum of mappings $h_t, t \in T$*. The direct sum of P-homomorphism is also a P-homomorphism. Indeed, we have the following theorem:

(3.6) *The direct sum h of P-homomorphisms $h_t, t \in T$, of $\boldsymbol{A}_t$ into $\boldsymbol{B}$, is the unique P-homomorphism of the direct sum $\boldsymbol{A}$ of quasi-algebras $\boldsymbol{A}_t, t \in T$, into $\boldsymbol{B}$ such that $h_t = hi_t$ for all $t \in T$, where i_t is the natural injection of A_t into A $\big(i_t(a) = (t, a)\big)$.*

C. $\mathfrak{B}$-P-homomorphisms. Let P be any $P_{F,G}$-mapping and let $\mathfrak{B}$ be an arbitrary class of quasi-algebras of the type G. Moreover, let $\boldsymbol{A}$ be an arbitrary quasi-algebra of the type F. Any pair $(h, \boldsymbol{B})$, where $\boldsymbol{B} \in \mathfrak{B}$ and h is a P-homomorphism of $\boldsymbol{A}$ into $\boldsymbol{B}$, is called a *$\mathfrak{B}$-P-homomorphism* of $\boldsymbol{A}$. The one-to-one $\mathfrak{B}$-P-homomorphisms of $\boldsymbol{A}$ are called *$\mathfrak{B}$-P-extensions* of $\boldsymbol{A}$. A $\mathfrak{B}$-P-homomorphism $(h, \boldsymbol{B})$ of $\boldsymbol{A}$ is said to be *almost onto* if the set $h(A)$ generates $\boldsymbol{B}$. Now we introduce some relations between $\mathfrak{B}$-P-homomorphisms of $\boldsymbol{A}$. Let $(h, \boldsymbol{B})$ and $(h', \boldsymbol{B}')$ be $\mathfrak{B}$-P-homomorphisms of $\boldsymbol{A}$. We shall say that:

1. $(h, \boldsymbol{B}) \leqslant (h', \boldsymbol{B}')$ *if there exists exactly one ordinary homomorphism q of $\boldsymbol{B}$ into $\boldsymbol{B}'$ with $h' = qh$,*

2. $(h, \boldsymbol{B}) \equiv (h', \boldsymbol{B}')$ *if and only if there exists exactly one strong isomorphism q of $\boldsymbol{B}$ onto $\boldsymbol{B}'$ with $h' = qh$.*

A $\mathfrak{B}$-P-homomorphism $(h, \boldsymbol{B})$ of $\boldsymbol{A}$ is said to be *free* if for every $\mathfrak{B}$-P-homomorphism $(h', \boldsymbol{B}')$ of $\boldsymbol{A}$ we have $(h, \boldsymbol{B}) \leqslant (h', \boldsymbol{B}')$. Now we prove that

(3.7) *The free $\mathfrak{B}$-P-homomorphism of $\boldsymbol{A}$ (if it exists) is uniquely determined up to relation $\equiv$.*

Proof. Let $(h, \boldsymbol{B})$ and $(h', \boldsymbol{B}')$ be two free $\mathfrak{B}$-P-homomorphisms of $\boldsymbol{A}$. Then $h' = qh$ and $h = q'h'$, where q and q' are the homomorphisms of $\boldsymbol{B}$ into $\boldsymbol{B}'$ and of $\boldsymbol{B}'$ into $\boldsymbol{B}$, respectively. Hence we obtain $h' = q \cdot q' \cdot h'$ and $h = q' \cdot q \cdot h$. But we have also $h' = I'h'$ and $h = Ih$, where I' and I are the identity isomorphism of $\boldsymbol{B}'$ onto $\boldsymbol{B}'$ and of $\boldsymbol{B}$ onto $\boldsymbol{B}$ respectively, and thus $q \cdot q' = I'$ and $q' \cdot q = I$. Hence it follows that q and q' are one-to-one and "onto", and moreover $q' = q^{-1}$. Therefore q is a strong isomorphism of $\boldsymbol{B}$ onto $\boldsymbol{B}'$, whence $(h, B) \equiv (h', B')$ and thus (3.7) is proved.

A class $\mathfrak{B}$ of quasi-algebras of the type G is called *quasi-primitive* if it has the following properties:

1*) $\mathfrak{B}$ is closed with respect to direct products,

2*) $\mathfrak{B}$ is closed with respect to subquasi-algebras,

3*) $\mathfrak{B}$ is closed with respect to strong isomorphic images.

Now we prove a general existence theorem of quasi-algebras.

(3.8) *Let $\mathfrak{B}$ be an arbitrary quasi-primitive class of quasi-algebras of the type G and let $\boldsymbol{A}$ be an arbitrary quasi-algebra of the type F. Moreover, let P be any $P_{F,G}$-mapping. Then there exists the free $\mathfrak{B}$-P-homomorphism of quasi-algebra $\boldsymbol{A}$.*

Proof. By virtue of theorem (2.3) there exists a number $\overline{\mathfrak{m}}$ such that $|\boldsymbol{B}| \leqslant \overline{\mathfrak{m}}$ for all quasi-algebras $\boldsymbol{B}$ of the type G generated by sets $\boldsymbol{M}$ with $|M| \leqslant |A|$. Let E be an arbitrary set with $|E| \geqslant \overline{\mathfrak{m}}$ and let $\boldsymbol{B}$ be an arbitrary quasi-algebra of the type G with $B \subset E$. Let us consider the family of quasi-algebras $\boldsymbol{B}_\lambda = \boldsymbol{B}$, where λ runs through all P-homomorphisms of $\boldsymbol{A}$ into $\boldsymbol{B}$, i. e. through the set P-Hom$(\boldsymbol{A}, \boldsymbol{B})$. Let $h_{\boldsymbol{B}}$ be the direct product of all P-homomorphisms of $\boldsymbol{A}$ into $\boldsymbol{B}$. By theorem (3.4) $h_{\boldsymbol{B}}$ is the unique P-homomorphism of $\boldsymbol{A}$ into $\boldsymbol{B}^{P\text{-Hom}(\boldsymbol{A},\boldsymbol{B})}$ such that $p_\lambda h_{\boldsymbol{B}} = \lambda$ for all $\lambda \in P$-Hom$(\boldsymbol{A}, \boldsymbol{B})$, where p_λ is the natural projection of $\boldsymbol{B}^{P\text{-Hom}(\boldsymbol{A},\boldsymbol{B})}$ onto $\boldsymbol{B}_\lambda = \boldsymbol{B}$. Let h be the direct product of all P-homomorphisms $h_{\boldsymbol{B}}$ with $\boldsymbol{B} \in \mathfrak{B}$ and $B \subset E$. By theorem (3.4) h is the unique P-homomorphism of $\boldsymbol{A}$ into the direct product $P\boldsymbol{B}^{P\text{-Hom}(\boldsymbol{A},\boldsymbol{B})}$ of all direct powers $\boldsymbol{B}^{P\text{-Hom}(\boldsymbol{A},\boldsymbol{B})}$, where $\boldsymbol{B} \in \mathfrak{B}$ and $B \subset E$, such that $h_{\boldsymbol{B}} = q_{\boldsymbol{B}} h$ for all $\boldsymbol{B} \in \mathfrak{B}$ with $B \subset E$, where $q_{\boldsymbol{B}}$ is the natural projection of $P\boldsymbol{B}^{P\text{-Hom}(\boldsymbol{A},\boldsymbol{B})}$ onto $\boldsymbol{B}^{P\text{-Hom}(\boldsymbol{A},\boldsymbol{B})}$. Let $\boldsymbol{C}$ be the subquasi-algebra of $P\boldsymbol{B}^{P\text{-Hom}(\boldsymbol{A},\boldsymbol{B})}$ generated by the set $h(A)$, i. e. $\boldsymbol{C} = \overline{h(A)}$. By 1* and 2* of the definition of quasi-primitive class the quasi-algebra $\boldsymbol{C}$ belongs to $\mathfrak{B}$. The pair $(h, \boldsymbol{C})$ is the free $\mathfrak{B}$-P-homomorphism of $\boldsymbol{A}$. Indeed, let $(h_1, \boldsymbol{B}_1)$ be an arbitrary $\mathfrak{B}$-P-homomorphism of $\boldsymbol{A}$. Let us denote by $\boldsymbol{D} = \overline{h_1(A)}$ the subquasi-algebra of $\boldsymbol{B}_1$ generated by $h_1(A)$. Since $|h_1(A)| \leqslant |A|$, obviously $\boldsymbol{D} \in \mathfrak{B}$ and $|D| \leqslant \overline{\mathfrak{m}}$, Hence it follows that there exists a quasi-algebra $\boldsymbol{B}$ with $B \subset E$ such that $\boldsymbol{B}$ is strongly isomorphic to $\boldsymbol{D}$. Let i be a strong isomorphism of $\boldsymbol{D}$ onto $\boldsymbol{B}$. By 2* and 3* of the definition of a quasi-primitive class, $\boldsymbol{B} \in \mathfrak{B}$. Then $h_1 = q \cdot h$, where $q = ip_\lambda q_{\boldsymbol{B}} | C$ with $\lambda = i^{-1} h_1$, and therefore $(h, \boldsymbol{C}) \leqslant (h_1, \boldsymbol{B}_1)$ (the homomorphism q is unique since $h(A)$ generates $\boldsymbol{C}$). Thus we have proved that the pair $(h, \boldsymbol{C})$ is the free $\mathfrak{B}$-P-homomorphism of $\boldsymbol{A}$, i. e. the theorem (3.8) is proved.

If $F \subset G$ and P is a proper $P_{F,G}$-mapping such that $P(f) = f$ for all $f \in F$, then from (3.8) we obtain the existence theorem contained in paper [7] of Schmidt. Let us assume that $\mathfrak{B}$ is a quasi-primitive class. Then we observe that

(3.9) *The quasi-algebra* $\boldsymbol{A}$ *has an* $\mathfrak{B}$-P-*extension if and only if the free* $\mathfrak{B}$-P-*homomorphism of* $\boldsymbol{A}$ *is a* $\mathfrak{B}$-P-*extension of* $\boldsymbol{A}$.

