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PROPERTIES OF THE FAMILY OF INDEPENDENT SUBSETS OF A GENERAL ALGEBRA *

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1. Introduction and summary of results. One of the important problems met by Marczewski's independence theory [1], [2] was that of characterizing the class of independent subsets of an abstract algebra in terms of general set theory (see [3]). The problem is: given an abstract, finite or infinite, set A and a family $\boldsymbol{J}$ of subsets of A, what are the conditions on $\boldsymbol{J}$ necessary and sufficient in order that $\boldsymbol{J}$ be the class of independent subsets of an abstract algebra whose set of elements is A?

It is fairly obvious that the following two conditions are necessary:

(i) $\boldsymbol{J}$ is hereditary,

(ii) $\boldsymbol{J}$ is of finite character (i. e. if $X \in \boldsymbol{J}$, for any finite $X \subset Y$, then $Y \in \boldsymbol{J}$).

Examination of simple examples shows that it is too much to expect that it will be possible to find simple conditions both necessary and sufficient, though it was not before Urbanik's results [6] when the complexity of the situation became clear. On the other hand, there are some easy conditions which, added to (i) and (ii), form a system of sufficient conditions for the existence of an algebra with the prescribed family $\boldsymbol{J}$ of independent sets; these, at the first place, are due to S. Świerczkowski [5].

In what follows, partly for completeness and partly to take advantage of the situation and to clarify its peculiarities, we give a brief sketch of the results of Świerczkowski [5] and Urbanik [6] adding some improvements to them.

Our notation is the following: $\langle A; \boldsymbol{F}\rangle$ denotes an algebra in which $\boldsymbol{F}$ is a set of fundamental operations; $\boldsymbol{A}^{(n)}$ is the family of algebraic operations of n variables. $C(X)$ denotes the subalgebra generated by a set $X \subset A$. In virtue of (ii) it is sufficient to consider only finite independent sets. $\mathrm{Ind} = \mathrm{Ind}\langle A; \boldsymbol{F}\rangle$ is the family of all finite independent sets in the

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algebra $\langle A; \boldsymbol{F}\rangle$. $(A, \boldsymbol{J})$ is a pair in which A is an arbitrary set and $\boldsymbol{J}$ a hereditary family of finite subsets of A. $\langle A; \boldsymbol{F}\rangle$ is called an *$(A, \boldsymbol{J})$-algebra* if $\operatorname{Ind}\langle A; \boldsymbol{F}\rangle = \boldsymbol{J}$. For a fixed pair $(A; \boldsymbol{J})$, we put

$$\boldsymbol{S} = \{x: \{x\} \notin \boldsymbol{J}\}.$$

Suppose that there exists a maximal number m such that every subset of m elements of A disjoint with $\boldsymbol{S}$ belongs to $\boldsymbol{J}$ (if such an m does not exist, then there is an $(A, \boldsymbol{J})$-algebra; such is of course the algebra $\langle A; \boldsymbol{S}\rangle$). We put

$$\boldsymbol{J}^* = \{X: X \in \boldsymbol{J}, |X| > m\} \ (^1),$$

$$U = \bigcup_{X \in \boldsymbol{J}} X, \qquad U^* = \bigcup_{X \in \boldsymbol{J}^*} X.$$

THEOREM (S. Świerczkowski). *If $|\boldsymbol{S}| \geqslant |\boldsymbol{J}|$, then an $(A, \boldsymbol{J})$-algebra exists.*

It can be seen at first glance that the set $\boldsymbol{S}$ of "self-dependent" elements is depressingly large and one would like to find weaker conditions satisfied in some important cases. We present here two conditions of that kind.

THEOREM 1. *Suppose there exists a mapping $f: \boldsymbol{J} \to \boldsymbol{S}$ such that if $X \neq Y$ and $X \cup Y \in \boldsymbol{J}$, then $f(X) \neq f(Y)$. Then an $(A, \boldsymbol{J})$-algebra exists.*

The proof of Theorem 1 is an easy modification of the proof of the theorem of Świerczkowski.

