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*INDEPENDENCE IN ABSTRACT ALGEBRAS RESULTS AND PROBLEMS**

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Introduction. The purpose of this paper is to review some results obtained mainly in Wrocław that concern the notion of independence in abstract algebras.

The starting point of the investigations was the search for a common scheme of the notions of independence in various branches of mathematics, that is, for a general notion whose special cases were such notions as: linear independence of numbers, linear independence of vectors, linear independence of points, algebraic independence of numbers, independence of polynomials and, more generally, of continuous functions, set theoretical independence, logical independence, etc.

Roughly speaking, there exist two such schemes: set-theoretical and general-algebraic. The former is based on the notion of (generalized) closure operation, which is connected with the names of Alfred Tarski (1) and Garret Birkhoff (see (1), p. 49)(1) and was investigated in the fifties by Jürgen Schmidt ((1) and (2)). This is not included in the present paper and only some relationships between the two notions are dealt with in Section 4.

The proper subject of the work is the scheme expressed in terms of the general algebra. This general algebraic notion has been the theme of a series of papers listed in the part B of the Bibliography. The first of them was my paper [1] written in 1958. The notion of independence formulated in it cannot be considered essentially new. First, it is closely related to the notion of free algebra introduced by Garret Birkhoff in the thirties, then, as we have learned from Professor B. H. Neumann, Philip Hall formulated it in a lecture in 1949.

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* This paper is a revised and extended version of a lecture delivered on the Conference on General Algebra held in Warsaw, September 7-11, 1964. It also contains some results obtained after the Conference.

(1) In the sequel the numbers in square brackets refer to the part B of the Bibliography and the numbers in parantheses — to the part C which contains various references.

This notion turned out to be useful and fruitful. It not only fulfilled the task mentioned before, but it also produced further interesting notions and theorems. Various trends in these researches are reviewed in the following sections of this paper and listed in the part A of the Bibliography ([2]).

1. General algebraic independence. Notation and definitions. By an *operation* in a set A we mean every mapping $f: A^n \to A$ $(n = 1, 2, \ldots)$. Operations of the form $e_j^{(n)}(x_1, \ldots, x_n) = x_j$ are called *trivial*. By an abstract algebra or a general algebra or, shortly, *algebra*, we mean any system $\mathfrak{A} = (A; \boldsymbol{F})$, where A is an arbitrary set (called *fundamental*) and $\boldsymbol{F}$ any family of operations (called *fundamental*) in A. If $A = \{a, b, \ldots\}$ and $\boldsymbol{F} = \{f, g, \ldots\}$ we write $\mathfrak{A} = (A; \boldsymbol{F}) = (a, b, \ldots; f, g, \ldots)$. The smallest family of operations in $\mathfrak{A}$ containing all trivial and fundamental operations, and closed with respect to superposition is called the family of algebraic operations and denoted by $\boldsymbol{A} = \boldsymbol{A}(\boldsymbol{F}) = \boldsymbol{A}(\mathfrak{A})$. By $\boldsymbol{A}^{(n)}$ we denote the family of algebraic n-ary operations. The set of all values of constant algebraic operations will be denoted by $\boldsymbol{A}^{(0)}$ and its elements will be called *algebraic constants*. If every algebraic operation is trivial, we say that algebra is *trivial*.

The subalgebra of $\mathfrak{A}$ generated by a non void set $E \subset A$ is denoted by $C(E)$. Moreover, we put $C(\emptyset) = \boldsymbol{A}^{(0)}$. The operation $C: 2^A \to 2^A$ is a closure operation in the sense of Birkhoff ((1), p. 40).

We can give now three equivalent definitions of independence in the algebra $\mathfrak{A} = (A; \boldsymbol{F})$ (see my papers [1] and [4]): A set $I \subset A$ is called a *set of independent elements*, or, shortly, an *independent set* if

(a) for any pairwise different $a_1, \ldots, a_n \in I$ and $f, g \in \boldsymbol{A}^{(n)}$ the equation $f(a_1, \ldots, a_n) = g(a_1, \ldots, a_n)$ implies $f = g$ $(n = 1, 2, \ldots)$,

or, equivalently,

(b) for any $a_1, \ldots, a_{m+n} \in I$, $f \in \boldsymbol{A}^{(m)}$, $g \in \boldsymbol{A}^{(n)}$ the equation $f(a_1, \ldots, a_m) = g(a_{m+1}, \ldots, a_{m+n})$ implies $f(x_1, \ldots, x_m) = g(x_{m+1}, \ldots, x_{m+n})$ for each sequence $x_1, \ldots, x_{m+n}$ such that $x_i = x_j$ whenever $a_i = a_j$ $(m, n = 1, 2, \ldots)$,

or

(c) any mapping of I into A may be extended to a homomorphism of $C(I)$ into A.

We denote by $\boldsymbol{Ind} = \boldsymbol{Ind}(\mathfrak{A}) = \boldsymbol{Ind}(A; \boldsymbol{F})$ the family of all independent sets.

We call the notion defined in this way briefly independence, or, a bit longer, *general algebraic* (*g. a.*) *independence*, when it is necessary to distinguish it from other related notions, notably from particular notions of independence applied in different mathematical theories.

([2]) The Roman figures refer to that list.

We must emphasize three points:

1° We consider always an individual algebra and not a class of algebras, e. g. not an equationally definable class of algebras. Thus we define the independence in a single algebra only.

This is a more primitive aspect than the investigations of equationally definable classes of algebras and the freeness in them. But it is in its right and throws some light on different questions of the general algebra.

2° In the definition of an algebra we consider a family of fundamental operations and not an indexed system of these operations.

