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LINEAR INDEPENDENCE IN ABSTRACT ALGEBRAS

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1. Introduction. E. Marczewski introduced some classes of abstract algebras in which the notion of independence (see [5]) has fundamental properties of linear independence in vector spaces. These classes are: v-algebras (called also *Marczewski's algebras*; see [4] and [11]), v^*-algebras (see [1], [7], [8], [12] and [13])and v_*^*-algebras (called also v^{**}-algebras; see [9]). W. Narkiewicz has recently defined a class of v_*-algebras being an extension of the class of v-algebras (see [14]). This expository paper* is devoted to the representation problem for algebras belonging to these four classes. Theorems proved in earlier papers will be presented here without proof.

For terminology and notation used here see [5]. Let $\mathfrak{A} = (A; \boldsymbol{F})$ be an *algebra*, i. e. a set A of elements and a class $\boldsymbol{F}$ of fundamental operations consisting of A-valued functions of several variables running over A. If $A = \{a, b, \ldots\}$ and $\boldsymbol{F} = \{f, g, \ldots\}$, we shall sometimes write $(a, b, \ldots; f, g, \ldots)$ or $(A; f, g, \ldots)$ instead of $(A; \boldsymbol{F})$. The n-ary operations

$$e_k^{(n)}(x_1, x_2, \ldots, x_n) = x \qquad (k = 1, 2, \ldots, n;\ n = 1, 2, \ldots)$$

will be called *trivial*. We denote by $\boldsymbol{A}$ the class of all *algebraic operations*, i. e. the smallest class containing trivial operations and closed under the composition with fundamental operations. The subclass of all n-ary algebraic operations will be denoted by $\boldsymbol{A}^{(n)}$ ($n \geqslant 1$). Further, by $\boldsymbol{A}^{(0)}$ we shall denote the class of all *algebraic constants*, i. e. the class of values of constant algebraic operations. If $1 \leqslant k \leqslant n$, then $\boldsymbol{A}^{(n,k)}$ will denote the subclass of $\boldsymbol{A}^{(n)}$ consisting of all operations essentially depending on at most k variables. Thus $f \in \boldsymbol{A}^{(n,k)}$ if there is an operation $g \in \boldsymbol{A}^{(k)}$ such that $f(x_1, x_2, \ldots, x_n) = g(x_{i_1}, x_{i_2}, \ldots, x_{i_k})$ for a system of indices i_1, $i_2, \ldots, i_k$. Two algebras $(A; \boldsymbol{F}_1)$ and $(A; \boldsymbol{F}_2)$ having the same class of algebraic operations will be treated here as identical. In particular, we have the equality $(A; \boldsymbol{F}) = (A; \boldsymbol{A})$. If a non-void subset B of A is closed

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with respect to $\boldsymbol{F}$, then the algebra $(B; \boldsymbol{F})$ is called a *subalgebra* of the algebra $(A; \boldsymbol{F})$. If E is a non-void subset of A, then the smallest subalgebra containing E will be denoted by $[E]$. If $E = \{a_1, a_2, \ldots, a_n\}$, then we shall sometimes write $[E] = [a_1, a_2, \ldots, a_n]$. Moreover, for the empty set $\emptyset$ we put $[\emptyset] = (\boldsymbol{A}^{(0)}, \boldsymbol{F})$.

Following Marczewski [5], we say that elements of a non-void set I $(I \subset A)$ are *independent* if for each system of different elements $a_1, a_2, \ldots, a_n$ from I and for each pair of operations $f, g \in \boldsymbol{A}^{(n)}$ the equation

$$f(a_1, a_2, \ldots, a_n) = g(a_1, a_2, \ldots, a_n)$$

implies that f and g are identical. A set whose elements are not independent will be called a set of *dependent* elements. An element $a \in A$ is said to be *self-dependent* if the one-point set containing a is a set of dependent elements.