THEOREM 2. *Suppose that $\boldsymbol{S} \neq 0$ and that there exists a mapping $f: \boldsymbol{J}^* \to U \setminus U^*$ such that if $X, Y, X \cup Y \in \boldsymbol{J}^*$, then $f(X) \neq f(Y)$. Then an $(A, \boldsymbol{J})$-algebra exists.*

Though Theorems 1 and 2 are similar, we have not found any reasonable common generalization of them.

In order to obtain some necessary conditions, K. Urbanik has found a wide class of hereditary families $\boldsymbol{J}$ of finite subsets of a set A with the property that if for an algebra whose set of elements is A and the family $\boldsymbol{J}$ is contained in the family of the independent subsets, then the algebra is trivial and so any subset of it is independent. Let us quote a typical theorem of Urbanik and one of its corollaries.

THEOREM (K. Urbanik). *Let n and m be integers satisfying $n > m$, $n > 3$, and let $\langle A; \boldsymbol{F}\rangle$ be a finite algebra without algebraic constants containing at least $n+m$ elements. Suppose there exists an m-element subset M of A such that each n-element subset of $A \setminus M$ is independent. Then each n-element subset of A is independent.*

(1) $|X|$ denotes the power of X.

This clearly implies

COROLLARY (K. Urbanik). *Let n and m be integers satisfying the inequalities $n > m$, $n > 3$, and let A be a finite set containing at least $n+m$ elements and M an m-element subset of A. For any hereditary family $\boldsymbol{J}$ of subsets of A containing all one-point sets and all n-elements subsets of $A \setminus M$ which does not contain all n-elements subsets of A, an $(A, \boldsymbol{J})$-algebra does not exist.*

We present here another type of "bad" families $\boldsymbol{J}$. First we will prove another theorem analogous to a theorem of Świerczkowski ([4], th. 2).

THEOREM 3. *Let $|A| \geqslant n > 4$ and $\langle A; \boldsymbol{F}\rangle$ be an algebra which has some sets $B \subset D \subset A$ with $|B| = n-1$, $|A \setminus D| \leqslant n-2$, and such that for any $x \in D \setminus B$ the set $B \cup \{x\}$ is a base (i. e. a set of independent generators of $\langle A; \boldsymbol{F}\rangle$), and there is no constant in the subalgebra $C(B)$. Then $\langle A; \boldsymbol{F}\rangle$ is the trivial algebra.*

And then the negative results follows.

COROLLARY. *Let $\aleph_0 > |A| \geqslant n > 4$, $B \subset D \subset A$, $|B| = n-1$ and $|A \setminus D| \leqslant n-2$. For any hereditary family $\boldsymbol{J}$ of finite subsets of A which contains all one-element subsets of A, all sets of the form $B \cup \{x\}$ with $x \in D \setminus B$ and not all n-element subsets of A, an $(A, \boldsymbol{J})$-algebra does not exist.*

In [1] Marczewski treated the following problem: Let μ denote a finitely additive measure over some set $\mathscr{X}$, let $\mu(\mathscr{X}) = 1$ and B be the family of μ-measurable subsets of $\mathscr{X}$. Does there exist a $(B, \boldsymbol{J})$-algebra in which $\boldsymbol{J}$ is a family of finite subsets of B stochastically μ-independent? Marczewski gives a negative solution of this problem under the additional assumption that the union of two elements of B is an algebraic operation in that algebra (see [1], p. 736). We can remove the last supposition.