3° And what more: the fundamental operations play in our consideration only an auxiliary rôle: they are, to be sure, useful in definitions and notation but it is only the algebraic operations that are essential. Consequently we identify two algebras $\mathfrak{A} = (A; \boldsymbol{F})$ and $\mathfrak{A}^* = (A; \boldsymbol{F}^*)$ with the same fundamental sets if their families of algebraic operations coincide: $\boldsymbol{A}(\mathfrak{A}) = \boldsymbol{A}(\mathfrak{A}^*)$. Thus, the rôle of fundamental operations here is similar to that of bases of neighbourhoods in the general topology and the rôle of algebraic operations similar to that of open sets (or of the closure). So e. g. the Boolean algebra $\mathfrak{B}$ of all subsets of a set X can be noted in each of the following forms: $(2^X; \cup, ') = (2^X; ', \cup) = (2^X; \cup, \cap, ')$ etc.

By the aid of the family of algebraic operations we define some numerical constants associated with algebras. For any algebra $\mathfrak{A}$ we denote by $\beta = \beta(\mathfrak{A})$ the smallest positive integer with the property: there exists a family $\boldsymbol{F}$ of at most β-ary operations such that $\mathfrak{A} = (A; \boldsymbol{F})$. We say then that the algebra $\mathfrak{A}$ is *β-ary*. Thus algebras usually treated in mathematics, as groups, rings, lattices etc. are binary.

By $\tau = \tau(\mathfrak{A})$ we denote the greatest number with the following property: every algebraic τ-ary operation in $\mathfrak{A}$ is trivial. Obviously τ is defined only for non-trivial algebras; if there are in $\mathfrak{A}$ algebraic constants, we put $\tau = -1$ by definition, and if there are no algebraic constants, but there is an algebraic non-trivial unary operation, we put $\tau = 0$. Obviously $\tau \leqslant \beta$ for every non trivial algebra. Algebras with $\tau \geqslant 1$, or, in other words, in which $f(x, \ldots, x) = x$ for every algebraic f, will be called *idempotent* (see Urbanik [4], [4a]).

Various algebraic systems will be treated here as algebras, e. g. any group as an algebra with fundamental operations xy and x^{-1}, and any vector space as an algebra in which fundamental operations are: addition of vectors as one binary operation and multiplication by each scalar separately as (perhaps infinitely many) unary operations.

Also some recently defined classes of algebras will be considered in the sequel, e. g. diagonal algebras, introduced by Płonka ([1], [2], see also [3], [4], [5] and Urbanik [4], [4a] and [10]).

The n-dimensional *diagonal algebra* is, by definition, of the form $(D; d)$, where d is an idempotent n-ary operation depending on each variable and satisfying the condition:

$$d\big(d(x_1^1, \ldots, x_n^1), \ldots, d(x_1^n, \ldots, x_n^n)\big) = d(x_1^1, x_2^2, \ldots, x_n^n).$$

Some individual algebras are of special interest for our considerations, e. g. two-element ternary algebras $\mathfrak{P}_*$, $\mathfrak{P}^*$ and $\mathfrak{P}$ defined by Post and, four-element ternary algebra $\mathfrak{S}$ defined by Świerczkowski (see Section 6).

2. Particular cases and variants. The first of trends in our research (number XII in the Bibliography) already mentioned at the beginning aimed at *subordinating various particular notions of independence to the general-algebraic notion.* Indeed, this is possible for all the particular notions cited earlier in the Introduction. The majority of the cases were dealt with in my paper [1] and [3].

The task is in each case the following. Considering any of the known particular notions of independence we search for such an abstract algebra that the g. a. independence in it be identical with the given particular notion. So it appears that the set-theoretical independence is identical with the g. a. independence in the Boolean algebra $\mathfrak{B}$ just considered, that the linear independence of real numbers is identical with the g. a. independence in the algebra $(R^1; +)$, the classical algebraic independence of real numbers is identical with the g. a. independence in the algebra $(R^1; +, \cdot)$, etc.

The next problem (number XIII in our list) is converse: we consider a class of algebras, e. g. lattices, distributive lattices, various classes of groups etc. and we ask what g. a. independence is in these algebras and what its properties are. The first question here is: since the g. a. independence is defined by algebraic operations, how can it be expressed in the given case by fundamental operations? So e. g. it turns out, that in a distributive lattice a set I is independent if and only if for any sequence $a_1, \ldots, a_m, b_1, \ldots, b_n$ of pairwise different elements of I we have

$$\bigcap a_j \ \mathit{non} \leqslant \bigcup b_j$$

(see Marczewski [6]). Similarly the g. a. independence was studied in Post algebras (Traczyk [1]), in diagonal algebras (Płonka [1], [2] and [5]), etc.

Coming back to the first problem, there are, as is seen, numerous particular notions of independence that naturally fall within the g. a. notion. On the other hand, there are particular notions which can only heavily be subordinated to it, as the independence of real continuous functions of many variables (a deep result of Świerczkowski [5]). And

the independence in the sense of probability theory cannot be subordinated at all (Marczewski [1], Fajtlowicz [1]).

The linear independence of vectors in R^n, or more generally, in vector spaces is obtained when they are treated as algebras as in Section 1.

The linear independence of points in the Euclidean space R^n coincides with the g. a. independence when we admit as fundamental operations the binary operations which can be noted in vector symbols so: $\lambda x_1 + (1-\lambda)x_2$ (Marczewski [10]; cf. also Urbanik [7], [1], [1a] and [2]), but I do not know so far of an algebra where the g. a. independence would be identical with the linear independence in a projective space (**P 522**).

