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ON WEAK ISOMORPHISMS AND WEAK HOMOMORPHISMS OF ABSTRACT ALGEBRAS

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1. Usually, when a homomorphism or isomorphism of two algebras is considered, it is defined with respect to a given correspondence between the fundamental operations of the algebras in question. From such a point of view two algebras $(R, +, \times)$ and $(R, \times, +)$, where R denotes the set of real numbers, $+$ the operation of addition, and $\times$ the multiplication, with the correspondence of operations $+ \leftrightarrow \times$, are not isomorphic. However, when abstract algebras are handled as in Marczewski's considerations concerning independence, there is no reason for establishing such a correspondence a priori, since all algebraic operations are equally treated. The two algebras mentioned above are indistinguishable now, since they have the same set of elements R and the same set of algebraic operations.

Therefore Marczewski and the author have suggested notions of isomorphism and homomorphism which do not make use of a prescribed correspondence of operations. We call it a *weak isomorphism* and *weak homomorphism*. The purpose of this paper* is to give an account of what is known about these notions, completing, on this special topic, Marczewski's review [2].

2. Consider two algebras $\mathfrak{A} = (A, \boldsymbol{A})$ and $\mathfrak{B} = (B, \boldsymbol{B})$, where $\boldsymbol{A}$ and $\boldsymbol{B}$ denote the classes of algebraic operations of the first and second algebra respectively.

A one-to-one mapping $h: A \to B$ gives rise to a natural one-to-one mapping of the class of all operations (algebraic as well as non-algebraic) in A onto the class of all operations in B, namely an n-ary operation f in A is mapped onto the n-ary operation $f^* = h \circ f \circ h^{-1}$ in B defined by formula

$$f^*(y_1, \ldots, y_n) = h\Big(f\big(h^{-1}(y_1), \ldots, h^{-1}(y)\big)\Big). \tag{1}$$

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The mapping h is called a *weak isomorphism* of $\mathfrak{A}$ onto $\mathfrak{B}$ if the induced mapping of operations maps the class $\boldsymbol{A}$ of all algebraic operations of $\mathfrak{A}$ onto the class $\boldsymbol{B}$ od all algebraic operations of $\mathfrak{B}$.

This definition can be reformulated as follows:

A mapping $h: A \to B$ is a weak isomorphism of algebras $\mathfrak{A}$ and $\mathfrak{B}$ if and only if there exists a one-to-one correspondence $f \leftrightarrow f^*$ between $\boldsymbol{A}$ and $\boldsymbol{B}$ such that

$$h \circ f = f^* \circ h,$$

where $(h \circ f)(x_1, \ldots, x_n) = h\big(f(x_1, \ldots, x_n)\big)$ *and* $(f^* \circ h)(x_1, \ldots, x_n) = f^*\big(h(x_1), \ldots, h(x_n)\big)$.

It is obvious that a weak isomorphism h of $\mathfrak{A}$ onto $\mathfrak{B}$ restricted to a subalgebra of $\mathfrak{A}$ is a weak isomorphism of this subalgebra onto a subalgebra of $\mathfrak{B}$.

3. Every isomorphism in the classical sense of two algebras is, of course, also a weak isomorphism, but there exist also other weak isomorphisms. In particular for groups not only an isomorphism but also an inverse isomorphism, i. e. a mapping satisfying the condition $h(xy) = h(y)h(x)$, is a weak isomorphism of algebras.

A natural question has arisen whether there exist other weak isomorphisms of groups than these two.

4. Given two groups G and G^* and a one-to-one mapping $h: G \to G^*$, we can identify the elements of G with their images in G^*. After such identification we have in G two group structures: the original one and that obtained from the group structure of G^*. Denote the group multiplications by xy and $x \times y$ and the inverse operations by x^{-1} and $x^{\dot{-}1}$ respectively. Symbol x^a with an integer a will be meant in the sense of the first group structure. If the mapping h is a weak isomorphism, the class of algebraic operations of the algebras $(G, \cdot, ^{-1})$ and $(G, \times, ^{\dot{-}1})$ must coincide. In other words, $x \times y$ and $x^{\dot{-}1}$ are algebraic functions in the algebra $(G, \cdot, ^{-1})$, i. e.

(2) $$x \times y = x^{a_1} y^{\beta_1} x^{a_2} y^{\beta_2} \ldots x^{a_n} y^{\beta_n} \quad (a_j, \beta_j \text{ — integers}),$$

(3) $$x^{\dot{-}1} = x^a,$$

(4) the units of both group structures coincide (since the unit is a unique constant of the group),

and vice versa, the operations $\cdot$ and $^{-1}$ can be expressed in terms of $\times$ and $^{\dot{-}1}$ in a similar way.