We say that a set B $(B \subset A)$ is a *basis* of the algebra $(A; \boldsymbol{F})$ if it is a set of independent elements and $[B] = (A; \boldsymbol{F})$. If an algebra $\mathfrak{A}$ has a basis and all bases have the same cardinal number, then this cardinal number will be called the *dimension* of the algebra $\mathfrak{A}$ and denoted by $\dim \mathfrak{A}$. Furthermore, if all elements of the algebra $\mathfrak{A}$ are algebraic constants or, in other words, $[\emptyset] = \mathfrak{A}$, then we put $\dim \mathfrak{A} = 0$.

Let $f, g \in \boldsymbol{A}^{(n)}$ $(n \geqslant 1)$. We say that the equation

$$f(x_1, x_2, \ldots, x_n) = g(x_1, x_2, \ldots, x_n)$$

depends on the variable x_n if there exists a system $a_1, a_2, \ldots, a_n, a_n'$ of elements of A for which

$$f(a_1, a_2, \ldots, a_{n-1}, a_n) = g(a_1, a_2, \ldots, a_{n-1}, a_n)$$

and

$$f(a_1, a_2, \ldots, a_{n-1}, a_n') \neq g(a_1, a_2, \ldots, a_{n-1}, a_n').$$

Now we shall give the definition of v-algebras, v_*-algebras, v^*-algebras and v_*^*-algebras.

An algebra $\mathfrak{A}$ is called a *v-algebra* if for every integer n $(n \geqslant 1)$ and for every pair of operations $f, g \in \boldsymbol{A}^{(n)}$ for which the equation

$$f(x_1, x_2, \ldots, x_n) = g(x_1, x_2, \ldots, x_n) \tag{1.1}$$

depends on the variable x_n there exists an operation $h \in \boldsymbol{A}^{(n-1)}$ such that the equation (1.1) is equivalent to the equation

$$x_n = h(x_1, x_2, \ldots, x_{n-1}).$$

An algebra $\mathfrak{A}$ is called a *v_*-algebra* if for every integer n $(n \geqslant 1)$ and for every pair of operations $f, g \in \boldsymbol{A}^{(n)}$ for which the equation

$$f(x_1, x_2, \ldots, x_n) = g(x_1, x_2, \ldots, x_n) \tag{1.2}$$

depends on the variable x_n there exist an index j $(1 \leqslant j \leqslant n)$ and an operation $h \in \boldsymbol{A}^{(n-1)}$ such that the equation (1.2) is equivalent to the equation

$$x_j = h(x_1, x_2, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n).$$

An algebra $\mathfrak{A}$ is called a *v^*-algebra* if it satisfies the following conditions:

(i) each self-dependent element is an algebraic constant,

(ii) if the elements $a_1, a_2, \ldots, a_n$ $(n \geqslant 1)$ are independent and the elements $a_1, a_2, \ldots, a_n, a_{n+1}$ are dependent, then $a_{n+1} \in [a_1, a_2, \ldots, a_n]$.

Condition (i) may be treated as a degenerated case $(n = 0)$ of (ii). The v^*-algebras can be also defined as follows. We say that the elements of a non-void subset $E \subset A$ are *C-independent* if $a \notin [E \setminus \{a\}]$ for every $a \in E$. Then the v^*-algebras are the algebras in which the following two axioms hold:

EQUIVALENCE AXIOM. Independence coincides with C-independence.

EXCHANGE AXIOM. If $a \notin [E]$ and $a \in [E \cup \{b\}]$, then $b \in [E \cup \{a\}]$.

An algebra $\mathfrak{A}$ is called a *v_*^*-algebra* if it satisfies the following conditions:

$(*)$ each self-dependent element is an algebraic constant,

$(**)$ if the elements $a_1, a_2, \ldots, a_n$ $(n \geqslant 1)$ are independent and the elements $a_1, a_2, \ldots, a_n, a_{n+1}$ are dependent, then there exists an index j $(1 \leqslant j \leqslant n+1)$ such that $a_j \in [a_1, a_2, \ldots, a_{j-1}, a_{j+1}, \ldots, a_{n+1}]$.

The v_*^*-algebras can be also defined as algebras in which the equivalence axiom holds.