The so-called linear independence in abelian groups coincides with the g. a. independence only in the case of torsion-free groups. In order to subordinate it fully it is necessary to modify the general notion. This has been done by G. Grätzer [1]. And so we approach the next direction of research: the *variants of g. a. independence* (III). Three different variants that are known to me: those of Grätzer [1], Schmidt [1] and Świerczkowski [7] can all be subordinated to the same scheme: in the definition (c) of independence we replace the family of all mappings $p : I \to A$ by a certain family $\boldsymbol{P}$ of such mappings (let us add that the definitions (a) and (b) can be modified in a respective way, too). So e. g. if $\boldsymbol{P}$ is the family of all mappings $p : I \to C(I)$, we obtain the "Unabhängigkeit in sich" of Schmidt, and if $\boldsymbol{P} = \{p : I \to I\}$ — the weak independence of Świerczkowski.

Recently Schmidt ([3], [4]) has worked out the theory of *independence in the algebras with infinitary operations* which is a further direction in the field (II). The indications of differences between finitary and infinitary cases is of particular interest here.

3. Independence in topological algebras. A new trend in our research is the investigation of *g. a. independence in topological algebras*, i. e. in algebras which are simultaneously Hausdorff spaces and in which every fundamental (and, consequently, every algebraic) operation is continuous (XI).

Of special interest seems to be the following result of Świerczkowski [7]:

THEOREM 1. *If F is a weakly independent subset of an algebra A, then every completely regular topology in F can be so extended that A becomes a completely regular topological algebra and F a closed set in A.*

The well-known von Neumann's result on the existence of a perfect set of algebraically independent (in the classical sense) reals has been generalized for topological algebras by Mycielski [1]. He says that a topological algebra $(A; \boldsymbol{F})$ is *analytic* whenever for each pair $f, g \in \boldsymbol{A}^{(n)}$

$(n = 1, 2, \ldots)$ and each sequence $V_1, \ldots, V_n$ of open subsets of A if $f(x_1, \ldots, x_n) = g(x_1, \ldots, x_n)$ in $V_1 \times \ldots \times V_n$, then $f = g$.

THEOREM 2. *If $\mathfrak{A} = (A; \boldsymbol{F})$ is an analytic algebra whose space is complete metric and dense in itself and $\mathfrak{A}$ has an at most denumerable system of fundamental operations, then there exists in A a perfect independent set.*

Examples of algebras satisfying the conditions of this theorem are the ring of real or complex numbers and connected Lie groups.

The quoted paper [1] of Mycielski contains moreover some related results.

4. Generating and independence. Exchange of independent sets. Coming back to general algebras let us note some simple relations between g. a. independence and some notions defined by the aid of generators, i. e. in terms of the operation C (see Section 2). It is easy to see that if I is a g. a. independent set, then I is *C-independent*, i. e. $a \notin C(I \setminus a)$ for each $a \in I$. We have also a stronger relation (Marczewski [4], p. 56):

THEOREM 1. *If I is independent, then*

$$I \cap C(\emptyset) = \emptyset \quad \textit{and} \quad C(E \cap F) = C(E) \cap C(F)$$

for any subsets E and F of I.

Some other relations of this kind are formulated and discussed in my paper [10]. Algebras in which g. a. independence and C-independence coincide are called *v_*^*-algebras* and are treated by Narkiewicz [3] and Urbanik [7].

Of all investigations concerning the relations between independence, generation and homomorphisms (see I and II, especially Marczewski [4]), a special subject called *exchange of independence sets* (IV) is outstanding. The following theorem on exchange is generally true (Marczewski [4], p. 58):

THEOREM 2. *If P, Q and R are subsets of an algebra such that $P \cup Q \in \boldsymbol{Ind}$, $P \cap Q = \emptyset$, $R \in \boldsymbol{Ind}$ and $C(Q) = C(R)$, then $P \cup R \in \boldsymbol{Ind}$.*

It seems at first glance that the identity $C(Q) = C(R)$ might be replaced by a weaker relation, namely by $R \subset C(Q)$. But this is not so: there are some algebras which do not satisfy this stronger "condition of the exchange of independent sets" (EIS). Here is the list of results concerning this condition:

THEOREM 3. *The following algebras satisfy EIS:*

1) *algebras having at most* 6 *elements* (Płonka [1], [4]),

2) *algebras without algebraic constants having at most* 11 *elements* (Płonka [7]),

3) *groups having at most* $3^6 - 1 = 728$ *elements* (Hulanicki, Marczewski and Mycielski [1]),

4) *abelian groups and more generally "separable variables algebras"* (ibidem),

5) *Boolean algebras and more generally Post algebras* (ibidem and Traczyk [1]),

6) *distributive lattices* (Płonka [7]),

7) *unary algebras* (Płonka [1], [4]),

8) *diagonal and generalized diagonal algebras* (Płonka [1], [4], [7]).

On the other hand, *there exists*

1*) *precisely one 7-element algebra* (Płonka [1], [4]),

2*) *12-element algebras without algebraic constants* (Płonka [7]),

3*) *precisely one 729-element (nilpotent) group* (Hulanicki, Marczewski and Mycielski [1]),

which do not satisfy EIS.

I do not know if there exists a (non-distributive) lattice not satisfying EIS (**P 523**).

Let us mention finally that Theorem 2 was strengthened by Schmidt [5]:

THEOREM 2a. *If* $(P_t)_{t\in T}$ *and* $(R_t)_{t\in T}$ *are two families of disjoint independent sets such that*

$$\bigcup_{t\in T} P_t \in \boldsymbol{Ind} \quad \text{and} \quad C(P_t) = C(R_t) \quad \text{for} \quad t\in T,$$

then $\bigcup_{t\in T} R_t \in \boldsymbol{Ind}$.

5. Set-theoretical properties of the family of independent sets. The next topic is the *set-theoretical characterization of the family of all independent sets* (V).