From the construction of the free $\mathfrak{B}$-P-homomorphism $(h, \boldsymbol{C})$ of the quasi-algebra $\boldsymbol{A}$, which is given in the proof of theorem (3.8), it follows that the equivalence relation R_h in A induced by h is the least among all equivalence relations R induced by P-homomorphisms $\lambda \in P$-Hom$(\boldsymbol{A}, \boldsymbol{B})$, where $\boldsymbol{B} \in \mathfrak{B}$. Hence we obtain

$$R_h = \bigcap_{\boldsymbol{B} \in \mathfrak{B}} \bigcap_{\lambda \in P\text{-Hom}(\boldsymbol{A},\boldsymbol{B})} R_\lambda. \tag{8}$$

Defining

$$R_{\boldsymbol{B}} = \bigcap_{\lambda \in P\text{-Hom}(\boldsymbol{A},\boldsymbol{B})} R_\lambda,$$

we have

$$R_h = \bigcap_{\boldsymbol{B} \in \mathfrak{B}} R_{\boldsymbol{B}} = R_{\mathfrak{B}}.$$

The elements of $R_{\boldsymbol{B}}$ are called *$\boldsymbol{A}$-P-equations of quasi-algebra $\boldsymbol{B}$.* Obviously $R_{\boldsymbol{C}} = R_{\mathfrak{B}} = R_h$, where the pair $(h, \boldsymbol{C})$ is the free $\mathfrak{B}$-P-homomorphism of $\boldsymbol{A}$. Extending the notion introduced by Tarski [12] for algebras, the quasi-algebra $\boldsymbol{C}$ is *$\boldsymbol{A}$-P-functionally* or *$\boldsymbol{A}$-P-equationally free in the class $\mathfrak{B}$.* By (8), we obtain

(3.10) *The free $\mathfrak{B}$-P-homomorphism $(h, \boldsymbol{C})$ of $\boldsymbol{A}$ is a $\mathfrak{B}$-P-extension of $\boldsymbol{A}$ if and only if $R_h \subset \mathrm{id}_A$, i. e. if and only if*

$$(A \times A - \mathrm{id}_A) \subset \bigcup_{\boldsymbol{B} \in \mathfrak{B}} \; \bigcup_{\lambda \in P\text{-}\mathrm{Hom}(\boldsymbol{A}, \boldsymbol{B})} (A \times A - R_\lambda),$$

or, in other words, if and only if for any different elements $a, b \in A$ there exists a $\mathfrak{B}$-P-homomorphism $(h, \boldsymbol{B})$ of $\boldsymbol{A}$ such that $h(a) \neq h(b)$.

D. Free quasi-algebras in quasi-primitive classes. Let $\mathfrak{B}$ be an arbitrary class of quasi-algebras of the type F and let Y be any set. A quasi-algebra $\boldsymbol{B}$ is said to be *$\mathfrak{B}$-free freely generated by Y* if it has the following properties: 1. $\boldsymbol{B} \in \mathfrak{B}$, 2. the set Y is a set of generators for $\boldsymbol{B}$ and 3. for every quasi-algebra $\boldsymbol{B}_1 \in \mathfrak{B}$ and for every mapping $\varphi: Y \to B_1$ there exists exactly one homomorphism of $\boldsymbol{B}$ into $\boldsymbol{B}_1$ being an extension of φ. The $\mathfrak{B}$-free quasi-algebra freely generated by Y, if it exists, is uniquely determined up to isomorphisms by the cardinal number of the set Y. Now we prove the existence theorem

(3.11) *Let $\mathfrak{B}$ be an arbitrary non-trivial quasi-primitive class of quasi-algebras of the type F and let Y be an arbitrary set. Then there exists the $\mathfrak{B}$-free quasi-algebra freely generated by Y.*

Proof. Let us consider proper $P_{F,F}$-mapping P such that $P(f) = f$ for $f \in F$. Then the $\mathfrak{B}$-P-homomorphisms of a quasi-algebra $\boldsymbol{A}$ of the type F are called briefly $\mathfrak{B}$-homomorphisms of $\boldsymbol{A}$, since in this case the P-homomorphisms are ordinary homomorphisms. Let us assume that $\boldsymbol{A}$ is the *discrete* quasi-algebra of the type F, i. e. an abstract set (in discrete quasi-algebra $\boldsymbol{A}$ the partial operations $f_{\boldsymbol{A}}$ are empty for all $f \in F$) such that $|A| = |Y|$. Let $(h, \boldsymbol{C})$ be the free $\mathfrak{B}$-homomorphism of $\boldsymbol{A}$. It exists by theorem (3.8). Then the quasi-algebra $\boldsymbol{C}$ is $\mathfrak{B}$-free freely generated by the set $h(A)$ which may be considered as the set Y. Indeed, since $\boldsymbol{A}$ is discrete, then every mapping of A into any quasi-algebra $\boldsymbol{B}$ is a homomorphism of $\boldsymbol{A}$ into $\boldsymbol{B}$. But the class $\mathfrak{B}$ being non-trivial, it contains a quasi-algebra $\boldsymbol{B}$ with $|B| \geqslant 2$, and thus h is one-to-one by (3.10). Let $\boldsymbol{B}$ be an arbitrary quasi-algebra belonging to $\mathfrak{B}$ and let φ be any mapping of the set $h(A) = Y$ into B. The pair $(\varphi h, \boldsymbol{B})$ is a $\mathfrak{B}$-homomorphism of $\boldsymbol{A}$ and therefore there exists a homomorphism q of $\boldsymbol{C}$ into $\boldsymbol{B}$ such that $\varphi h = qh$ and is an extension of φ; thus the set $h(A)$ $= Y$ is $\mathfrak{B}$-free. The set $h(A) = Y$ generates the quasi-algebra $\boldsymbol{C}$, since the pair $\big(h, \overline{h(A)}\big)$ is a $\mathfrak{B}$-homomorphism of $\boldsymbol{A}$ such that $\big(h, \overline{h(A)}\big) \leqslant (h, \boldsymbol{C})$

with $h = i \cdot h$, where $\overline{h(A)}$ is the subquasi-algebra of $\boldsymbol{C}$ generated by $h(A)$ and i is the identity mapping of $\overline{h(A)}$ into C, and thus $(h, \overline{h(A)}) = (h, \boldsymbol{C})$, because $(h, \boldsymbol{C})$ is the free $\mathfrak{B}$-homomorphism of $\boldsymbol{A}$. Hence $\boldsymbol{C} = \overline{h(A)}$ and theorem (3.11) is proved.

Let us remark that the free $\mathfrak{B}$-P-homomorphism $(h, \boldsymbol{C})$ of an arbitrary quasi-algebra $\boldsymbol{A}$ of the type F is always almost onto, i. e. $\boldsymbol{C} = \overline{h(A)}$ for every quasi-primitive class $\mathfrak{B}$ of quasi-algebras of the type G and for any $P_{F,G}$-mapping P. The proof of this fact is the same as the proof of theorem (3.11).

E. P-homomorphisms of quasi-algebras into algebras. Let $\mathfrak{B}$ be an arbitrary *primitive* class of algebras of the type G, that is let $\mathfrak{B}$ be a quasi-primitive class of algebras of the type G closed with respect to homomorphic images. Let $\boldsymbol{A} = \langle A, (f_{\boldsymbol{A}}, f \in F)\rangle$ be any quasi-algebra of the type F and let P be an arbitrary $P_{F,G}$-mapping. The purpose of this section is to determine the form of all almost onto $\mathfrak{B}$-P-homomorphism of $\boldsymbol{A}$. For a solution of this problem, let us denote by $\boldsymbol{W} = \langle W, (g_{\boldsymbol{W}}, g \in G)\rangle$ the $\mathfrak{B}$-free algebra freely generated by A. Since $\mathfrak{B}$ is a primitive class, the algebra $\boldsymbol{W} = \mathrm{Free}(\mathfrak{B}, A)$ exists by (3.11). The $\mathfrak{B}$-P-homomorphisms of $\boldsymbol{A}$ are determined by the $\mathfrak{B}$-P-regularizers of $\boldsymbol{A}$ which are certain congruences of the algebra $\boldsymbol{W}$ (an equivalence relation $\sim$ in W is said to be a *congruence* of $\boldsymbol{W}$ if it preserves all operations $g_{\boldsymbol{W}}$ for $g \in G$, i. e. if it has the following property: for all $g \in G$ and for any elements a_ξ, b_ξ in W, where $\xi < n(g)$, the conditions $a_\xi \sim b_\xi$ for $\xi < n(g)$ imply the condition $g_{\boldsymbol{W}}(a_\xi, \xi < n(g)) \sim g_{\boldsymbol{W}}(b_\xi, \xi < n(g))$.

The following is the precise definition of $\mathfrak{B}$-P-regularizers of $\boldsymbol{A}$.

Definition (3.a). Let $\sim_0$ be the least congruence $\sim$ of the algebra $\boldsymbol{W}$ such that, for all $f \in F$, all sequences $(a_\xi, \xi < n(f)) \in A^{n(f)}$, and all $\tau \in P(f)$ we have

$$f_{\boldsymbol{A}}(a_\xi, \xi < n(f)) \sim \tau_{\boldsymbol{W}}(a_\xi, \xi < n(f)),$$

provided $f_{\boldsymbol{A}}$ is defined for $(a_\xi, \xi < n(f))$. The congruences $\sim$ of the algebra $\boldsymbol{W}$ with $\sim_0 \subset \sim$ are called the *$\mathfrak{B}$-P-regularizers* of $\boldsymbol{A}$. The relation $\sim_0$ is the minimal $\mathfrak{B}$-P-regularizer of $\boldsymbol{A}$. $\mathfrak{B}$-P-regularizer $\sim$ of $\boldsymbol{A}$ is said to be *proper* if the relation $a \sim b$ implies $a = b$ for all $a, b \in A$.