Putting $y = e$ in (2), we get $x \times e = x^{a_1 + a_2 + \ldots + a_n}$. Similarly we have $e \times y = y^{\beta_1 + \beta_2 + \ldots + \beta_n}$, whence, in virtue of (4), it follows that

(5) $$x^{a_1 + a_2 + \ldots + a_n} = x \quad \text{and} \quad x^{\beta_1 + \beta_2 + \ldots + \beta_n} = x.$$

Further

$$e = x \times x^{\dot{-}1} = x \times x^{a} = x^{a_1} x^{a\beta_1} x^{a_2} x^{a\beta_2} \ldots x^{a_n} x^{a\beta_n}$$
$$= x^{a_1+a_2+\ldots+a_n} (x^{\beta_1+\beta_2+\ldots+\beta_n})^{a}$$

or, in virtue of (5), $xx^{a} = e$, which shows that

(6) $$x^{\dot{-}1} = x^{-1}.$$

So the question of finding weak isomorphisms of groups is reduced to the question of finding group structures in a given group with the same inverse operation and binary operations mutually expressible by a formula of form (2) with a's and β's satisfying condition (5).

5. Using the notation of section 4 and assuming that the algebras $(G, \cdot, {}^{-1})$ and $(G, \times, {}^{\dot{-}1})$ are weakly isomorphic, we obtain the following two theorems:

THEOREM 1. *If the square x^2 of every element of the group G belongs to its centre, then either $x \times y = xy$ or $x \times y = yx$. Hence the only possible kind of weak isomorphisms of such groups are usual isomorphisms and inverse isomorphisms.*

Remark. In the particular case of Abelian groups this was shown by K. Urbanik.

Proof. Since the squares are permutable with any element, an expression $x^{a_1} y^{\beta_1} \ldots x^{a_n} y^{\beta_n}$ can always be reduced to the form $x^{a}(yx)^{\varepsilon} y^{\beta}$, where $\varepsilon = 0$ or 1. Consequently $x \times y = x^{a}(yx)^{\varepsilon} y^{\beta}$.

Using condition (5), in the case $\varepsilon = 0$ we get $x^{a} = x$, $y^{\beta} = y$, whence $x \times y = xy$; in the case $\varepsilon = 1$ we have $x^{a+1} = x$ or $x^{a} = e$ and $y^{\beta+1} = y$ or $y^{\beta} = e$, whence $x \times y = yx$.

THEOREM 2. *The lattices of subgroups and lattices of invariant subgroups for both group structures of section 4 coincide. Consequently, a weak isomorphism of groups induces an isomorphism of the lattices of subgroups and an isomorphism of the lattices of invariant subgroups.*

Proof. It is quite evident that each subgroup of $(G, \cdot)$ is a subgroup of $(G, \times)$, and vice versa. If a subgroup H is a normal divisor for the group structure $(G, \cdot)$, then for any $g \in G$ and $h_0, h_1, \ldots, h_n \in H$ and any integers $\gamma_1, \ldots, \gamma_n$ there exists an $h^* \in H$ such that

(7) $$h_0 g^{\gamma_1} h_1 g^{\gamma_2} h_2 \ldots h_{n-1} g^{\gamma_n} h_n = g^{\gamma_1+\gamma_2+\ldots+\gamma_n} h^*.$$

Hence we have, in virtue of (2) and (5),

$$h \times g = h^{a_1} g^{\beta_1} h^{a_2} g^{\beta_2} \ldots h^{a_n} g^{\beta_n} = g^{\beta_1+\ldots+\beta_n} h' = gh'$$

for some $h' \in H$, and

$$g^{-1}\times(h\times g) = g^{-1}\times(gh') = g^{-\alpha_1}(gh')^{\beta_1}g^{-\alpha_2}(gh')^{\beta_2}\dots g^{-\alpha_n}(gh')^{\beta_n}$$
$$= g^{\beta_1-\alpha_1}h_1' g^{\beta_2-\alpha_2}h_2'\dots g^{\beta_n-\alpha_n}h_n' = g^{-(\alpha_1+\dots+\alpha_n)+(\beta_1+\dots+\beta_n)}h^{**}$$
$$= g^{-1}gh^{**} = h^{**}$$

for due elements $h_1', h_2', \dots, h_n', h^{**} \in H$.

It follows from a result by Hanna Neumann [3] that the same result as in Theorem 1 holds for free groups.

However, Hulanicki and Świerczkowski have given, in [2], an example of a group which is nilpotent of order two and such that there exists a group operation $x\times y$ with the required properties different from xy and yx. This shows that the identity mapping $G\to G$ is a weak isomorphism of $(G, \cdot, ^{-1})$ and $(G, \times, ^{-1})$, which is neither an isomorphism nor an inverse isomorphism. Nevertheless it is not known whether the groups with these two structures are isomorphic or not (of course if such isomorphism exists it must be different from the identity).