It follows directly from the definition of independence that for every algebra $\mathfrak{A} = (A; \boldsymbol{F})$ the family $\boldsymbol{Ind}(A)$ is hereditary and (since we consider only finitary operation) of finite character (i. e. if exery finite subset of a set I belongs to it, then so does I.) Hence in the study of the set-theoretical properties of the family $\boldsymbol{Ind}$ it is sufficient to consider only finite sets belonging to it.

The main question: for which hereditary families $\boldsymbol{J}$ of finite subsets of a given set A there exists an algebra $\mathfrak{A} = (A; \boldsymbol{F})$ such that $\boldsymbol{J}$ is the family of the finite independent sets of the algebra $\mathfrak{A}$?

Not for all. Let us consider for instance the class $\boldsymbol{J}$ of all subsets E of the three-element set $A = \{a, b, c\}$ such that $a \notin E$ or $b \notin E$. One can prove that there exists no algebra $\mathfrak{A} = (A; \boldsymbol{F})$ such that $\boldsymbol{Ind}(\mathfrak{A}) = \boldsymbol{J}$. (Analogous example in paper [4] of Świerczkowski, p. 39 is not exact).

It seems that the problem of finding a simple necessary and sufficient condition is hopeless. We know only some sufficient conditions and some necessary conditions.

Świerczkowski [4] has given a sufficient condition:

THEOREM 1. *Let A be a set with $|A| \geqslant 2$, $\boldsymbol{J}$ an hereditary family of finite subsets of A and $D(\boldsymbol{J}) = \{x : \{x\} \notin \boldsymbol{J}\}$. If $|D(\boldsymbol{J})| \geqslant |\boldsymbol{J}|$, then there exists an algebra $\mathfrak{A} = (A; \boldsymbol{F})$ such that $\boldsymbol{J}$ coincides with the family of all finite independent sets in $\mathfrak{A}$.*

With the aid of Świerczkowski's result and fololwing an idea of J. Schmidt [1] we may give a complete answer to the weakened problem of the set-theoretical characterization (in this variant of the problem we do not fix the set A in advance, i. e. we may adjoin arbitrarily many new elements to the sum of the sets belonging to $\boldsymbol{J}$):

COROLLARY 1a. *Let $\boldsymbol{J}$ be a family of finite sets. There exists an algebra $\mathfrak{A} = (A; \boldsymbol{F})$ such that $\boldsymbol{J}$ is the family of all finite independent sets in $\mathfrak{A}$ if and only if $\boldsymbol{J}$ is hereditary.*

The necessity is obvious and in order to prove the sufficiency it is enough to adjoin a set of the power $|\boldsymbol{J}|$ to the union of all sets belonging to $\boldsymbol{J}$ — and apply Świerczkowski's Theorem 1.

J. Anusiak observed that Theorem 1 permits to construct easily an algebra in which every independent set is finite while there exists an n-element independent set for every $n = 1, 2, \ldots$

Returning to our main problem let us mention some new results of Fajtlowicz [1] to this respect.

Some necessary conditions in the case of finite algebras (or, by transposition, some recipes for construction of counter-examples) have been found by K. Urbanik [3]:

THEOREM 2. *Let $m < n > 3$, let $\mathfrak{A} = (A; \boldsymbol{F})$ be a finite algebra without algebraic constants with $|A| \geqslant m+n$, and let M be an m-element set such that each n-element subset of $A \setminus M$ is independent. Then each n-element subset of A is independent.*

All hypotheses are here essential. For $n = 2$ or 3 the Theorem is true under an additional assumption:

THEOREM 3. *Let $n = 2$ or 3, $n > m$, and let $(A; \boldsymbol{F})$ be a finite algebra without algebraic constants, such that $|A| \geqslant m+n$. Suppose that there exists an m-element subset M of A such that every n-element subset of $A \setminus M$ is independent. Moreover, suppose that M is contained in an n-element independent set. Then every n-element subset of A is independent.*

Obviously the hypothesis of the non-existence of algebraic constants may be replaced in Theorems 2 and 3 by a stronger one that no element is self-dependent. So, those theorems of Urbanik give necessary conditions that are in question in our main problem.

Some new necessary conditions have been obtained recently by Fajtlowicz [1].

Analogous problems for the "Unabhängigkeit in sich" have been treated by Schmidt [1].

6. Algebras in which all elements are independent and related algebras (see IX). For every trivial algebra the whole fundamental set is independent. S. Świerczkowski raised in 1959 the problem: is the converse of this proposition true? He has proved ([2] and [2a]) the

THEOREM 1. *Every algebra* $\mathfrak{A} = (A; \boldsymbol{F})$ *such that* $|A| \geqslant 3$ *and* $A \in \boldsymbol{Ind}(\mathfrak{A})$ *is trivial.*

On the other hand, in the three following non-trivial two-element algebras defined by E. L. Post the fundamental set is independent:

$$\mathfrak{P}_* = (a, b; p_*), \quad \mathfrak{P}^* = (a, b; p^*), \quad \mathfrak{P} = (a, b; p_*, p^*),$$

where p_* and p^* are symmetrical ternary operations in $\{a, b\}$ and satisfy the conditions

$$p_*(x, x, y) = y = p^*(y, y, x)$$

(p^* gives the result of a "vote by majority" and p_* the result of a "vote by minority or unanimity").

Since there exist no other two-element algebras with the same property (Marczewski-Urbanik [1]), we have finally the following theorem, stronger than Theorem 1.