For every congruence $\sim$ of the algebra $\boldsymbol{W}$ by $\boldsymbol{W}/\!\sim\, = \langle W/\!\sim, (g_{\boldsymbol{W}/\sim}, g \in G)\rangle$ will be denoted the *quotient algebra* formed by dividing the algebra $\boldsymbol{W}$ by the congruence $\sim$, i. e. an algebra of the type G such that $W/\!\sim$ is the set of all abstraction classes $w/\!\sim$ of $\sim$, where $w \in W$ and $w/\!\sim$ is the set of all elements v in W such that $v \sim w$, and $g_{\boldsymbol{W}/\sim}$ is defined by the formula

$$g_{\boldsymbol{W}/\sim}(w_\xi/\!\sim, \xi < n(g)) = g_{\boldsymbol{W}}(w_\xi, \xi < n(g))/\!\sim.$$

The mapping $g_\sim : W \to W/\sim$ with $g_\sim(w) = w/\sim$ is the *natural homomorphism of algebra* $\boldsymbol{W}$ *onto quotient algebra* $\boldsymbol{W}/\sim$ induced by the congruence $\sim$. By $j_\sim$ will be denoted the restriction of $g_\sim$ to the set A, i.e. $j_\sim$ is a mapping of A into $W/\sim$ such that $j_\sim(a) = a/\sim$ for all $a \in A$. Using the definition (3.a) it is easy to verify analogically to my paper [10] the following theorem

(3.12) *For every* $\mathfrak{B}$*-*P*-regularizer* $\sim$ *of the quasi-algebra* $\boldsymbol{A}$ *the pair* $(j_\sim, \boldsymbol{W}/\sim)$ *is an almost onto* $\mathfrak{B}$*-*P*-homomorphism of* $\boldsymbol{A}$. *This pair is a* $\mathfrak{B}$*-*P*-extension of* $\boldsymbol{A}$ *if and only if the* $\mathfrak{B}$*-*P*-regularizer* $\sim$ *of* $\boldsymbol{A}$ *is proper.*

Proof. Using Definition (3.a) we have

$$\begin{aligned} j_\sim\big(f_{\boldsymbol{A}}(a_\xi,\ \xi < n(f))\big) &= f_{\boldsymbol{A}}(a_\xi,\ \xi < n(f))/\sim\ = \tau_{\boldsymbol{W}}(a_\xi,\ \xi < n(f))/\sim \\ &= \tau_{\boldsymbol{W}/\sim}(a_\xi/\sim,\ \xi < n(f)) = \tau_{\boldsymbol{W}/\sim}\big(j_\sim(a_\xi),\ \xi < n(f)\big), \end{aligned}$$

if $f \in F$ and $\tau \in P(f)$ and if $f_{\boldsymbol{A}}$ is defined for $(a_\xi,\ \xi < n(f)) \in A^{n(f)}$. Hence the pair $(j_\sim, \boldsymbol{W}/\sim)$ is a $\mathfrak{B}$-P-homomorphism of $\boldsymbol{A}$, since the class $\mathfrak{B}$ is, as a primitive one, closed with respect to homomorphic images and whence $\boldsymbol{W}/\sim\ \in \mathfrak{B}$. The set $\{j_\sim(a) = a/\sim,\ a \in A\}$ generates the algebra $\boldsymbol{W}/\sim$, since A generates the algebra $\boldsymbol{W}$, and thus the pair $(j_\sim, \boldsymbol{W}/\sim)$ is an almost onto $\mathfrak{B}$-P-homomorphism of $\boldsymbol{A}$. The mapping $j_\sim$ is one-to-one if and only if the $\mathfrak{B}$-P-regularizer $\sim$ of $\boldsymbol{A}$ is proper. Thus we have proved (3.12).

Now let $(h, \boldsymbol{B})$ be any $\mathfrak{B}$-P-homomorphism of the quasi-algebra $\boldsymbol{A}$. Since $\boldsymbol{W}$ is a $\mathfrak{B}$-free algebra freely generated by A, there exists exactly one homomorphism $\bar{h}$ of $\boldsymbol{W}$ into $\boldsymbol{B}$ with $\bar{h}|A = h$. By $\sim_h$ we shall denote the congruence of $\boldsymbol{W}$ induced by $\bar{h}$ (i. e. $w \sim_h v$ if and only if $\bar{h}(w) = \bar{h}(v)$). Hence we obtain the following theorem:

(3.13) *For every* $\mathfrak{B}$*-*P*-homomorphism* $(h, \boldsymbol{B})$ *of quasi-algebra* $\boldsymbol{A}$ *the congruence* $\sim_h$ *defined above is a* $\mathfrak{B}$*-*P*-regularizer of* $\boldsymbol{A}$ *and we have* $(j_{\sim h}, \boldsymbol{W}/\sim_h) \leqslant (h, \boldsymbol{B})$ *with* $h = i_h j_{\sim h}$, *where* i_h *is the natural isomorphism of* $\boldsymbol{W}/\sim_h$ *into* $\boldsymbol{B}$ *with* $i_h(v/\sim_h) = \bar{h}(v)$ *for* $v \in W$. *If* h *maps* $\boldsymbol{A}$ *almost onto* $\boldsymbol{B}$, *i. e. if* $h(A)$ *generates* $\boldsymbol{B}$, *then* $(j_{\sim h}, \boldsymbol{W}/\sim_h) \equiv (h, \boldsymbol{B})$. *The pair* $(h, \boldsymbol{B})$ *is a* $\mathfrak{B}$*-*P*-extension of* $\boldsymbol{A}$ *if and only if the* $\mathfrak{B}$*-*P*-regularizer* $\sim_h$ *is proper.*

Proof. Since $h = \bar{h}|A$, and h is a P-homomorphism, and $\bar{h}$ is a homomorphism, we have for $f \in F$ and $\tau \in P(f)$:

$$\begin{aligned} \bar{h}\big(f_{\boldsymbol{A}}(a_\xi,\ \xi < n(f))\big) &= h\big(f_{\boldsymbol{A}}(a_\xi,\ \xi < n(f))\big) = \tau_{\boldsymbol{B}}\big(h(a_\xi),\ \xi < n(f)\big) \\ &= \tau_{\boldsymbol{B}}\big(\bar{h}(a_\xi),\ \xi < n(f)\big) = \bar{h}\big(\tau_{\boldsymbol{W}}(a_\xi,\ \xi < n(f))\big); \end{aligned}$$

hence

$$f_{\boldsymbol{A}}(a_\xi,\ \xi < n(f)) \sim_h \tau_{\boldsymbol{W}}(a_\xi,\ \xi < n(f)),$$

provided $f_{\boldsymbol{A}}$ is defined for $(a_\xi,\ \xi < n(f))$; and thus $\sim_h$ is $\mathfrak{B}$-P-regularizer of $\boldsymbol{A}$. Obviously, $h = i_h g_{\sim h}$ and hence in view of $h = \bar{h}|A$ and $j_{\sim h} = g_{\sim h}|A$

we obtain $h = i_h j_{\sim h}$; so $(j_{\sim h}, \boldsymbol{W}/\sim_h) \leqslant (h, \boldsymbol{B})$. If h maps $\boldsymbol{A}$ almost onto $\boldsymbol{B}$, then i_h maps onto $\boldsymbol{B}$ and $(j_{\sim h}, \boldsymbol{W}/\sim_h) \equiv (h, \boldsymbol{B})$. Since i_h is one-to-one, h is one-to-one if and only if $j_{\sim h}$ is one-to-one. Hence by theorem (3.12), (h, B) is a $\mathfrak{B}$-P-extension of $\boldsymbol{A}$ if and only if the $\mathfrak{B}$-P-regularizer $\sim_h$ is proper. Thus theorem (3.13) is proved.

The relation $\sim_h$ and the isomorphism i_h defined above are called the *$\mathfrak{B}$-P-regularizer* and the *$\mathfrak{B}$-P-specialization of $\boldsymbol{A}$ induced by the $\mathfrak{B}$-P-homomorphism $(h, \boldsymbol{B})$ of $\boldsymbol{A}$.*

Theorems (3.12) and (3.13) easily imply

(3.14) *The pairs $(j_{\sim}, \boldsymbol{W}/\sim)$, where $\sim$ is an $\mathfrak{B}$-P-regularizer of $\boldsymbol{A}$, are all almost onto $\mathfrak{B}$-P-homomorphisms of $\boldsymbol{A}$, up to the relation $\equiv$.*

Proof. The pairs $(j_{\sim}, \boldsymbol{W}/\sim)$ are, by (3.12), almost onto $\mathfrak{B}$-P-homomorphisms of $\boldsymbol{A}$. If $(h, \boldsymbol{B})$ is an almost onto $\mathfrak{B}$-P-homomorphism of $\boldsymbol{A}$, then by (3.13) we have $(h, \boldsymbol{B}) \equiv (j_{\sim h}, \boldsymbol{W}/\sim_h)$, where $\sim_h$ is the $\mathfrak{B}$-P-regularizer of $\boldsymbol{A}$ induced by $(h, \boldsymbol{B})$, and thus theorem (3.14) is proved.

Let us observe that

(3.15) *For any two $\mathfrak{B}$-P-regularizers $\sim_1$ and $\sim_2$ of $\boldsymbol{A}$ we have:*

1° $(j_{\sim 1}, \boldsymbol{W}/\sim_1) \leqslant (j_{\sim 2}, \boldsymbol{W}/\sim_2)$ *if and only if* $\sim_1 \subset \sim_2$.

2° $(j_{\sim 1}, \boldsymbol{W}/\sim_1) \equiv (j_{\sim 2}, \boldsymbol{W}/\sim_2)$ *if and only if* $\sim_1 = \sim_2$.

Indeed, the mapping $j_{\sim 1 \sim 2}\colon W/\sim_1 \to W/\sim_2$ such that $j_{\sim 1\sim 2}(v/\sim_1)$, $= v/\sim_2$ is a homomorphism of $\boldsymbol{W}/\sim_1$ onto $\boldsymbol{W}/\sim_2$ if and only if $\sim_1 \subset \sim_2$.

The next theorem determines the form of the free $\mathfrak{B}$-P-homomorphism of $\boldsymbol{A}$.

(3.16) *Let $\sim_0$ be the minimal $\mathfrak{B}$-P-regularizer of a quasi-algebra $\boldsymbol{A}$. Then the pair $(j_{\sim 0}, \boldsymbol{W}/\sim_0)$ is the free $\mathfrak{B}$-P-homomorphism of $\boldsymbol{A}$.*

Proof. By theorem (3.12) the pair $(j_{\sim 0}, \boldsymbol{W}/\sim_0)$ is an almost onto $\mathfrak{B}$-P-homomorphism of $\boldsymbol{A}$. For each $\mathfrak{B}$-P-homomorphism $(h, \boldsymbol{B})$ of $\boldsymbol{A}$ we have, by theorems (3.14) and (3.15), $(j_{\sim 0}, \boldsymbol{W}/\sim_0) \leqslant (j_{\sim h}, \boldsymbol{W}/\sim_h) \leqslant (h, \boldsymbol{B})$; then $(j_{\sim 0}, \boldsymbol{W}/\sim_0)$ is the free $\mathfrak{B}$-P-homomorphism of $\boldsymbol{A}$, and thus theorem (3.16) is proved.