It is very easy to prove that the class of v_*^*-algebras contains the classes of v-algebras, v_*-algebras and v^*-algebras. Moreover, both classes of v_*-algebras and v^*-algebras contain the class of v-algebras. In the sequel it will be shown that all these classes are different.

W. Narkiewicz proved in [9] (Theorem I) that *if a v_*^*-algebra has a basis, then all bases have the same cardinal number.* Thus for any v_*^*-algebra with a basis the concept of dimension is well defined. Moreover, *each v^*-algebra and, consequently, v-algebra either has a basis or consists of algebraic constants* (see [4], p. 616, and [7], p. 334). Thus for any v^*-algebra the concept of dimension is defined.

2. v-algebras. First we shall give some examples of v-algebras.

2.1. Let A be a linear space over a field $\mathscr{K}$ (i. e. an associative division ring not necessarily commutative) and let A_0 be a linear subspace of A. If $\boldsymbol{A}$ is the class of all operations f defined as

$$f(x_1, x_2, \ldots, x_n) = \sum_{k=1}^{n} \lambda_k x_k + a,$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathscr{K}$ and $a \in A_0$, then $\mathfrak{A} = (A; \boldsymbol{A})$ is a v-algebra. In this case we have the relations $\boldsymbol{A}^{(0)} \neq \emptyset$ and $\boldsymbol{A}^{(n)} \neq \boldsymbol{A}^{(n,1)}$ $(n \geqslant 2)$ whenever A contains at least two elements. Moreover, denoting by $\operatorname{lin}\dim B$ the linear dimension of the linear space B and by B/B_0 the quotient space with respect to a linear subspace B_0 of B we have the formula

$$\dim \mathfrak{A} = \operatorname{lin}\dim A/A_0.$$

2.2. Let A be a linear space over a field $\mathscr{K}$ and let A_0 be a linear subspace of A. If $\boldsymbol{A}$ is the class of all operations f defined as

$$f(x_1, x_2, \ldots, x_n) = \sum_{k=1}^{n} \lambda_k x_k + a,$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathscr{K}$, $\sum_{k=1}^{n} \lambda_k = 1$ and $a \in A_0$, then $\mathfrak{A} = (A; \boldsymbol{A})$ is a v-algebra. In this case we have the relations $\boldsymbol{A}^{(0)} = \emptyset$ and $\boldsymbol{A}^{(n)} \neq \boldsymbol{A}^{(n,1)}$ $(n \geqslant 3)$ whenever the set A contains at least two elements. Let us remark that $\boldsymbol{A}^{(2)} = \boldsymbol{A}^{(2,1)}$ if the field $\mathscr{K}$ consists of two elements. Moreover,

$$\dim \mathfrak{A} = 1 + \operatorname{lin}\dim A/A_0.$$

2.3. Let $\mathscr{G}$ be a group of permutations of a set A. We suppose that every permutation that is not the identity has at most one fixed point in A. Let A_0 be a subset of A containing all fixed points of permutations that are not the identity and which is invariant under all permutations from $\mathscr{G}$. If $\boldsymbol{A}$ is the class of all operations f defined as

$$f(x_1, x_2, \ldots, x_n) = g(x_j) \qquad (1 \leqslant j \leqslant n),$$

or

$$f(x_1, x_2, \ldots, x_n) = a,$$

where $g \in \mathscr{G}$ and $a \in A_0$, then $\mathfrak{A} = (A; \boldsymbol{A})$ is a v-algebra. In this case we have the relation $\boldsymbol{A}^{(n)} = \boldsymbol{A}^{(n,1)}$ $(n \geqslant 1)$. Let $B \subset A$. By $t(B)$ we shall denote the cardinal number of transitive constituents of B with respect to the permutation group $\mathscr{G}$. It is very easy to prove the formula

$$\dim \mathfrak{A} = t(A \setminus A_0).$$

REPRESENTATION THEOREM FOR v-ALGEBRAS. *Let $\mathfrak{A}$ be a v-algebra.*

(i) *If $\boldsymbol{A}^{(0)} \neq \emptyset$ and $\boldsymbol{A}^{(3)} \neq \boldsymbol{A}^{(3,1)}$, then there is a field $\mathscr{K}$ such that A is a linear space over $\mathscr{K}$ and, further, there exists a linear subspace A_0 of A such that all algebraic operations are given by the formula*

$$f(x_1, x_2, \ldots, x_n) = \sum_{k=1}^{n} \lambda_k x_k + a,$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathscr{K}$ and $a \in A_0$.