THEOREM 2. *There are only three algebras (with more than one element), namely* $\mathfrak{P}_*$, $\mathfrak{P}^*$, *and* $\mathfrak{P}$, *in which the fundamental set is independent.*

A more general problem raised also by Świerczkowski is the following: in which algebras is every k-element set a basis (i. e. a set of independent generators), k being a given positive integer? Here is the answer due to Świerczkowski ([2] and [2a]):

THEOREM 3. *If* $k > 3$ *and in an at least* k*-element algebra every* k*-element set is a basis, then this algebra is trivial (and, consequently, has* k *elements).*

Next Świerczkowski has defined an algebra, which will be denoted here by $\mathfrak{S}$, as follows: $\mathfrak{S} = (A; s)$, where $|A| = 4$, and $s(x, y, z)$ is different from x, y and z if they are pairwise different, and $s(x, y, z) = p_*(x, y, z)$ otherwise. Every three-element subset of A is a basis of S. Moreover, we have (Świerczkowski [2] and (2a]):

THEOREM 4. *There is only one at least three-element non-trivial algebra, namely* $\mathfrak{S}$, *in which every three-element set is a basis.*

Świerczkowski [2a] and Grätzer [1] discussed also the case of $k = 1$ and $k = 2$ (see also Urbanik [5] and [7]).

The main problem of this section may be modified as follows. If every k-ary algebraic operation in an algebra $\mathfrak{A} = (A; \boldsymbol{F})$ is trivial, then obviously every k-element subset of A is independent. Is the converse of this proposition true? Świerczkowski [6] has proved the

THEOREM 5. *If* $\mathfrak{A} = (A; \boldsymbol{F})$ *is a finite algebra,* $|A| \geqslant k > 3$, *and every* k*-element subset of* A *is independent, then every* k*-ary algebraic operation in* $\mathfrak{A}$ *is trivial.*

The algebra $\mathfrak{S}$ shows that the hypothesis that $k > 3$ is essential.

7. Bases and number of their elements. The following theorem, by the way not difficult to prove, is important especially in investigations of bases (Świerczkowski [1]):

THEOREM 1. *Let* A *be a finite algebra in which there exist an* n*-element independent set and a set of* n *generators. Then every* n*-element subset of* A *is independent if and only if it is a set of generators.*

Let us add that this theorem is true also for vector spaces and more generally, for algebras called v^*-algebras. It would be interesting to know for which other classes of algebras this is true (**P 524**).

It follows from Theorem 1 that any two bases of a finite algebra have the same number of elements. This is false for infinite algebras. Jónsson and Tarski ((1) and (2)) have proved the existence of an algebra having one-, two-, three-, etc. element bases. Generalizing a result of Świerczkowski [1], we proved with Ryll-Nardzewski (see my paper [4], p. 59)

THEOREM 2. *If an algebra has bases with different cardinal numbers, then all these numbers are finite and form an infinite arithmetical progression.*

The converse theorem is also true. After some partial results of Goetz and Ryll-Nardzewski [1], Świerczkowski ([3] and [3a]) proved

THEOREM 3. *Any arithmetical progression is the set of numbers of elements of all bases of a certain algebra.*

The quoted paper of Świerczkowski contains some more general formulations concerning free algebras in equationally definable classes (see also Schmidt [5]).

Let us add that there exists a module having one-, two-, three-, etc. element bases (Everett (1); see also Rédei (1), p. 255-257). We do not know if for modules, and, more generally, for binary algebras the progression $1, 2, 3, \ldots$ is the only possible one (**P 525**).

An element a of an algebra is called *self-dependent*, if $\{a\} \notin \boldsymbol{Ind}$. Some properties of self-dependent elements have been proved by Nitka [1].

In connection with theorems 2 and 3 Goetz and Ryll-Nardzewski [1] proved

THEOREM 4. *If there exist simultaneously a basis consisting of one element and a basis consisting of* $n > 1$ *elements of an algebra* A, *then for every positive integer* k *there exists a minimal set of* k *self-dependent generators of* A.

We do not know whether an analogous thesis may be deduced from a weakened assumption, namely, that the considered algebra possesses bases of different number of elements (**P 526**).

8. Maximal numbers of independent elements, minimal number of generators and some related numerical constants. For each finite algebra $\mathfrak{A} = (A; \boldsymbol{F})$ with $2 \leqslant |A|$ we define the following integers:

$\alpha = |A|$,

$\gamma^* =$ the smallest number with the property: every set $G \subset A$ with $|G| = \gamma^*$ is a set of generators of $\mathfrak{A}$,

$\gamma =$ the minimal number of generators of $\mathfrak{A}$,

$\iota =$ the maximal number of independent elements in $\mathfrak{A}$,

$\iota_* =$ the greatest number with the following property: every set $I \subset A$ with $|I| = \iota_*$ is a set of independent elements of $\mathfrak{A}$.

Let us remark that if each element of A is an algebraic constant of A, then the empty set is treated by definition as a set of generators of A, whence $\gamma = \gamma^* = 0$.

There are various relations between defined numerical constants (Marczewski [5], Świerczkowski [6]). At first let us note

THEOREM 1. *We have for every non-trivial finite algebra*:

$$\alpha \geqslant \gamma^* \geqslant \gamma \geqslant \iota \geqslant \iota_* \quad \textit{and} \quad \tau = \iota_* \quad \textit{or} \quad \tau = \iota_* - 1.$$

While proving these properties of τ one must base on deep properties of algebras $\mathfrak{P}_*$, $\mathfrak{P}^*$ and $\mathfrak{P}$ (Theorem 2 of Section 6). The inequality $\gamma \geqslant \iota$ is a consequence of Theorem 1 of Section 7 on finite algebras, and other inequalities are trivial.

There are also other relations, e. g. *if, in a non-trivial algebra,* $\alpha > \tau$, *then* $\alpha > \iota$ *and* $\gamma > \tau$.