By (3.9), (3.16) and (3.13) we obtain immediately

(3.17) *The quasi-algebra $\boldsymbol{A}$ has a $\mathfrak{B}$-P-extension if and only if the minimal $\mathfrak{B}$-P-regularizer $\sim_0$ of $\boldsymbol{A}$ is proper.*

Let $(h_1, \boldsymbol{B}_1)$ and $(h_2, \boldsymbol{B}_2)$ be two $\mathfrak{B}$-P-homomorphisms of $\boldsymbol{A}$. By $\sim_1, i_1$ and $\sim_2, i_2$ we shall denote the $\mathfrak{B}$-P-regularizers and $\mathfrak{B}$-P-specializations of $\boldsymbol{A}$ induced by $(h_1, \boldsymbol{B}_1)$ and $(h_2, \boldsymbol{B}_2)$. Moreover, let $\boldsymbol{W}_1 = \boldsymbol{W}/\sim_1$ and $\boldsymbol{W}_2 = \boldsymbol{W}/\sim_2$. Then we have

(3.18) $(h_1, \boldsymbol{B}_1) \leqslant (h_2, \boldsymbol{B}_2)$ *if and only if* $\sim_1 \subset \sim_2$ *and the diagram*

$$\begin{tikzcd} \boldsymbol{W}_1 \arrow[r, "j_{\sim 1\sim 2}"] \arrow[d, "i_1"'] & \boldsymbol{W}_2 \arrow[d, "i_2"] \\ \boldsymbol{B}_1 & \boldsymbol{B}_2 \end{tikzcd}$$

may be completed to a commutative diagram

(*)
$$\begin{tikzcd} W_1 \arrow[r, dashed, "j_{\sim 1 \sim 2}"] \arrow[d, "i_1"'] & W_2 \arrow[d, "i_2"] \\ B_1 \arrow[r, "q"] & B_2 \end{tikzcd}.$$

If (h_1, B_1) *is almost onto, then* $(h_1, B_1) \leqslant (h_2, B_2)$ *if and only if* $\sim_1 \subset \sim_2$.

Proof. If $h_1 \leqslant h_2$, then also $\bar{h}_1 \leqslant \bar{h}_2$, where $\bar{h}_1$ and $\bar{h}_2$ are homomorphisms of W into B_1 such that $h_1 = \bar{h}_1 | A$ and $h_2 = \bar{h}_2 | A$. Hence $\sim_1 \subset \sim_2$, because $\sim_1$ and $\sim_2$ are the congruences of W induced by homomorphisms h_1 and h_2, and thus by theorems (3.13) and (3.15) the following diagram is commutative:

(**)
$$\begin{tikzcd} W_1 \arrow[rr, "j_{\sim 1 \sim 2}"] \arrow[dd, "i_1"'] & & W_2 \arrow[dd, "i_2"] \\ & A \arrow[dl, "h_1"'] \arrow[dr, "h_2"] & \\ B_1 \arrow[rr, "q"] & & B_2 \end{tikzcd}$$

where q is a homomorphism of B_1 into B_2 such that $\bar{h}_2 = q\bar{h}_1$.

If the diagram (*) is commutative, then by (3.13) diagram (**) is also commutative, and thus $(h_1, B_1) \leqslant (h_2, B_2)$. If (h_1, B_1) is almost onto and if $\sim_1 \subset \sim_2$, then, by (3.13) and (3.15), we have $(h_1, B_1) \equiv (j_{\sim 1}, W/\sim_1) \leqslant (j_{\sim 2}, W/\sim_2) \leqslant (h_2, B_2)$. Thus theorem (3.18) is proved.

If $F \subset G$ and P is such a proper $P_{F,G}$-mapping that $P(f) = f$ for all $f \epsilon F$, then from theorems (3.12), (3.13), (3.14), (3.15), (3.16), (3.17) and (3.18) we obtain Theorems 1, 2, 3, 4, 7, 8, and Theorems 5 and 6 contained in my paper [10].

F. The $\mathfrak{B}$-P-direct sums of quasi-algebras. Let $\mathfrak{B}$ be an arbitrary quasi-primitive class of quasi-algebras of the type G. Let T be any set and let A_t, $t \epsilon T$, be any family of quasi-algebras of the type F. Moreover, let P be an arbitrary $P_{F,G}$-mapping. Let $A = \underset{t \epsilon T}{\mathrm{S}} A_t$ be the direct sum of quasi-algebras A_t, $t \epsilon T$ (see § 1, section D, and § 2, the ending of introduction). By theorem (3.8) there exists the free $\mathfrak{B}$-P-homomorphism of A. Let (h, C) be the free $\mathfrak{B}$-P-homomorphism of A. The quasi-algebra C is called the *$\mathfrak{B}$-P-direct sum of quasi-algebras* A_t, $t \epsilon T$. We write $C = \mathfrak{B}\text{-}P \underset{t \epsilon T}{\mathrm{S}} A_t$. The $\mathfrak{B}$-P-direct sum of quasi-algebras A_t, $t \epsilon T$, is uniquely determined up to strong isomorphisms. Now we prove the following theorem:

(3.19) *Putting, for all* $t \in T$, $h_t = hi_t$, *where* i_t *is the natural injection of* A_t *into* $A = \mathop{\mathrm{S}}\limits_{t \in T} A_t$, *we obtain a family* $h_t, t \in T$, *of P-homomorphisms of* $\boldsymbol{A}_t$ *into* $\boldsymbol{C} = \mathfrak{B}\text{-}P \mathop{\mathrm{S}}\limits_{t \in T} \boldsymbol{A}_t$ *which has the following property:*

for each quasi-algebra $\boldsymbol{B} \in \mathfrak{B}$ *and each family of P-homomorphisms* $\chi_t, t \in T$, *of* $\boldsymbol{A}_t$ *into* $\boldsymbol{B}$, *there exists one and only one homomorphism* ψ *of* $\boldsymbol{C}$ *into* $\boldsymbol{B}$ *such that* $\chi_t = \psi h_t$ *for all* $t \in T$.

Proof. By theorem (3.3), the mappings $h_t, t \in T$, are P-homomorphisms. Let h' be the direct sum of P-homomorphisms $\chi_t, t \in T$ (see § 2, sec. B). Hence, by theorem (3.6), h' is the unique P-homomorphism of $\boldsymbol{A}$ into $\boldsymbol{B}$ such that $\chi_t = h' i_t$ for all $t \in T$. But the pair $(h', \boldsymbol{B})$ is an $\mathfrak{B}$-P-homomorphism of $\boldsymbol{A}$. Therefore $(h, \boldsymbol{C}) \leqslant (h', \boldsymbol{B})$, and thus there exists exactly one homomorphism ψ of $\boldsymbol{C}$ into $\boldsymbol{B}$ with $h' = \psi h$. Hence we have $\chi_t = h' i_t = \psi h i_t = \psi h_t$ for all $t \in T$, and theorem (3.19) is proved.

If h is one-to-one, then the $\mathfrak{B}$-P-direct sum $\boldsymbol{C}$ of quasi-algebras $\boldsymbol{A}_t$ is said to be proper (and in this case all $h_t, t \in T$, are one-to-one).

Hence and from (3.9) we obtain immediately theorem

(3.20) *The proper* $\mathfrak{B}$-P-*direct sum of quasi-algebras* $\boldsymbol{A}_t$, $t \in T$, *exists if and only if the direct sum* $\boldsymbol{A}$ *of quasi-algebras* $A_t, t \in T$, *has a* $\mathfrak{B}$-P-*extension.*

The pairs $\langle \{\chi_t\}_{t \in T}, \boldsymbol{B} \rangle$, where $\boldsymbol{B} \in \mathfrak{B}$ and $\{\chi_t\}_{t \in T}$ is a family of the type T of P-homomorphisms χ_t of quasi-algebras $\boldsymbol{A}_t$ into quasi-algebra $\boldsymbol{B}$, are called *common* $\mathfrak{B}$-P-homomorphisms of quasi-algebras $\boldsymbol{A}_t, t \in T$. If, moreover, all $\chi_t, t \in T$, are one-to-one, then this pair is said to be a *common* $\mathfrak{B}$-P-*extension* of quasi-algebras $\boldsymbol{A}_t, t \in T$. Let $H' = \langle \{h_t'\}_{t \in T}, \boldsymbol{B}' \rangle$ and $H'' = \langle \{h_t''\}_{t \in T}, \boldsymbol{B}'' \rangle$ be two common $\mathfrak{B}$-P-homomorphisms of quasi-algebras $\boldsymbol{A}_t, t \in T$. We say that:

1. $H' \leqslant H''$ if and only if there exists exactly one homomorphism q of $\boldsymbol{B}'$ into $\boldsymbol{B}''$ such that $h_t'' = q \cdot h_t'$ for all $t \in T$.

2. $H' \equiv H''$ if and only if there exists exactly one strong isomorphism q of $\boldsymbol{B}'$ onto $\boldsymbol{B}''$ with $h_t'' = q \cdot h_t'$ for all $t \in T$.

A common $\mathfrak{B}$-P-homomorphism H of quasi-algebras $\boldsymbol{A}_t, t \in T$, is called *free* if, for each common $\mathfrak{B}$-P-homomorphism H' of quasi-algebras $\boldsymbol{A}_t$, $t \in T$, we have $H \leqslant H'$. The free common $\mathfrak{B}$-P-homomorphism of $\boldsymbol{A}_t, t \in T$, is uniquely determined up to relation $\equiv$.

(3.21) *Let* $\mathfrak{B}$ *be any quasi-primitive class of quasi-algebras of the type* G *and let* $\boldsymbol{A}_t, t \in T$, *be an arbitrary family of quasi-algebras of the type* F. *Moreover, let* P *be any* $P_{F,G}$-*mapping. Then there exists the free common* $\mathfrak{B}$-P-*homomorphism of quasi-algebras* $\boldsymbol{A}_t, t \in T$.

Proof. Let $(h, \boldsymbol{C})$ be the free $\mathfrak{B}$-P-homomorphism of the direct sum $\boldsymbol{A}$ of quasi-algebras $\boldsymbol{A}_t, t \in T$, which exists by theorem (3.8). The quasi-algebra $\boldsymbol{C}$ is the $\mathfrak{B}$-P-direct sum of quasi-algebras $\boldsymbol{A}_t, t \in T$. By theorem (3.19) the pair $H = \langle \{h_t\}_{t \in T}, \boldsymbol{C} \rangle$, where $h_t = h \cdot i_t$, is the free common $\mathfrak{B}$-P-homomorphism of quasi-algebras $\boldsymbol{A}_t, t \in T$, and thus theorem (3.21) is proved. From (3.20) results

(3.22) *The proper $\mathfrak{B}$-P-direct sum of quasi-algebras $\boldsymbol{A}_t, t \in T$, exists if and only if there exists a common $\mathfrak{B}$-P-extension of $\boldsymbol{A}_t, t \in T$.*

Theorem (3.22) may be considered as a generalization of Theorem VIII in Sikorski's paper [8].