(ii) *If* $\boldsymbol{A}^{(0)} = \emptyset$ *and* $\boldsymbol{A}^{(3)} \neq \boldsymbol{A}^{(3,1)}$, *then there is a field* $\mathscr{K}$ *such that* A *is a linear space over* $\mathscr{K}$ *and, further, there exists a linear subspace* A_0 *of* A *such that all algebraic operations are given by the fromula*

$$f(x_1, x_2, \ldots, x_k) = \sum_{k=1}^{n} \lambda_k x_k + a,$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathscr{K}$, $\sum_{k=1}^{n} \lambda_k = 1$ *and* $a \in A_0$.

(iii) *If* $\boldsymbol{A}^{(3)} = \boldsymbol{A}^{(3,1)}$, *then there is a permutation group* $\mathscr{G}$ *of* A *such that. each permutation that is not the identity has at most one fixed point in* A*. Moreover, there is a subset* A_0 *of* A *containing all fixed points of permutations from* $\mathscr{G}$ *that are not the identity and which is invariant under all permutations from* $\mathscr{G}$ *such that all algebraic operations are given by the formulas*

$$f(x_1, x_2, \ldots, x_n) = g(x_j) \qquad (1 \leqslant j \leqslant n),$$

or

$$f(x_1, x_2, \ldots, x_n) = a,$$

where $g \in \mathscr{G}$ *and* $a \in A_0$.

The proof of this theorem is presented in [11].

An algebra is said to be *trivial* if all algebraic operations are trivial. Obviously, all trivial algebras are v-algebras (the case (iii) of the representation theorem for one-element group $\mathscr{G}$). Consider the two-element set $T = \{0, 1\}$ with the addition mod 2. From the representation theorem it follows that the following ten algebras are the only v-algebras over the set T: the trivial algebra $\mathfrak{T}$, $\mathfrak{P}_* = (T; x+y+z)$, $\mathfrak{Q}_1 = (T; x+y+z+1)$, $\mathfrak{Q}_2 = (T; x+y)$, $\mathfrak{Q}_3 = (T; x+1)$, $\mathfrak{Q}_4 = (T; 0)$, $\mathfrak{Q}_5 = (T; 1)$, $\mathfrak{R}_1 = (T; x+y+1)$, $\mathfrak{R}_2 = (T; x+1, 0)$, $\mathfrak{R}_3 = (T; 0, 1)$. Moreover, $\dim \mathfrak{T} = \dim \mathfrak{P}_* = 2$, $\dim \mathfrak{Q}_1 = \dim \mathfrak{Q}_2 = \dim \mathfrak{Q}_3 = \dim \mathfrak{Q}_4 = \dim \mathfrak{Q}_5 = 1$ and $\dim \mathfrak{R}_1 = \dim \mathfrak{R}_2 = \dim \mathfrak{R}_3 = 0$.

3. v_***-algebras.** We start with some examples of v_*-algebras. In this section $\mathscr{R}$ will denote an associative ring with the unit element, without divisors of zero, such that for every pair α, β of elements of $\mathscr{R}$ there exists an element $\gamma \in \mathscr{R}$ satisfying the equation $\alpha = \beta\gamma$ or the equation $\beta = \alpha\gamma$.