Some theorems quoted before can be expressed by the aid of the defined constants, e. g. Świerczkowski's Theorem 4 of Section 6: *if* $\gamma^* = \iota_* \geqslant 3$, *then the algebra is trivial or identical with* $\mathfrak{S}$, and, analogously, his Theorem 5 of Section 6: *if, for a non trivial algebra,* $\iota_* \geqslant 4$, *then* $\tau = \iota_*$.

Świerczkowski [6] has given an interesting discussion of algebras with $\tau = \iota_* - 1$. Namely the problem arises which values are admissible for the number α of elements of such an algebra with $\iota_* = 0, 1, 2$ or 3? It is easy to see that for $\iota_* = 0$ or 1 all positive integers are admissible. For $\iota_* = 2$ and 3 the problem was reduced by Świerczkowski to some questions about families of finite sets, called Steiner's systems. By their aid Świerczkowski obtains

THEOREM 2. *A positive integer* $n > 3$ *is the number of elements of an algebra with* $\iota_* = 3$ *and* $\tau = 2$ *if and only if* $n \equiv 2$ *or* $4 \pmod 6$.

An analogous problem for $\iota_* = 2$ is open, because the respective question for Steiner's system is open too.

In view of these results, it seems that the problem of the complete characterization of all possible 6-tuples of numbers $(\alpha, \gamma^*, \gamma, \iota, \iota_*, \tau)$ is hopeless. But the situation is quite different if we restrict the consideration to algebras of some special kind. So, e. g. my paper [7] (p. 99 and 101) contains a list of possible values of the considered numbers for algebras called *homogeneous*, i. e. such that for every permutation $h: A \to A$ we have $fh = hf$ for every fundamental operation f.

In this connection the following notion is useful: an operation f in A is called *quasi-trivial*, if $f(a_1, \ldots, a_n)$ always belongs to the set of elements $a_1, \ldots, a_n$. If all fundamental (and, consequently, all algebraic) operations of an algebra are quasi-trivial, we say that the algebra is quasi-trivial. A finite algebra is quasi-trivial if and only if $\alpha = \gamma$.

THEOREM 3. *The list of all systems $(\alpha, \gamma^*, \gamma, \iota, \iota_*, \tau)$ for finite homogeneous algebras with $\alpha > 1$ is the following:*

	α	$\gamma^* = \gamma$	$\iota = \iota_*$	τ	
1	2	2	2	2	
2	n	n	k	k	*where* $n > k > 1$
3	n	$n-1$	k	k	*where* $n-1 > k > 1$
4	n	$n-1$	$n-1$	$n-2$	*where* $n = 2, 3$ or 4

where rows 1 *and* 2 *correspond to quasi-trivial homogeneous algebras, and rows* 3 *and* 4 *to non quasi-trivial ones.*

The analogous list for all quasi-trivial algebras is the following:

$\alpha = \gamma^* = \gamma$	$\iota = \iota_* = \tau$	
n	k	*where* $n > k > 1$
2	2	

One of the fundamental questions concerning the constants considered is the evaluation of the number ι by α or conversely. As we know, the identity $\iota = \alpha$ holds only for three algebras: $\mathfrak{P}_*$, $\mathfrak{P}^*$ and $\mathfrak{P}$ (Theorem 2 of Section 6). There are also some algebras with $\iota = \alpha - 1$ or $\iota = \alpha - 2$ (and with arbitrary finite α): see e. g. the table in my paper [7]. Algebras of this kind are defined there by fundamental α- or $(\alpha-1)$-ary operations and they have $\tau = \alpha - 1$ or $\tau = \alpha - 2$. Since in algebras, treated usually in mathematics, there is always an algebraic binary operation depending on both variables, the problem arises of finding the required

evaluation under this assumption. Such an evaluation has been found, aftersome partial results contained in my paper [8], by Płonka ([1] and [3]):

THEOREM 4. *Let us suppose that in the algebra $\mathfrak{A}$ there exists a binary algebraic operation f depending on every variable. Therefore:*

(a) *if there are algebraic constants in $\mathfrak{A}$, then*

$$a \geqslant \frac{1}{2}\iota(\iota+1)+1;$$

(b) *if there are no algebraic constants in $\mathfrak{A}$ and f is symmetrical, then*

$$a \geqslant 2^{\iota}-1;$$

(c) *if there are no algebraic constants in $\mathfrak{A}$ and f is non-symmetrical, then*

$$a \geqslant \iota^2.$$

All these inequalities are strong for $\iota \geqslant 2$.

Let us denote by ω_k^* the number of all k-ary algebraic operations in $\mathfrak{A}$ and by ω_k the number of k-ary algebraic operations in $\mathfrak{A}$ depending on every variable. Inequalities (a) and (c) follow easily from the fundamental formula:

$$(*) \qquad a \geqslant \omega_\iota^* = \sum_{k=0}^{\iota} \binom{\iota}{k} \omega_k$$

(see e. g. Marczewski [8]). In the proof of (b) one uses $(*)$ and the following result of Płonka: if the hypothesis of (b) is fulfilled, then

$$(\overset{*}{*}) \qquad \omega_k \geqslant 1 \quad \text{for} \quad k = 2, 3, \ldots$$

(a stronger result is contained in my paper [9]).

For every $\iota \geqslant 2$ there are algebras which realize the equality in the formulas (a), (b) and (c). In the case (b) it is the algebra $(2^X \setminus \{\emptyset\}; \cup)$, where $\iota = |X|$, $\omega_0 = 0$ and $\omega_k = 1$ for $k = 1, 2, 3, \ldots$, and in the case (c) the two-dimensional diagonal algebra, where $\omega_0 = 0$, $\omega_1 = 1$, $\omega_2 = 2$ and $\omega_k = 0$ for $k \geqslant 3$.

The investigations of Płonka, completed recently by Fajtlowicz, give also the complete answer to the more complicated converse problem: evaluation of ι by the aid of a (Płonka [3]).