G. $\mathfrak{B}$-P-bilinears and $\mathfrak{B}$-P-tensor products of quasi-algebras. Let $\boldsymbol{A} = \langle A, (f_{\boldsymbol{A}}, f \in F) \rangle$ and $\boldsymbol{B} = \langle B, (f_{\boldsymbol{B}}, f \in F) \rangle$ be two quasi-algebras of the type F and let $\boldsymbol{C} = \langle C, (g_{\boldsymbol{C}}, g \in G) \rangle$ be any quasi-algebra of the type G. Moreover, let P be an arbitrary $P_{F,G}$-mapping. A mapping φ of cartesian product $A \times B$ into C is called a *P-bilinear of $\boldsymbol{A}$* and *$\boldsymbol{B}$ into $\boldsymbol{C}$* if, for all $f \in F$, all elements $(a_\xi, \xi < n(f))$ belonging to the domain of $f_{\boldsymbol{A}}$, all elements $(b_\xi, \xi < n(f))$ belonging to the domain of $f_{\boldsymbol{B}}$, all $a \in A$, all $b \in B$, and for all $\tau \in P(f)$, we have:

$$1^\circ \quad \varphi\big(\langle f_{\boldsymbol{A}}(a_\xi, \xi < n(f)), b \rangle\big) = \tau_{\boldsymbol{C}}\big(\varphi(\langle a_\xi, b \rangle), \xi < n(f)\big),$$

$$2^\circ \quad \varphi\big(\langle a, f_{\boldsymbol{B}}(b_\xi, \xi < n(f)) \rangle\big) = \tau_{\boldsymbol{C}}\big(\varphi(\langle a, b_\xi \rangle), \xi < n(f)\big).$$

If the left-hand sides of 1° and 2° exist, then the right-hand sides of 1° and 2° exist according to definition of a P-bilinear. Let $\mathfrak{B}$ be any class of quasi-algebras of the type G. Any pair $(\varphi, \boldsymbol{C})$, where $\boldsymbol{C} \in \mathfrak{B}$ and φ is a P-bilinear of $\boldsymbol{A}$ and $\boldsymbol{B}$ into $\boldsymbol{C}$ is said to be an *$\mathfrak{B}$-P-bilinear* of $\boldsymbol{A}$ and $\boldsymbol{B}$. Let $\boldsymbol{D}$ be the quasi-tensor product of quasi-algebras $\boldsymbol{A}$ and $\boldsymbol{B}$ (see § 1, sec. D, and § 2, the ending of introduction). If $(\varphi, \boldsymbol{C})$ is a $\mathfrak{B}$-P-bilinear of $\boldsymbol{A}$ and $\boldsymbol{B}$, then $(\varphi, \boldsymbol{C})$ is a $\mathfrak{B}$-P-homomorphism of $\boldsymbol{D}$. But the converse is not always true and we have only the following theorem:

(3.23) *A $\mathfrak{B}$-P-homomorphism $(\varphi, \boldsymbol{C})$ of the quasi-tensor product $\boldsymbol{D}$ of quasi-algebras $\boldsymbol{A}$ and $\boldsymbol{B}$ is a $\mathfrak{B}$-P-bilinear of $\boldsymbol{A}$ and $\boldsymbol{B}$ if and only if, for all $f \in F$, all $\tau \in P(f)$, all $a \in A$, and all $b \in B$, we have*

$$1' \quad \varphi\big(\langle f_{\boldsymbol{A}}(a, a, \ldots,), b \rangle\big) = \tau_{\boldsymbol{C}}\big(\varphi(\langle a, b \rangle), \varphi(\langle a, b \rangle), \ldots\big),$$

$$2' \quad \varphi\big(\langle a, f_{\boldsymbol{B}}(b, b, \ldots) \rangle\big) = \tau_{\boldsymbol{C}}\big(\varphi(\langle a, b \rangle), \varphi(\langle a, b \rangle, \ldots\big),$$

provided the left-hand sides of $1'$ and $2'$ exist.

The $\mathfrak{B}$-P-bilinears of $\boldsymbol{A}$ and $\boldsymbol{B}$ are the $\mathfrak{B}$-P-homomorphisms of the quasi-tensor product $\boldsymbol{D}$ of $\boldsymbol{A}$ and $\boldsymbol{B}$, which fulfill the relations $1'$ and $2'$. A $\mathfrak{B}$-P-bilinear $(\varphi, \boldsymbol{C})$ of $\boldsymbol{A}$ and $\boldsymbol{B}$ such that for every $\mathfrak{B}$-P-bilinear $(\varphi', \boldsymbol{C}')$ of $\boldsymbol{A}$ and $\boldsymbol{B}$ we have $(\varphi, \boldsymbol{C}) \leqslant (\varphi', \boldsymbol{C}')$ is called *free* [4].

If $(\varphi, \boldsymbol{C})$ is the free $\mathfrak{B}$-P-bilinear of $\boldsymbol{A}$ and $\boldsymbol{B}$, then the quasi-algebra $\boldsymbol{C}$ is said to be the *$\mathfrak{B}$-P-tensor product of $\boldsymbol{A}$ and $\boldsymbol{B}$*. The free $\mathfrak{B}$-P-bilinear

[4] The relation $(\varphi, \boldsymbol{C}) \leqslant (\varphi, \boldsymbol{C}')$ between two $\mathfrak{B}$-P-bilinears of $\boldsymbol{A}$ and $\boldsymbol{B}$ is understood as the relation $\leqslant$ between the $\mathfrak{B}$-P-homomorphisms of the quasi-tensor product of $\boldsymbol{A}$ and $\boldsymbol{B}$.

and the $\mathfrak{B}$-P-tensor product of A and B (if they exist) are uniquely determined up to relation $\equiv$ and strong isomorphisms, respectively. Let us observe that

(3.24) *The direct product of P-bilinears of* A *and* B *is also a P-bilinear of* A *and* B. *Let* φ_t, $t \in T$, *be any P-bilinear of* A *and* B *into* C_t. *Then the direct product of* φ_t, $t \in T$, *i. e. the mapping* $\varphi: A \times B \to C$, *where* $C = \mathop{\mathrm{P}}_{t \in T} C_t$ *is the direct product of* C_t *such that for all* $a \in A$, $b \in B$ *and all* $t \in T$ *we have* $\varphi(\langle a, b\rangle)(t) = \varphi_t(\langle a, b\rangle)$, *is the unique P-bilinear of* A *and* B *into* $C = \mathop{\mathrm{P}}_{t \in T} C_t$ *with* $\varphi_t = p_t\varphi$ *for all* $t \in T$, *where* p_t *is the natural projection of* C *onto* C_t.

Now we prove an existence theorem:

(3.25) *Let* $\mathfrak{B}$ *be an arbitrary quasi-primitive class of quasi-algebras of the type* G *and let* A *and* B *be any quasi-algebras of the type* F. *Moreover, let* P *be an arbitrary* $P_{F,G}$*-mapping. Then there exist the free* $\mathfrak{B}$-P*-bilinear of* A *and* B *and the* $\mathfrak{B}$-P*-tensor product of* A *and* B.

Proof. By theorem (2.3) there exists a cardinal number $\overline{\mathfrak{m}}$ such that $|C| \leqslant \overline{\mathfrak{m}}$ for every quasi-algebra C of type G generated by a set M with $|M| \leqslant |A \times B|$. Let E be any set with $|E| \geqslant \overline{\mathfrak{m}}$. Let C be any quasi-algebra of the type G with $C \subset E$, where C is the set of C. We denote by P-$H(A, B, C)$ the set of all P-bilinears of A and B into C. Let φ_C be the direct product of all P-bilinears $\lambda \in P$-$H(A, B, C)$. By (3.24), φ_C is the unique P-bilinear of A and B into $C^{P\text{-}H(A,B,C)}$ such that $\lambda = p_\lambda\varphi_C$ for all $\lambda \in P$-$H(A, B, C)$, where p_λ are the natural projections. Now let φ be the direct product of all P-bilinears φ_C, where $C \in \mathfrak{B}$ and $C \subset E$. The mapping φ is the unique P-bilinear of A and B into $R = PC^{P\text{-}H(A,B,C)}$, where $C \in \mathfrak{B}$ and $C \subset E$, with $q_C\varphi = \varphi_C$ for all $C \in \mathfrak{B}$ with $C \subset E$, where q_C are the natural projections. Let us denote by $U = \overline{\varphi(A \times B)}$ the subquasi-algebra of R generated by the image of φ. Then the pair (φ, U) is the free $\mathfrak{B}$-P-bilinear of A and B, and U is the $\mathfrak{B}$-P-tensor product of A and B. Indeed, since $\mathfrak{B}$ is quasi-primitive, $U \in \mathfrak{B}$ and thus (φ, U) is an $\mathfrak{B}$-P-bilinear of A and B. Now let (ψ, V) be any $\mathfrak{B}$-P-bilinear of A and B. Let us denote by $V_0 = \overline{\psi(A \times B)}$ the subquasi-algebra of V generated by the image of ψ. Obviously $|V_0| \leqslant \overline{\mathfrak{m}}$, because $|\psi(A \times B)| \leqslant |A \times B|$. Hence it follows that there exists a quasi-algebra $C \in \mathfrak{B}$ with $C \subset E$ strongly isomorphic to V_0. Let i be the strong isomorphism of C onto V_0. Then we have $\psi = q\varphi$, where $q = i \cdot p_\lambda q_C | U$ with $\lambda = i^{-1}\psi$, and thus $(\varphi, U) \leqslant (\psi, V)$. Therefore we have proved that the pair (φ, U) is the free $\mathfrak{B}$-P-bilinear of A and B, and thus also that U is the $\mathfrak{B}$-P-tensor product of A and B. Theorem (3.25) is proved.

Let us assume that $\mathfrak{B}$ is an arbitrary primitive class of algebras of the type G. Let $\boldsymbol{A} = \langle A, (f_{\boldsymbol{A}}, f \epsilon F)\rangle$ and $\boldsymbol{B} = \langle B, (f_{\boldsymbol{B}}, f \epsilon F)\rangle$ be any two quasi-algebras of the type F. Moreover, let P be an arbitrary $P_{F,G}$-mapping. In this special case the free $\mathfrak{B}$-P-bilinear and the $\mathfrak{B}$-P-tensor product of $\boldsymbol{A}$ and $\boldsymbol{B}$ may be obtained by the use of the algebra $\boldsymbol{W} =$ Free $(\mathfrak{B}, A \times B)$, $\mathfrak{B}$-free freely generated by the set $A \times B$. Indeed, let $\sim_0$ be the least congruence $\sim$ of the algebra $\boldsymbol{W}$ such that for all $f \epsilon F$, all $\tau \epsilon P(f)$, all sequences $(a_\xi, \xi < n(f)) \epsilon A^{n(f)}$, $(b_\xi, \xi < n(f)) \epsilon B^{n(f)}$, all $a \epsilon A$, and all $b \epsilon B$ we have:

1. $\langle f_{\boldsymbol{A}}(a_\xi, \xi < n(f)), b\rangle \sim \tau_{\boldsymbol{W}}(\langle a_\xi, b\rangle, \xi < n(f))$,
2. $\langle a, f_{\boldsymbol{B}}(b_\xi, \xi < n(f))\rangle \sim \tau_{\boldsymbol{W}}(\langle a, b_\xi\rangle, \xi < n(f))$

provided the left-hand sides exist. Now let us denote by $\boldsymbol{W}/{\sim_0}$ the quotient algebra formed by dividing $\boldsymbol{W}$ by the congruence $\sim_0$, and by φ_0 the natural mapping such that $\varphi_0(\langle a, b\rangle) = \langle a, b\rangle/{\sim_0}$ for all $a \epsilon A$ and all $b \epsilon B$. Then the following theorem may be proved:

(3.26) *The pair* $(\varphi_0, \boldsymbol{W}/{\sim_0})$ *is the free* $\mathfrak{B}$-P-*bilinear of* $\boldsymbol{A}$ *and* $\boldsymbol{B}$ *and the algebra* $\boldsymbol{W}/{\sim_0}$ *is the* $\mathfrak{B}$-P-*tensor product of* $\boldsymbol{A}$ *and* $\boldsymbol{B}$.