THEOREM 4. *If $0 < \mu(\mathscr{E}) < 1$ for a certain $\mathscr{E} \in B$, then there exists no $(B, \boldsymbol{J})$-algebra.*

This theorem will be shown to be an easy consequence of the following

PROPOSITION. *If in the algebra $\langle A; \boldsymbol{F}\rangle$ there is a set P with $n > 0$ elements such that, for any $x \in A \setminus P$, $P \cup \{x\}$ is independent, then every operation of n variables is trivial.*

2. Proofs. Proof of Theorem 1, as in [5].

For the proof of Theorem 2 we need first the following

LEMMA. *In the algebra $\langle A; F\rangle$, let $B \subset A$ be a set such that $|A \setminus B| \geqslant n > 1$. If for every non-trivial operation f of $n-1$ variables and an arbitrary system $a_1, \ldots, a_{n-1} \in A$ we have $f(a_1, \ldots, a_{n-1}) \in B$, then there exists an $(A, \boldsymbol{N})$-algebra, where $\boldsymbol{N}$ is the set $\{X: X \in \mathrm{Ind}\langle A; \boldsymbol{F}\rangle$ and $|X| < n\}$.*

Proof. We define an operation by

$$g(x_1, \ldots, x_n) = \begin{cases} x_n & \text{if } x_i \text{ are different and all } x_i \notin B, \\ x_1 & \text{in the other case.} \end{cases}$$

In the algebra $\langle A; \boldsymbol{F} \cup \{g\}\rangle$ every n-element set is dependent, because g has the same values as e_n^n or e_1^n but is not equal to any of them. Therefore for any n-element set there are different operations having the same values on this set.

Now we show that every operation of $n-1$ variables which is algebraic in $\langle A; \boldsymbol{F} \cup \{g\}\rangle$ is algebraic in $\langle A; \boldsymbol{F}\rangle$ too, i.e. we must verify that

$$h(x_1, \ldots, x_{n-1}) = g\big(f_1(x_1, \ldots, x_{n-1}), \ldots, f_n(x_1, \ldots, x_{n-1})\big)$$

is an algebraic operation in $\langle A; \boldsymbol{F}\rangle$ if f_i are algebraic operations in $\langle A; \boldsymbol{F}\rangle$.

If some f_i is non-trivial, then its values on each system $x_1, \ldots, x_{n-1}$ belong to B and in view of the definition of g

$$(+) \qquad h(x_1, \ldots, x_{n-1}) = f_1(x_1, \ldots, x_{n-1}).$$

And if all f_i are trivial, there must be repetitions among them, because there are only $n-1$ trivial operations of $n-1$ variables. Hence we have also $(+)$.

Since the algebras $\langle A; \boldsymbol{F}\rangle$ and $\langle A; \boldsymbol{F} \cup \{g\}\rangle$ have the same operations of $n-1$ variables, then each at most $(n-1)$-element set is independent in one of them iff it is independent in the other.

Proof of theorem 2. Let $c \in \boldsymbol{S}$ and f be a mapping as supposed in our theorem. We take the notation

$$(*) \quad f^n_{\{i_1, \ldots, i_k\}}(x_1, \ldots, x_n) = \begin{cases} f\{x_{i_1}, \ldots, x_{i_k}\} & \text{if } \{x_{i_1}, \ldots, x_{i_k}\} \in \boldsymbol{J}^* \text{ and all } x_{i_j} \\ & \text{are different,} \\ c & \text{in the other case.} \end{cases}$$

We will show that the algebra $\langle A; \boldsymbol{F}\rangle = \langle A; \boldsymbol{S} \cup$

$$\big\{f^n_{\{i_1, \ldots, i_k\}}\colon n > m,\ \{i_1, \ldots, i_k\} \subset \{1, \ldots, n\}, \text{ and } |\{i_1, \ldots, i_k\}| > m\big\}$$

has no non-trivial algebraic operations with the exception of fundamental operations.