9. The family of algebraic operations: cardinality and extensions. Investigations reported in the final part of the preceding section suggest an examination of numbers ω_k and ω_k^* (see VI). Świerczkowski says that the algebra $\mathfrak{A}$ is *quasi-finite*, if the family $A^{(n)}(\mathfrak{A})$ is finite for every n and considers for every quasi-finite $\mathfrak{A}$ the sequences $\omega = (\omega_1, \omega_2, \ldots)$ and $\omega^* = (\omega_1^*, \omega_2^*, \ldots)$. So, when $\mathfrak{A}$ runs over the collection of all quasi-finite algebras he obtains sets Ω and Ω^* of sequences and proves

THEOREM 1. *The sets Ω and Ω^* are closed in the space* $\{0, 1, 2, \ldots\}^{\aleph_0}$.

Of special interest is the set $S(\mathfrak{A})$ defined as follows: a non negative integer $k \neq 1$ belongs to $S(\mathfrak{A})$ if $\omega_k > 0$, and $1 \in S(\mathfrak{A})$ if $\omega_1 > 1$. In other words, $k \in S(\mathfrak{A})$ if there is in $\mathfrak{A}$ an algebraic non-trivial k-ary operation depending on every variable. Some simple construction (Urbanik [4a], p. 130) gives

THEOREM 2. *For any set E of non-negative integers, containing* 0 *or* 1, *there exists an algebra $\mathfrak{A}$ such that $S(\mathfrak{A}) = E$.*

Consequently, in order to obtain the complete characterization of sets $S(\mathfrak{A})$, it suffices to consider the case $S(\mathfrak{A}) \subset \{2, 3, \ldots\}$, i. e. the case of idempotent algebras. Just this case turns out to be difficult. Urbanik gives in his recent papers [4] and [4a] the complete discussion of the set $S(\mathfrak{A})$ for idempotent algebras, and therefore, in view of Theorem 2, for all algebras. For idempotent algebras the set $S(\mathfrak{A})$ sometimes characterizes the algebraic structure of $\mathfrak{A}$ and in any case it is of a special form, as is shown by the following theorem (resulting from a more detailed theorem of Urbanik):

THEOREM 3. *If $S(\mathfrak{A}) \subset \{2, 3, \ldots\}$, i. e. if $\mathfrak{A}$ is idempotent, then either $S(\mathfrak{A})$ is finite and $\mathfrak{A}$ is diagonal, or $S(\mathfrak{A}) = \{3, 5, 7, \ldots\}$ or else $S(\mathfrak{A})$ contains all sufficiently great integers.*

Some recent papers deal with the set $S(\mathfrak{A})$ for algebras satisfying some conditions, e. g. having symmetrical or quasi-symmetrical fundamental operations (Marczewski [9], Urbanik [8], [9]), or having simultaneously a one-element basis and a many-element basis (Narkiewicz [4]). The problem of characterizing of the set $S(\mathfrak{A})$ in the more general case, when $\mathfrak{A}$ has two bases with distinct cardinalities, is open (**P 527**) The analogous problem for the class of all binary algebras is open too (**P 528**).

Some problems treated in preceding sections are connected with extensions of the family of algebraic operations. Let us consider an algebra $\mathfrak{A} = (A; \boldsymbol{F})$ and suppose that it is not complete, i. e. that there exists in A an operation which is non algebraic in $\mathfrak{A}$. We call the *degree of extendability* of $\mathfrak{A}$ (and we denote it by $\varepsilon = \varepsilon(\mathfrak{A})$) the supremum of such numbers n for which there exists a non algebraic operation f in $\mathfrak{A}$ with the property that $\boldsymbol{A}^{(n-1)}(\boldsymbol{F}) = \boldsymbol{A}^{(n-1)}(\boldsymbol{F} \cup \{f\})$. It may be defined also as follows. Let $\boldsymbol{G}$ run over all families of operations in A; then

$$\varepsilon(\mathfrak{A}) = \inf\{n : \bigwedge_{\boldsymbol{G}} [\boldsymbol{A}^{(n)}(\boldsymbol{F}) = \boldsymbol{A}^{(n)}(\boldsymbol{G})] \Rightarrow [\boldsymbol{A}(\boldsymbol{F}) \supset \boldsymbol{A}(\boldsymbol{G})]\}.$$

It is interesting that there exists a duality between the numbers ε and β (see Section 1), namely, it is easy to see that

$$\beta(\mathfrak{A}) = \inf\{n : \bigwedge_{\boldsymbol{G}} [\boldsymbol{A}^{(n)}(\boldsymbol{F}) = \boldsymbol{A}^{(n)}(\boldsymbol{G})] \Rightarrow [\boldsymbol{A}(\boldsymbol{F}) \subset \boldsymbol{A}(\boldsymbol{G})]\}.$$

The constant ε is still little examined, but it seems to be very important. For instance the most difficult part of the proof of Theorems 2 and 4 of Section 6 consists virtually in asserting that for algebras $\mathfrak{P}_*$, $\mathfrak{P}^*$, $\mathfrak{P}$ and $\mathfrak{S}$ we have $\varepsilon \leqslant 3$. In general for different algebras the knowledge of ε permits to prove that they are unique algebras having certain properties. So is e. g. in the case of algebras realizing the equalities in the formulas of Płonka's Theorem 4 of Section 8 and in the case of some algebras without EIS-property. Urbanik [10] found recently the values of ε for some vast class of algebras and showed some connection between ε and γ. At first he proved

THEOREM 4. *For any finitely generated algebra* $\varepsilon \geqslant \gamma - 1$.