Proof. From the definition easily follows that the pair $(\varphi_0, \boldsymbol{W}/{\sim_0})$ is a $\mathfrak{B}$-P-bilinear of $\boldsymbol{A}$ and $\boldsymbol{B}$. Now let $(\psi, \boldsymbol{C})$ be an arbitrary $\mathfrak{B}$-P-bilinear of $\boldsymbol{A}$ and $\boldsymbol{B}$. Let us denote by h a homomorphism of the algebra $\boldsymbol{W}$ into $\boldsymbol{C}$ such that $h(\langle a, b\rangle) = \psi(\langle a, b\rangle)$ for all $\langle a, b\rangle \epsilon A \times B$, i. e. h is an extension of ψ. The congruence $\sim_h$ of $\boldsymbol{W}$ induced by h fulfils the relations 1 and 2, i. e. $\sim_0 \subset \sim_h$. Hence it results that the mapping q: $W/{\sim_0} \to C$ such that $q(v/{\sim_0}) = h(v)$ for $v \epsilon W$ is a homomorphism of $\boldsymbol{W}/{\sim_0}$ into $\boldsymbol{C}$ with $\psi = q\varphi_0$, and thus $(\varphi_0, \boldsymbol{W}/{\sim_0}) \leqslant (\psi, \boldsymbol{C})$. Therefore $(\varphi_0, \boldsymbol{W}/{\sim_0})$ is the free $\mathfrak{B}$-P-bilinear of $\boldsymbol{A}$ and $\boldsymbol{B}$ and $\boldsymbol{W}/{\sim_0}$ is the $\mathfrak{B}$-P-tensor product of $\boldsymbol{A}$ and $\boldsymbol{B}$. Thus theorem (3.26) is proved.

H. P-independence in algebras. Now let us consider the notion of independence with respect to P-homomorphisms, i. e. P-independence.

Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be any quasi-algebras of the type F and G, respectively, and let P be an arbitrary $P_{F,G}$-mapping. A subset $M \subset A$ is said to be $\boldsymbol{B}$-P-*independent* (or $\boldsymbol{B}$-P-*free*) if each mapping of M into B can be extended to a P-homomorphism of the subquasi-algebra $\overline{\boldsymbol{M}}$ of $\boldsymbol{A}$ generated by M into $\boldsymbol{B}$. Now we prove the following theorems.

(3.27) *The class* P-ind*M *of all quasi-algebras* $\boldsymbol{B}$ *of the type* G *such that* M *is* $\boldsymbol{B}$-P-*independent and the class* P-ind M *of all algebras* $\boldsymbol{B}$ *of the type* G *for which* M *is* $\boldsymbol{B}$-P-*independent are primitive* (*i. e. quasi primitive closed with respect to homomorphic images*) *for any quasi-algebra* $\boldsymbol{A}$ *of the type* F *and any subset* M *of* A.

Proof. Let $\boldsymbol{B} \epsilon P$-ind*$M$ (resp. $\boldsymbol{B} \epsilon P$-ind M). Then obviously we have $\boldsymbol{B}_0 \epsilon P$-ind*$M$ (resp. $\boldsymbol{B}_0 \epsilon P$-ind M) for any subquasi-algebra $\boldsymbol{B}_0$ of $\boldsymbol{B}$. Let h be a homomorphism of $\boldsymbol{B}$ onto $\boldsymbol{C}$ and let φ be any mapping of M

into C. Then there exists a mapping ψ of M into B such that $\varphi = h\psi$. Let h_ψ be the P-homomorphism of $\bar{M}$ into $\boldsymbol{B}$ being an extension of ψ. Defining $h_\varphi = h \cdot h_\psi$ we obtain, by (3.2), a P-homomorphism of $\bar{M}$ into $\boldsymbol{C}$ being an extension of φ, and thus $\boldsymbol{C} \in P\text{-ind}^* M$ (resp. $\boldsymbol{C} \in P\text{-ind} M$). Let T be any set and let $\boldsymbol{B}_t \in P\text{-ind}^* M$ (resp. $\boldsymbol{B}_t \in P\text{-ind} M$) for $t \in T$. Let us consider the quasi-algebra (resp. algebra) $\boldsymbol{B} = \mathop{\mathrm{P}}_{t \in T} \boldsymbol{B}_t$ being the direct product of $\boldsymbol{B}_t$, $t \in T$. Let φ be any mapping of M into B. Let us denote by φ_t, $t \in T$, the mapping of M into B_t being the projection of φ, i. e. $\varphi_t = p_t \varphi$. Let h_t, $t \in T$, be the P-homomorphism of $\bar{M}$ into $\boldsymbol{B}_t$ being an extension of φ_t and let h be the direct product of P-homomorphisms h_t, $t \in T$. By theorem (3.4), h is a P-homomorphism of $\bar{M}$ into $\boldsymbol{B}$, and, obviously, h is an extension of φ. Therefore $\boldsymbol{B} \in P\text{-ind}^* M$ (resp. $\boldsymbol{B} \in P\text{-ind}\, M$ and theorem (3.27) is proved.

(3.28) *A subset M of a quasi-algebra $\boldsymbol{A}$ of the type F is $\boldsymbol{B}$-P-independent, where $\boldsymbol{B}$ is any quasi-algebra of the type G and P is any proper $P_{F,G}$-mapping, if and only if M is $P(\boldsymbol{B})$-independent (i. e. independent with respect to ordinary homomorphisms in a quasi-algebra $P(\boldsymbol{B})$ similar to $\boldsymbol{A}$), where $P(\boldsymbol{B})$ is the P-quasi-algebra over $\boldsymbol{B}$.*

Proof. This follows from (3.1).

(3.29) *If M is an absolutely free of the type F subset of an algebra $\boldsymbol{A}$ of the type F (i. e. M is $\boldsymbol{D}$-independent for all algebra $\boldsymbol{D}$ of the type F), then M is $\boldsymbol{B}$-P-independent for each algebra $\boldsymbol{B}$ of the type G and each proper $P_{F,G}$-mapping P.*

Proof. This results immediately from (3.28).

In the sequel we shall consider the P-independence with respect to a proper $P_{F,G}$-mapping and for algebras only. Let us assume that P is an arbitrary proper $P_{F,G}$-mapping. By (3.29) the identity mapping $x \to x$ for $x \in X$ can be extended to the unique P-homomorphism h_P of the absolutely free algebra $\boldsymbol{F}^* = \mathrm{Free}(F, X)$ of the type F freely generated by X (i. e. $\boldsymbol{F}^*$ is the Peano-algebra of the type F generated by X; see § 2, sec. C) into the absolutely free algebra $\boldsymbol{G}^* = \mathrm{Free}(G, X)$ of the type G freely generated by X ($\boldsymbol{G}^*$ is the Peano-algebra of the type G generated by X). The elements of $\boldsymbol{F}^*$ and $\boldsymbol{G}^*$ will be considered as F-terms and G-terms respectively. The elements of X will be considered as variables. In the following theorems (3.31), (3.32) and (3.33) we shall assume that $|X| \geqslant \bar{\varrho}$, where ϱ is the rank of F. For all F-terms $\tau \in F^*$ we put $P(\tau) = h_P(\tau)$. Moreover, according to (2.11), for every F-term τ, τ' will denote the F-term obtained from τ by the substitution x_{β_ξ} by x_ξ for $\xi < n(\tau)$, where $\left(x_{\beta_\xi}, \xi < n(\tau)\right)$ is the support of τ. Let us observe that $P(\tau')' = P(\tau')$.

(3.30) *Let h be any P-homomorphism of an algebra $\boldsymbol{D}$ of the type F into an algebra $\boldsymbol{B}$ of the type G. Then for all $\tau \in F^*$ we have:*

1° $h(\tau_{\boldsymbol{D}}(m_\xi, \xi < k)) = P(\tau)_{\boldsymbol{B}}(h(m_\xi), \xi < k)$ *for* $k \geqslant n(\tau)$ *and all sequences* $(m_\xi, \xi < k) \in D^k$,

2° $h({}_{\boldsymbol{D}}\tau(\varphi)) = {}_{\boldsymbol{B}}P(\tau)(h(\varphi)$ *for all* $\varphi \in D^X$.

Proof. By (3.1) h is an ordinary homomorphism of $\boldsymbol{D}$ into $P(\boldsymbol{B})$, where $P(\boldsymbol{B})$ is a P-algebra over $\boldsymbol{B}$. From the definition of a P-algebra it follows that $\tau_{P(\boldsymbol{B})} = P(\tau)_{\boldsymbol{B}}$ for all F-terms $\tau \in F^*$. Hence, using (2.10), we obtain $h(\tau_{\boldsymbol{D}}(m_\xi, \xi < k)) = \tau_{P(\boldsymbol{B})}(h(m_\xi), \xi < k) = P(\tau)_{\boldsymbol{B}}(h(m_\xi), \xi < k)$. Similarly, we can obtain the relation 2°, and thus (3.30) is proved.

The next theorem gives the necessary and sufficient conditions for the $\boldsymbol{B}$-P-independence of a subset M of $\boldsymbol{A}$.