Let

$$g(x_1, \ldots, x_n) = f^m_{\{i_1, \ldots, i_k\}}\big(f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n)\big).$$

If some f_{i_j} is of the form $(*)$, then for each sequence $a_1, \ldots, a_n$ we have $f_{i_j}(a_1, \ldots, a_n) \in U \setminus U^* \cup \{c\}$, whence, consequently, $f_{i_1}(a_1, \ldots, a_n), \ldots$ $\ldots, f_{i_k}(a_1, \ldots, a_n) \notin \boldsymbol{J}^*$ and $g(x_1, \ldots, x_n) = c$. Of course, the last equation holds if either there are repetitions among f_{i_j} or some of them are constant. If all f_{i_j} are trivial, i. e. if $f_{i_j} = e^n_{p_j}$, then

$$g(x_1, \ldots, x_n) = \begin{cases} \{f\{x_{p_1}, \ldots, x_{p_k}\} \text{ if } \{x_{p_1}, \ldots, x_{p_k}\} \in \boldsymbol{J}^* \text{ and all } x_{p_j} \text{ are different,} \\ c \text{ in the other case,} \end{cases}$$

i. e. $g(x_1, \ldots, x_n) = f^m_{\{p_1, \ldots, p_k\}}(x_1, \ldots, x_n)$.

In the algebra $\langle A; \boldsymbol{F}\rangle$ there are no algebraic operations of m variables but the trivial operations. Thus all at most m-element sets disjoint with $\boldsymbol{S}$ are independent.

Let $\{a_1, \ldots, a_n\} \in \boldsymbol{J}^*$. Different algebraic operations of n variables have different values on this set since

$$f^n_{\{i_1, \ldots, i_k\}}(a_1, \ldots, a_n) = f\{a_{i_1}, \ldots, a_{i_k}\}$$

and f is a one-to-one mapping on sets $\in \boldsymbol{J}^*$. Therefore $f^n_{\{i_1, \ldots, i_k\}}(a_1, \ldots, a_n) \in (U \setminus U^*)$, $a_i \in U^*$ and $S \cap U = 0$ imply $\{a_1, \ldots, a_n\} \in \mathrm{Ind}\langle A; \boldsymbol{F}\rangle$.

Let $0 < n \leqslant \sup\limits_{X \in \boldsymbol{J}} |X| = l$ (if $n = 0$, $\langle A; \boldsymbol{F}\rangle$ is of course an $\langle A; \boldsymbol{J}\rangle$-algebra). If $\{a_1, \ldots, a_n\} \notin \boldsymbol{J}$, then $f^n_{\{1,2,\ldots,n\}}(a_1, \ldots, a_n) = c$. Because there exists an n-element independent set, then

$$f^n_{\{1,2,\ldots n\}}(x_1, \ldots, x_n) \neq c,$$

and, consequently, $\{a_1, \ldots, a_n\} \notin \mathrm{Ind}$.

If $x \in \boldsymbol{S}$, then $x \notin \mathrm{Ind}\langle A; \boldsymbol{F}\rangle$, because it is a constant.

By these facts, if $l = \aleph_0$, then $\langle A; \boldsymbol{F}\rangle$ is an $(A, \boldsymbol{J})$-algebra. Let $l < \aleph_0$. If $|U^*| = l$, then $\langle A; \boldsymbol{F}\rangle$ is an $(A, \boldsymbol{J})$-algebra in view of the fact that only one l-element set belongs to $\boldsymbol{J}$ and, consequently, every $(l+1)$-element set is dependent since it contains dependent subsets. If $|U^*| > l$, then the set $(U \setminus U^*) \cup S$ satisfies the supposition of the lemma concerning B. Hence by that lemma there exists an $(A; \boldsymbol{J})$-algebra.

Proof of Theorem 3. For every $x \in C(B) \setminus B$ we have $x \in A \setminus D$, because for every $d \in D$ the set $B \cup \{d\}$ is independent. Since subalgebra $C(B)$ has no constants, $|B| = n-1 > 3$, $|A \setminus D| \leqslant n-2$, and B is independent, hence in view of the theorem of Urbanik stated in the introduction every $(n-1)$-element set of this algebra is independent. The algebra $C(B)$ being finite and having an $(n-1)$-element base, the independend sets generate subalgebras of the same powers and $n-1 > 3$. Hence in view of a result of Świerczkowski ([4], th. 2) the algebra $C(B)$ is trivial.

