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ON DISCRETE SUBSPACES OF A HILBERT SPACE

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Let (X, ϱ) be a metric space. We say that a subspace $Y \subset X$ is a *discrete subspace* of (X, ϱ) if $\varrho(x, y) = 1$ whenever $x, y \in Y$ and $x \neq y$. We say that Y is a *maximal discrete subspace* if it is maximal with respect to $\subset$.

Let l_2 denote, as usually, the Hilbert space of all infinite square-summable real sequences. Denote by F the subspace of l_2 containing all sequences which vanish for all but finitely many indices.

Observe that F can be treated, in a natural way, as the union of the monotone sequence $\boldsymbol{R}^1 \subset \boldsymbol{R}^2 \subset \boldsymbol{R}^3 \subset \ldots$

We construct an infinite sequence in F.

First choose $a_0 = (0, 0, \ldots)$ and $a_1 = (1, 0, \ldots)$. If we have already chosen $a_0, a_1, \ldots, a_n$, all contained in $\boldsymbol{R}^n$, then we take

$$b_n = \sum_{i=0}^{n} \frac{a_i}{n+1}$$

and choose a point a_{n+1} in $\boldsymbol{R}^{n+1}$ such that its projection on $\boldsymbol{R}^n$ is b_n and $\|a_{n+1} - a_i\| = 1$ for $i = 0, 1, \ldots, n$. It should be noted that a_{n+1} can always be chosen in two ways; we apply the one for which a_{n+1} has positive coordinates.

THEOREM. *The sequence $a_0, a_1, \ldots$ forms a maximal discrete subspace of F and, moreover, of l_2.*

Proof. Suppose that $\{c, a_0, a_1, \ldots\}$ is maximal in l_2 for some $c \in l_2$, $c \neq a_i$, $i = 0, 1, \ldots$ Denote by c_n the projection of c on $\boldsymbol{R}^n$. We have

$$\|c - a_k\|^2 = \|c - c_n\|^2 + \|c_n - a_k\|^2 = 1,$$

so

$$\|c_n - a_k\|^2 = \|c_n - a_l\|^2 \quad \text{for } k, l = 0, 1, \ldots, n.$$

Thus c_n equals b_n for $n = 1, 2, \ldots$ It can be shown that

$$\|b_n\|^2 = \frac{1}{2} - \frac{1}{2(n+1)},$$

so

$$\|c - a_0\| = \|c\| = \lim \|c_n\| = \left(\frac{1}{2}\right)^{1/2} \neq 1,$$

a contradiction.

Remark that, for $n > 1$, $a_n = (a_n^1, a_n^2, \ldots)$, where

$$a_n^k = \begin{cases} \big(2k(k+1)\big)^{-1/2} & \text{for } 1 \leqslant k < n, \\ \big((n+1)/2n\big)^{1/2} & \text{for } k = n, \\ 0 & \text{for } k > n. \end{cases}$$

Let A be an infinite set and let F_A be the subspace of $l_2(A)$ consisting of all square-summable functions on A that vanish for all but finitely many indices. Obviously, there exists a discrete subspace of cardinality $|A|$ in F_A. We show that there exists a countable maximal discrete subspace in F_A.

Choose a countable subset $B \subset A$. The subspace $l_2(B)$ of all functions in $l_2(A)$ which vanish beyond B can be identified with our previous space l_2. By the same construction, as $b_n \neq 0$, we get

COROLLARY. *The sequence $a_0, a_1, \ldots$ forms a maximal discrete subspace of F_A.*

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