(3.31) *A subset M of an algebra $\boldsymbol{A}$ of the type F is $\boldsymbol{B}$-P-independent, where $\boldsymbol{B}$ is any algebra of the type G, if and only if for all F-terms $\tau, \vartheta \in F^*$ and every mapping $\varphi : X \to M$ such that φ is one-to-one on the set $X_\tau \cup X_\vartheta$, where X_τ and X_ϑ are the supports of terms τ and ϑ, the equality ${}_{\boldsymbol{A}}\tau(\varphi) = {}_{\boldsymbol{A}}\vartheta(\varphi)$ implies the validity of the G-equation $\ulcorner P(\tau) = P(\vartheta)\urcorner$ in the algebra $\boldsymbol{B}$ (i. e. ${}_{\boldsymbol{B}}P(\tau) = {}_{\boldsymbol{B}}P(\vartheta)$).*

(3.32) *A subset M of an algebra $\boldsymbol{A}$ of the type F is $\boldsymbol{B}$-P-independent, where $\boldsymbol{B}$ is any algebra of the type G, if and only if for all F-terms $\tau, \vartheta \in F^*$ and all different elements $m_\xi \in M$, $\xi < k$, where $k \geqslant n(\vartheta) \geqslant n(\tau)$, the equality $\tau_{\boldsymbol{A}}(m_\xi, \xi < k) = \vartheta_{\boldsymbol{A}}(m_\xi, \xi < k)$ implies the validity of the G-equation $\ulcorner P(\tau') = P(\vartheta')\urcorner$ in the algebra $\boldsymbol{B}$ (resp. $P(\tau')_{\boldsymbol{B}} = P(\vartheta')_{\boldsymbol{B}}$).*

(3.33) *A subset $M = (m_0, m_1, \ldots, m_\xi, \ldots, \xi < k)$, where m_ξ are different elements of an algebra $\boldsymbol{A}$ of the type F, is $\boldsymbol{B}$-P-independent, where $\boldsymbol{B}$ is any algebra of the type G, if and only if for all F-terms $\tau, \vartheta \in F^*$ with $n(\tau) \leqslant n(\vartheta) \leqslant k$ the equality $\tau_{\boldsymbol{A}}(m_\xi, \xi < k) = \vartheta_{\boldsymbol{A}}(m_\xi, \xi < k)$ implies the validity of the G-equation $\ulcorner P(\tau') = P(\vartheta')\urcorner$ in the algebra $\boldsymbol{B}$ (resp. $P(\tau')_{\boldsymbol{B}} = P(\vartheta')_{\boldsymbol{B}}$).*

Proof of theorems (3.31)-(3.33). Let us assume that M is $\boldsymbol{B}$-P-independent.

The necessity of (3.31). Let ${}_{\boldsymbol{A}}\tau(\varphi) = {}_{\boldsymbol{A}}\vartheta(\varphi)$. Hence, by the definition, $\bar{\varphi}(\tau) = \bar{\varphi}(\vartheta)$, where $\bar{\varphi}$ is the homomorphism of $\boldsymbol{F}^* = \mathrm{Free}(F, X)$ into $\boldsymbol{A}$ which is an extension of φ. Let $\psi : X \to B$ be any mapping and let ψ_0 be a mapping such that the following diagram is commutative:

(1)
$$\begin{tikzcd} X \arrow[rr, "\varphi"] \arrow[dr, "\psi"'] & & M \arrow[dl, "\psi_0"] \\ & B & \end{tikzcd} .$$

Hence it follows that the next diagram is commutative:

(2)
$$\begin{tikzcd} \boldsymbol{F}^* \arrow[r, "h_p"] \arrow[d, "\bar{\varphi}"'] & \boldsymbol{G}^* \arrow[d, "\bar{\psi}"] \\ \boldsymbol{A} \arrow[r, "\bar{\psi}_0"'] & \boldsymbol{B} \end{tikzcd}$$

where $\bar{\psi}$ and $\bar{\psi}_0$ are the homomorphism of G^* into B and the P-homomorphism of A into B being extensions of ψ and ψ_0, respectively. Using the diagram (2) and the equality $\bar{\varphi}(\tau) = \bar{\varphi}(\vartheta)$ we have

$$\begin{aligned} {}_{\boldsymbol{B}}P(\tau)(\psi) &= \bar{\psi}(P(\tau)) = \bar{\psi}(h_P(\tau)) = \bar{\psi}_0\bar{\varphi}(\tau) = \bar{\psi}_0\bar{\varphi}(\vartheta) = \bar{\psi}(h_P(\vartheta)) \\ &= \bar{\psi}(P(\vartheta)) = {}_{\boldsymbol{B}}P(\vartheta)(\psi), \end{aligned}$$

and, therefore ${}_{\boldsymbol{B}}P(\tau) = {}_{\boldsymbol{B}}P(\vartheta)$, i. e. the G-equation $\ulcorner P(\tau) = P(\vartheta)\urcorner$ is valid in the algebra $\boldsymbol{B}$. For the existence of the commutative diagram (1) we always replace ψ by a mapping ψ_1 as follows.

Let $\psi: X \to B$ be any mapping and let $\psi_0 : M \to B$ be a mapping such that $\psi_0\varphi(x) = \psi(x)$ for all $x \in X_\tau \cup X_\vartheta$. If we define $\psi_1 = \psi_0\varphi$, then $\psi_1(x) = \psi(x)$ for all $x \in X_\tau \cup X_\vartheta$ and the diagram

(3)
$$\begin{array}{ccc} \boldsymbol{F}^* & \xrightarrow{h_P} & \boldsymbol{G}^* \\ {\scriptstyle\bar{\varphi}}\downarrow & & \downarrow{\scriptstyle\bar{\psi}_1} \\ \boldsymbol{A} & \xrightarrow[\bar{\psi}_0]{} & \boldsymbol{B} \end{array}$$

is commutative, where $\bar{\psi}_0$ and $\bar{\psi}_1$ are the homomorphism of G^* into $\boldsymbol{B}$ and the P-homomorphism of $\boldsymbol{A}$ into $\boldsymbol{B}$ being extensions of ψ_0 and ψ_1 respectively. The mappings ψ and ψ_1 coincide on supports of τ and ϑ, then ${}_{\boldsymbol{B}}P(\tau)(\psi) = {}_{\boldsymbol{B}}P(\tau)(\psi_1)$ and ${}_{\boldsymbol{B}}P(\vartheta)(\psi) = {}_{\boldsymbol{B}}P(\vartheta)(\psi_1)$. Using the diagram (3) and the equality $\bar{\varphi}(\tau) = \bar{\varphi}(\vartheta)$ we have

$$\begin{aligned} {}_{\boldsymbol{B}}P(\tau)(\psi) &= {}_{\boldsymbol{B}}P(\tau)(\psi_1) = \bar{\psi}_1(P(\tau)) = \bar{\psi}_1(h_P(\tau)) = \bar{\psi}_0\bar{\varphi}(\tau) = \bar{\psi}_0\bar{\varphi}(\vartheta) \\ &= \bar{\psi}_1(h_P(\vartheta)) = \bar{\psi}_1(P(\vartheta)) = {}_{\boldsymbol{B}}P(\vartheta)(\psi_1) = {}_{\boldsymbol{B}}P(\vartheta)(\psi) \end{aligned}$$

and therefore ${}_{\boldsymbol{B}}P(\tau) = {}_{\boldsymbol{B}}P(\vartheta)$, i. e. the G-equation $\ulcorner P(\tau) = P(\vartheta)\urcorner$ is valid in the algebra $\boldsymbol{B}$.

The necessity of (3.32) and (3.33). Let the equality $\tau_{\boldsymbol{A}}(m_\xi, \xi < k) = \vartheta_{\boldsymbol{A}}(m_\xi, \xi < k)$ hold. Then by (2.12) we have $\tau'_{\boldsymbol{A}}(m_\xi, \xi < k) = \vartheta'_{\boldsymbol{A}}(m_\xi, \xi < k)$, i. e. (by the definition of $\tau'_{\boldsymbol{A}}$ and $\vartheta'_{\boldsymbol{A}}$; see § 2, sec. D) $\bar{\varphi}(\tau') = \bar{\varphi}(\vartheta')$, where φ is a mapping of X into M such that $\varphi(x_\xi) = m_\xi$ for $\xi < n(\vartheta')$, and $\bar{\varphi}$ is the homomorphism of $\boldsymbol{F}^*$ into $\boldsymbol{A}$ which is an extension of φ. Let $\psi: X \to B$ be any mapping and let ψ_0, ψ_1 be the same as before. Then we have ${}_{\boldsymbol{B}}P(\tau')(\psi) = {}_{\boldsymbol{B}}P(\tau')(\psi_1)$ and ${}_{\boldsymbol{B}}P(\vartheta')(\psi) = {}_{\boldsymbol{B}}P(\vartheta')(\psi_1)$. Using the diagram (3) and the equality $\bar{\varphi}(\tau') = \bar{\varphi}(\vartheta')$ we have

$$\begin{aligned} {}_{\boldsymbol{B}}P(\tau')(\psi) &= {}_{\boldsymbol{B}}P(\tau')(\psi_1) = \bar{\psi}_1(P(\tau')) = \bar{\psi}_1(h_P(\tau')) = \bar{\psi}_0\bar{\varphi}(\tau') = \bar{\psi}_0\bar{\varphi}(\vartheta') \\ &= \bar{\psi}_1(h_P(\vartheta')) = \bar{\psi}_1(P(\vartheta')) = {}_{\boldsymbol{B}}P(\vartheta')(\psi_1) = {}_{\boldsymbol{B}}P(\vartheta')(\psi) \end{aligned}$$

and therefore ${}_{\boldsymbol{B}}P(\tau') = {}_{\boldsymbol{B}}P(\vartheta')$, i. e. the G-equation $\ulcorner P(\tau') = P(\vartheta')\urcorner$

is valid in $\boldsymbol{B}$ and, by (2.15), $P(\tau')_{\boldsymbol{B}} = P(\vartheta')_{\boldsymbol{B}}$, since $P(\tau')' = P(\tau')$ and $P(\vartheta')' = P(\vartheta')$. The necessity of (3.31)-(3.33) is proved.

Proof of sufficiency. Let us assume that the set M has the property given in either (3.31) or (3.32) or (3.33), and let $\psi : X \to B$ be any mapping of M into B. Moreover, let χ be the homomorphism of the absolutely free algebra $\boldsymbol{M}^* = \mathrm{Free}(F, M)$ of the type F freely generated by M onto the subalgebra $\overline{M}$ of $\boldsymbol{A}$ generated by M such that $\chi(m) = m$ for $m \in M$. The mapping ψ can be extended, by theorem (3.29), to the $\boldsymbol{P}$-homomorphism h_ψ of $\boldsymbol{M}^*$ into $\boldsymbol{B}$. Let us observe that

(u) if $\chi(w) = \chi(w')$, then $h_\psi(w) = h_\psi(w')$.