Let us note that from general Urbanik's considerations follows

THEOREM 5. *We have* $\varepsilon = \gamma$ *for*: 1° *any finitely generated unary algebra with* $\gamma \geqslant 3$, 2° *any finite Boolean algebra with more than four elements*, 3° *any finitely generated abelian group with* $\gamma \geqslant 2$.

Every trivial algebra is unary, thus Theorem 5 easily implies Świerczkowski's Theorem 1 of Section 6 for finite algebras. (For infinite algebras the last is easy to prove).

The considerations of this section, though indirectly connected with independence, actually belong to another chapter of general algebra (and simultaneously of general set theory), where, considering the algebra $(A; \boldsymbol{F})$, the aim of research is not the distinguishing of various subsets of A and the investigation of their properties but only the investigations of properties of the family $\boldsymbol{A}(\boldsymbol{F})$ of all algebraic operations. Thus that chapter constitutes a theory of superpositions of operations.

10. Final remarks. Turning back to problems of independence, we must emphasize that a separate and well developed line of research is that of *algebras in which the g. a. independence has the main properties of the linear independence* (X). Numerous deep results achieved in this field are presented in a separate review by Urbanik [7].

In these researches one defines some kinds of algebras, denoted v-algebras, v_*-algebras, v^*-algebras and v_*^*-algebras and examines the respective representation problems. In particular, the question is: to which extent these algebras are connected with vector spaces. This connection turns out to be especially clear for all v-algebras and for v^*-algebras with dimension not less than 3. What is of special interest in the representation theorems, proved by Urbanik, is that in these theorems repeatedly appears the division into the case of linear independence of vectors, the case of linear independence of points, and degenerated cases. In v-algebras and, more generally, in v^*-algebras, the g. a. independence may be treated as the abstract linear independence in the sense of Whitney (see my papers [2] and [10]).

In general the investigation of independence led to the determination of various interesting classes of algebras. For instance, S. Fajtlowicz, K. Głazek and K. Urbanik [1] have worked quite recently upon separable variable algebras introduced in a paper by Hulanicki, Marczewski and Mycielski [1] (see Theorem 3 of Section 4).

Now, one more example of questions to which the investigations of g. a. independence leads. Since in this research we treat, as identical, algebras with the same fundamental set and with the same family of algebraic (and not necessarily fundamental) operations, the appropriate notion of isomorphism is the notion proposed by Goetz, which has been called *weak isomorphism*, namely a one-one correspondence between fundamental sets which transforms the family of all algebraic operations of the first algebra onto the analogous family of the other. The essential difference between the ordinary and weak isomorphism in the case of groups was studied by Hulanicki and Świerczkowski [1]. The characterization of weak isomorphisms in Boolean and Post algebras is recently made by Traczyk [2]. The paper [1] by Goetz may be treated as the supplement to this paper, as regards weak isomorphism and a corresponding notion of weak homomorphism.

I want to finish with one reflexion. One is struck by the variety and irregularities of phenomena connected with small numbers. I am not thinking about troubles and subtleties of 0 and 1 or, strictly speaking, of the empty subalgebra and one-element subalgebras (these troubles have been well characterized and precisely overcome by Schmidt in [3]). I am thinking primarily about numbers 2 and 3. The exceptionality of these numbers comes out in a great part of our studies: it is often necessary to exclude the respective cases from consideration and to add further assumptions for their treatment (as in particular, in Sections 5, 6 and 8; see also Płonka [4] and Urbanik [7]).

As said before, the research I have reviewed here originated from the desire to find a general notion comprising various notions of independence in mathematics. When it was asserted that such a general-algebraic notion exists, it could be supposed that it is too general to allow any deeper consideration. It seems now, however, that the scope of problems involved here is vast and far to exhaust.

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II. Independence in algebras with infinitary operations; differences between finitary and infinitary cases: Schmidt [3], [4]. (See Section 2.)

III. Different variants of the g. a. notions of independence: Schmidt [1] ("Unabhängigkeit in sich"), Świerczkowski [7] ("weak independence"), Grätzer [1] and [3] (See Section 2.)

IV. Exchange of independent sets: Hulanicki-Marczewski-Mycielski [1], Płonka [4], [7], Traczyk [1]. (See Section 4.)

V. Set-theoretical characterization of the class of independent sets: Schmidt [1], Świerczkowski [4], Urbanik [3], Fajtlowicz [1]. (See Section 5.)

VI. Number of algebraic n-ary operations: Marczewski [8], [9], Płonka [1], [3], Urbanik [4], [4a], [8], [9], Narkiewicz [4]. (See Section 9.)

VII. Minimal number of generators, maximal number of independent elements, and some related numerical constants: Świerczkowski [1], [6], Marczewski [5], [7], [8], Płonka [1], [3]. (See Section 8.)

VIII. Bases and number of their elements: Świerczkowski [1], Marczewski [4], Goetz and Ryll-Nardzewski [1], Schmidt [5]. (See Section 7.)

IX. Algebras in which all elements are independent and algebras in which every k-element set forms a basis: Świerczkowski: [2], [2a], Marczewski and Urbanik [1], [1a], Grätzer [1]. (See Section 6.)

X. Algebras in which g. a. independence has the main properties of linear independence: Marczewski [2], Urbanik [1], [1a], [2], [5], [6], [7], Narkiewicz [1], [1a], [2], [3], Grätzer [1]. (See Section 10.)

XI. G. a. independence in topological algebras: Świerczkowski [7], Mycielski [1]. (See Section 3.)

XII. Different mathematical notions of independence as special cases of g. a. independence: Marczewski [1], [3], Świerczkowski [5]. (See Section 2.)

XIII. G. a. independence in Boolean algebras, lattices, etc.: Marczewski [3], [6], Szász [1], Traczyk [1]. (See Section 2.)

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