6. Similar questions were solved by Traczyk [4] for Post algebras and, in particular, Boolean algebras. Every weak isomorphism of Boolean algebra is either an isomorphism or a mapping h such that $h(a\cup b) = h(a)\cap h(b)$ and $h(a\cap b) = h(a)\cup h(b)$. More generally, in the case of Post algebras, every weak isomorphism is a composition of an usual isomorphism and some standard mapping which permutes the constants of the algebra.

7. Finally we shall define the weak homomorphism of algebras. Let be given two algebras $\mathfrak{A} = (A, \boldsymbol{A})$ and $\mathfrak{B} = (B, \boldsymbol{B})$ and a mapping h of A into B. Making use of the mapping h we define a relation ϱ_h between $\boldsymbol{A}$ and $\boldsymbol{B}$ setting for $f\in\boldsymbol{A}$ and $f^*\in\boldsymbol{B}$

$$f\varrho_h f^* \text{ if and only if } f^*\circ h = h\circ f.$$

This relation has obviously the following properties:

1° If $f\in\boldsymbol{A}^{(n)}$ and $f\varrho_h f^*$, then $f^*\in\boldsymbol{B}^{(n)}$.

2° If $f\in\boldsymbol{A}^{(n)}$, $g_i\in A$ $(i=1,\dots,n)$, $f\varrho_h f^*$ and $g_i\varrho_h g_i^*$, then

$$f(g_1,\dots,g_n)\varrho_h f^*(g_1^*, g_2^*, \dots, g_n^*).$$

3° Given three algebras $\mathfrak{A} = (A, \boldsymbol{A})$, $\mathfrak{B} = (B, \boldsymbol{B})$, $\mathfrak{C} = (C, \boldsymbol{C})$ and mappings $h_1: A\to B$ and $h_2: B\to C$, if $f\in\boldsymbol{A}$, $f^*\in\boldsymbol{B}$, $f^{**}\in\boldsymbol{C}$, $f\varrho_{h_1}f^*$ and $f^*\varrho_{h_2}f^{**}$, then $f\varrho_{h_2h_1}f^{**}$.

The mapping h is called a *weak homomorphism* of $\mathfrak{A}$ into $\mathfrak{B}$ if to every operation $f\in\boldsymbol{A}$ there exists an operation $f^*\in\boldsymbol{B}$ such that $f\varrho_h f^*$ and, vice versa, to each $f^*\in\boldsymbol{B}$ there exists such an $f\in\boldsymbol{A}$.

Using the properties 1°-3° it is easy to prove that the following theorems hold:

THEOREM 3. *If h_1 is a weak homomorphism of $\mathfrak{A}$ into $\mathfrak{B}$ and h_2 a weak homomorphism of $\mathfrak{B}$ into $\mathfrak{C}$, then $h_2 h_1$ is a weak homomorphism of $\mathfrak{A}$ into $\mathfrak{C}$.*

THEOREM 4. *If a weak homomorphism h of an algebra $\mathfrak{A}$ into $\mathfrak{B}$ is a one-to-one mapping of A onto B, then the inverse mapping h^{-1} is also a weak homomorphism $\mathfrak{B} \to \mathfrak{A}$ and h is a weak isomorphism of algebras $\mathfrak{A}$ and $\mathfrak{B}$.*

It follows from property 2° that if the algebras $\mathfrak{A}$ and $\mathfrak{B}$ are defined in terms of fundamental operations, then it suffices to require in the definition of a weak homomorphism that to each *fundamental* operation f of the algebra $\mathfrak{A}$ there exists *algebraic* operation f^* of the algebra $\mathfrak{B}$ such that $f \varrho_h f^*$ and to each *fundamental* operation g^* of $\mathfrak{B}$ there corresponds an *algebraic* operation g of $\mathfrak{A}$ such that $g \varrho_h g^*$. Therefore every homomorphism h of algebras in the usual sense is, at the same time, a weak homomorphism, since in this case there is given a fixed one-to-one correspondence between fundamental operations such that the corresponding fundamental operations f and f^* are in relation $f \varrho_h f^*$.

8. Let us observe that, in general, the relation ϱ_h is not one-to-one; to an operation $f \in \boldsymbol{A}$ can exist many operations $f^* \varrho \mathfrak{B}$ satisfying $f \varrho_h f^*$ and vice versa.

A weak homomorphism h of $\mathfrak{A}$ into $\mathfrak{B}$, when restricted to a subalgebra $\mathfrak{A}'$ of $\mathfrak{A}$, is also a weak homomorphism of $\mathfrak{A}'$ into $\mathfrak{B}$. However, h can cease to be a homomorphism when we change one or both algebras by reducing the classes $\boldsymbol{A}$ or $\boldsymbol{B}$ of algebraic functions to a subclass containing the trivial operations and closed under compositions.

The problem of a deeper investigations of the concept of weak homomorphism remains open.

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