Suppose that the algebra has non-trivial operations of n variables. A lemma of Urbanik [6] states that if

(α) *there are repetitions among* $x_1, \ldots, x_n$,

then for every operation f there exists an index $k = k_f$ such that $f(x_1, \ldots, x_n) = x_k$.

Now we will show that if (a) holds, then

(β) *for every* $x \in D \setminus B$ *there exists an* $y \neq x$, $y \in D \setminus B$ *and a function* $g \in \boldsymbol{A}^n$ *such that* $g(b_1, \ldots, b_{n-1}, x) = y$ *and* $g(x_1, \ldots, x_n) \neq x$.

In $\boldsymbol{A}^n$ there are n non-trivial function f_i such that for $i \neq j$ we have $k_{f_i} \neq k_{f_j}$. In fact, if for example f_1 is a non-trivial function and $k_{f_1} = 1$, we have to take for f_j a function $f_1(x_{i_1}, \ldots, x_{i_n})$ where $i_1, \ldots, i_n$ is a permutation such that $i_1 = j$. Functions f_i are different, since $k_{f_i} = i$, and non-trivial, since $f_1(b_1, \ldots, b_{n-1}, x)$ does not belong to the set $\{b_1, \ldots, b_{n-1}, x\}$ and, consequently, neither $f_j(b_1, \ldots, b_{n-1}, x)$ does.

In view of independence of the set $\{b_1, \ldots, b_{n-1}, x\}$ all f_i take different values on it and, on account of $|A \setminus D| \leqslant n-2$, at least two of them take values from $D \setminus B \cup \{x\}$. Choose from these two functions this f_i for which $k_{f_i} \neq n$ (different functions have different indices) and denote it by g. Taking $g(b_1, \ldots, b_{n-1}, x)$ as y we see that condition (β) is satisfied. Since $\{b_1, \ldots, b_{n-1}, y\}$ is a base, there exists a function h such that $h(b_1, \ldots, b_{n-1}, y) = x$. Substituting $g(b_1, \ldots, b_{n-1}, x)$ in place of y we get the equation

(γ) $x = h\big(b_1, \ldots, b_{n-1}, g(b_1, \ldots, b_{n-1}, x)\big)$.

The set of arguments is independent, thus also (γ) holds, if we do any substitution for them. Of course, $g(b_1, b_1, b_3, \ldots, b_{n-1}, x) = b_{k_g} \neq x$ since $k_g \neq n$. Replacing in (γ) the element b_1 by b_2, we get

$$x = h(b_1, b_1, b_3, \ldots, b_{n-1}, b_{k_g}),$$

which contradicts the triviality of $\boldsymbol{A}^{(n-1)}$.

Proof of Corollary. Suppose that such an algebra exists. But the subalgebra $C(B \cup \{x\})$ satisfies the assumptions of Theorem 3 and so must be trivial. Hence each algebraic operation of n variables must be trivial in view of independence of the set $B \cup \{x\}$. Consequently, each n-element set must be independent which contradicts our assumptions.

Proof of Proposition. Let $P = \{b_1, \ldots, b_n\}$. In view of the independence of $P \cup \{f(b_1, \ldots, b_n)\}$ for any operation f, we have $f(b_1, \ldots, \ldots, b_n) \in P$. Thus P is the set of a subalgebra. Since P is independent, each operation of n variables must be trivial.

Proof of Theorem 4. Suppose that a $(B; \boldsymbol{J})$-algebra exists. Any triple which contains the empty set and the full set is of course stochastically μ-independent. According to Proposition every two-element set of our $(B; \boldsymbol{J})$-algebra must be independent. But the pair $\{\mathscr{E}, \mathscr{X} \setminus \mathscr{E}\}$ is stochastically dependent which gives a contradiction.

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