Indeed, let $w = \tau'_{\boldsymbol{M}^*}(m_\xi, \xi < k)$ and let $w' = \vartheta'_{\boldsymbol{M}^*}(m_\xi, \xi < k)$ (see (2.12)), where $\chi(w) = \chi(w')$. Hence, by (2.10), $\tau'_{\boldsymbol{A}}(m_\xi, \xi < k) = \vartheta'_{\boldsymbol{A}}(m_\xi, \xi < k)$ and thus ${}_{\boldsymbol{A}}\tau'(\varphi) = \vartheta'_{\boldsymbol{A}}(\varphi)$, where φ is a mapping of X into M such that $\varphi(x_\xi) = m_\xi$ for $\xi < \max(n(\tau'), n(\vartheta'))$. Therefore, by the assumption and by (2.15), we have $P(\tau')_{\boldsymbol{B}} = P(\vartheta')_{\boldsymbol{B}}$ (since $P(\tau')' = P(\tau')$ and $P(\vartheta')' = P(\vartheta')$, and thus, by (3.30) for $\boldsymbol{D} = \boldsymbol{M}^*$, we obtain

$$h_\psi(w) = h_\psi(\tau'_{\boldsymbol{M}^*}(m_\xi, \xi < k) = P(\tau')_{\boldsymbol{B}}(h_\psi(m_\xi), \xi < k) = P(\vartheta')_{\boldsymbol{B}}(h_\psi(m_\xi, \xi < k)$$
$$= h_\psi\big(\vartheta'_{\boldsymbol{M}^*}(m_\xi, \xi < k)\big) = h_\psi(w').$$

Lemma (u) is proved.

From (u) it follows that the mapping $h : \overline{M} \to B$ such that for $a \in \overline{M}$, $h(a) = h_\psi(w)$, where $a = \chi(w)$, may be considered as a function. Obviously, h is a P-homomorphism of $\overline{M}$ into $\boldsymbol{B}$ being an extension of ψ, and thus M is $\boldsymbol{B}$-P-independent. The sufficiency of (3.31)-(3.33) is also proved. This completes the proof of theorems (3.31)-(3.33).

As it follows from my paper [9], it is sufficient for a study of algebras of type F by the use of the notion of validity to use a set X of variables of cardinal number $\bar{\varrho}$, where ϱ is the rank of F. The support X_τ of any F-term τ fulfils the relation $|X_\tau| < \bar{\gamma}$, where γ is the dimension of F. Hence, from (3.31)-(3.33), and from § 2, sec. D, we obtain

(3.34) $P\text{-ind}\, M = \bigcap\limits_{M' \subset M, |M'| < \bar{\gamma}} P\text{-ind}\, M'$ *for any algebra* $\boldsymbol{A}$ *of the type* F *and any subset* M *of* $\boldsymbol{A}$.

Let us consider two equationally definable classes $\mathfrak{A}$ and $\mathfrak{B}$ of algebras of the type F and G, respectively. From (3.31) immediately results

(3.35) *An* $\mathfrak{A}$*-free set is* $\mathfrak{B}$*-*P*-independent (or* $\mathfrak{B}$*-*P*-free), i. e.* $\boldsymbol{B}$*-*P*-independent for all* $\boldsymbol{B} \in \mathfrak{B}$*, if and only if for all* F*-terms* $\tau, \vartheta \in F^*$ *the validity of the* F*-equation* $\ulcorner \tau = \vartheta \urcorner$ *in the class* $\mathfrak{A}$ *(i. e. in all algebras belonging to* $\mathfrak{A}$*) implies the validity of the* G*-equation* $\ulcorner P(\tau) = P(\vartheta) \urcorner$ *in the class* $\mathfrak{B}$.

Since the G-equation $\ulcorner P(\tau) = P(\vartheta) \urcorner$ is valid in an algebra $\boldsymbol{B}$ of the type G if and only if the F-equation $\ulcorner \tau = \vartheta \urcorner$ is valid in $P(\boldsymbol{B})$, where $P(\boldsymbol{B})$ is P-algebra over $\boldsymbol{B}$, we obtain from (3.35)

(3.36) *An $\mathfrak{A}$-free set is $\mathfrak{B}$-P-independent if and only if $P(\boldsymbol{B}) \in \mathfrak{A}$ for all $\boldsymbol{B} \in \mathfrak{B}$, where $P(\boldsymbol{B})$ is the P-algebra over $\boldsymbol{B}$.*

The mapping P is said to be $(\mathfrak{A}, \mathfrak{B})$-*universal* if every $\mathfrak{A}$-free set is a $\mathfrak{B}$-P-free set. By (3.36) we have

(3.37) *The mapping P is $(\mathfrak{A}, \mathfrak{B})$-universal if and only if $P(\boldsymbol{B}) \in \mathfrak{A}$ for all $\boldsymbol{B} \in \mathfrak{B}$, where $P(\boldsymbol{B})$ is a P-algebra over $\boldsymbol{B}$.*

Let us denote by $\mathfrak{B}(P, \mathfrak{A})$ the class of all algebras $\boldsymbol{B} \in \mathfrak{B}$ such that the G-equation $\ulcorner P(\tau) = P(\vartheta) \urcorner$ is valid in $\boldsymbol{B}$ provided the F-equation $\ulcorner \tau = \vartheta \urcorner$ is valid in the class $\mathfrak{A}$. By $\mathfrak{B}(\mathfrak{A})$ will be denoted the intersection of all classes $\mathfrak{B}(P, \mathfrak{A})$, where P is a proper $P_{F,G}$-mapping, i. e. $\mathfrak{B}(\mathfrak{A}) = \bigcap_P \mathfrak{B}(P, \mathfrak{A})$.

(3.38) *For every pair $(\mathfrak{A}, \mathfrak{B})$ and every proper $P_{F,G}$-mapping P there exists the maximal subclass $\mathfrak{B}_P \subset \mathfrak{B}$ such that P is $(\mathfrak{A}, \mathfrak{B}_P)$-universal.*

Proof. By (3.35), the class $\mathfrak{B}_P = \mathfrak{B}(P, \mathfrak{A})$ fulfils the thesis of (3.38).

The pair $(\mathfrak{A}, \mathfrak{B})$ is said to be *universal* if every proper $P_{F,G}$-mapping P is $(\mathfrak{A}, \mathfrak{B})$-universal.

(3.39) *For every pair $(\mathfrak{A}, \mathfrak{B})$ there exists the maximal subclass $\mathfrak{B}_0 \subset \mathfrak{B}$ such that the pair $(\mathfrak{A}, \mathfrak{B}_0)$ is universal.*

Proof. The class $\mathfrak{B}_0 = \mathfrak{B}(\mathfrak{A})$ fulfils the thesis of (3.39).

A subset M of an algebra $\boldsymbol{A}$ of the type F is called *strongly $\boldsymbol{B}$-independent*, where $\boldsymbol{B}$ is any algebra of the type G, if the set M is $\boldsymbol{B}$-P-independent for any proper $P_{F,G}$-mapping P. The subset M is said to be *strongly $\mathfrak{B}$-independent* (*or strongly $\mathfrak{B}$-free*), where $\mathfrak{B}$ is a class of algebras of the type G, if it is strongly $\boldsymbol{B}$-independent for all $\boldsymbol{B} \in \mathfrak{B}$. It is easy to verify that

(3.40) *A pair $(\mathfrak{A}, \mathfrak{B})$ is universal if and only if any $\mathfrak{A}$-free set is a strongly $\mathfrak{B}$-free set.*

Let us assume that $F = G$.

(3.41) *For every class $\mathfrak{A}$ of algebras of the type F there exists the maximal subclass $\mathfrak{A}_0 \subset \mathfrak{A}$ such that the pair $(\mathfrak{A}_0, \mathfrak{A}_0)$ is universal.*

Proof. The class $\mathfrak{A}_0 = \mathfrak{A}(\mathfrak{A})$ fulfils the thesis of (3.41), since $\mathfrak{A}_0(\mathfrak{A}_0) = \mathfrak{A}_0$. The equality $\mathfrak{A}_0 = \mathfrak{A}_0(\mathfrak{A}_0)$ is true, because the superposition PQ of two proper $P_{F,F}$-mappings is also a proper $P_{F,F}$-mapping.

The classes $\mathfrak{A}$ such that the pairs $(\mathfrak{A}, \mathfrak{A})$ are universal are called *universal*. The strongly $\mathfrak{A}$-free sets are, obviously, $\mathfrak{A}$-free sets, but the converse is true if and only if $\mathfrak{A}$ is universal. A subset M of an algebra $\boldsymbol{A}$ of the type F is said to be *strongly independent in $\boldsymbol{A}$* provided it is strongly $\boldsymbol{A}$-independent. A strongly independent set of generators for $\boldsymbol{A}$ is called *a strong basis of* A. Let P be any proper $P_{F,F}$-mapping. If P is $(\mathfrak{A}, \mathfrak{A})$-universal, then P is called briefly $\mathfrak{A}$-*universal*. The mapping P is said to be *absolutely universal* if P is $\mathfrak{A}$-universal for every equationally definable class $\mathfrak{A}$ of algebras of the type F. Obviously, the natural proper $P_{F,F}$-mapping P such that $P(f) = f$ for all $f \in F$, is absolutely universal. By a result of Fujiwara [3], if P is absolutely universal, then P-homomorphisms are ordinary homomorphisms, i. e. P is the natural $P_{F,F}$-mapping, provided F is a set of finitary operator symbols only containing at least one non-unary operator symbol. For a set F containing infinitary operator symbols, this problem is open. **(P 521)**

Finally we remark that the considerations of this paper may be generalized in large part on arbitrary systems of P-mappings between quasi-algebras. For this generalization see my paper [11].

REFERENCES

[1] G. Birkhoff, *On the structure of abstract algebras*, Proceedings of the Cambridge Philosophical Society 31 (1935), p. 433-454.

[2] — *Subdirect unions in universal algebras*, Bulletin of the American Mathematical Society 50 (1944), p. 764-768.

[3] T. Fujiwara, *On mappings between algebraic systems*, Osaka Mathematical Journal 11 (1959), p. 153-172.

[4] L. Henkin, *On mathematical induction*, American Mathematical Monthly 67 (4), 1960.

[5] E. Marczewski, *Independence and homomorphisms in abstract algebras*. Fundamenta Mathematicae 50 (1961), p. 45-61.

[6] J. Schmidt, *Algebraic operations and algebraic independence in algebras with infinitary operations*, Mathematica Japonicae 6 (1963), p. 77-112.

[7] — *A general existence theorem on partial algebras and its special cases*, this volume, p. 00-00.

[8] R. Sikorski, *Products of abstract algebras*, Fundamenta Mathematicae 39 (1952), p. 212-228.

[9] J. Słomiński, *The theory of abstract algebras with infiniteary operations*. Rozprawy Matematyczne 18 (1959).

[10] — *A theory of extensions of quasi-algebras to algebras*, Rozprawy Matematyczne 40 (1964).

[11] — *On mappings between quasi-algebras*, Colloquium Mathematicum (in print).

[12] A. Tarski, *A remark on functionally free algebras*, Annals of Mathematics 47 (1946), p. 163-165.

DEPARTMENT OF MATHEMATICS, N. COPERNICUS UNIVERSITY

Reçu par la Rédaction le 12. 1. 1965