3.1. Let A be a *unital left-module over* $\mathscr{R}$ satisfying the *cancellation law*, i. e. a left-module satisfying for every $x \in A$ the condition $1x = x$ and such that for any $\alpha \in \mathscr{R}$ $(\alpha \neq 0)$ and $y \in A$ the relation $\alpha y = 0$ implies $y = 0$. A submodule A_0 of A is said to be *divisible* if for any $\alpha \in \mathscr{R}$ $(\alpha \neq 0)$ and $a \in A$ the relation $\alpha a \in A_0$ implies the relation $a \in A_0$. Given a divisible submodule A_0 of A we denote by $\boldsymbol{A}$ the class of all operations

$$f(x_1, x_2, \ldots, x_k) = \sum_{k=1}^{n} \lambda_k x_k + a,$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathscr{R}$ and $a \in A_0$. The algebra $\mathfrak{A} = (A; \boldsymbol{A})$ is a v_*-algebra (see [9] and [14]). Moreover, if A contains at least two elements, then $\boldsymbol{A}^{(0)} \neq \emptyset$ and $\boldsymbol{A}^{(n)} \neq \boldsymbol{A}^{(n,1)}$ $(n \geqslant 2)$. The algebra in question is an analogue of the algebra (2.1). However, it may have no basis. For instance, as a ring $\mathscr{R}$ we take the ring $\mathscr{R}_p$ of all rational numbers $n/(pm+1)$, where n, m are arbitrary integers and p a fixed prime, under usual addition and multiplication. As the set A we take the set of all rational numbers. Setting $A_0 = \{0\}$, we get an algebra $\mathfrak{A}_p$ in which every element different from zero is independent and every pair of elements is dependent. On the other hand, the algebra $\boldsymbol{A}_p$ is not generated by a finite number of elements. Consequently, it has no basis and zero is the only self-dependent element.

3.2. Consider a unital left-module A over $\mathscr{R}$ satisfying the cancellation law. A subset B of the Cartesian product $\mathscr{R} \times A$ is said to be *admissible* if it satisfies the following conditions:

(i) $\langle 1, 0 \rangle \in B$,

(ii) if $\langle \lambda, a \rangle \in B$, then the element λ is invertible in $\mathscr{R}$,

(iii) if $\mu_1, \mu_2, \ldots, \mu_n \in \mathscr{R}$, $\sum_{j=1}^{n} \mu_j = 1$ and $\langle \lambda_j, a_j \rangle \in B$ $(j = 1, 2, \ldots, n)$, then

$$\langle \sum_{j=1} \mu_j \lambda_j, \sum_{j=1} \mu_j a_j \rangle \in B,$$

(iv) if $a, \lambda \in \mathscr{R}$, $a \neq 0$, $a \in A$ and $\langle 1 + a\lambda - a, aa \rangle \in B$, then $\langle \lambda, a \rangle \in B$.

If A_0 is a divisible submodule of A, then the set $\{1\} \times A_0$ is admissible. Now we shall give a less trivial example of an admissible set. As the ring $\mathscr{R}$ and the module A we take the ring $\mathscr{R}_p$ considered in the preceding example. The set of all elements

$$\left\langle \frac{pn+1}{pm+1}, \frac{n-m}{pm+1} \right\rangle,$$

where n and m are arbitrary integers is admissible (see [14], Section 2).

Given an arbitrary admissible subset B of $\mathscr{R} \times A$ we denote by $\boldsymbol{A}$ the class of all operations

$$f(x_1, x_2, \ldots, x_n) = \sum_{k=1}^{n} \lambda_k x_k + a,$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathscr{R}$, $a \in A$ and $\langle \sum_{k=1}^{n} \lambda_k, a \rangle \in B$. The algebra $\mathfrak{A} = (A; \boldsymbol{A})$ is a v_*-algebra (see [14], Section 3). Moreover, if A contains at least two elements, then $\boldsymbol{A}^{(0)} = \emptyset$ and $\boldsymbol{A}^{(n)} \neq \boldsymbol{A}^{(n,1)}$ for $n \geqslant 3$. This algebra is an analogue of the algebra considered in 2.2.

3.3. Let $\mathscr{S}$ be a semigroup of one-to-one transformations of a non-void set A into itself containing the identical transformation and satisfying the following conditions:

($*$) each transformation that is not the identical transformation has at most one fixed point in A,

($**$) if $g_1, g_2 \in \mathscr{S}$ and $g_1(A) \cap g_2(A) \neq \emptyset$, then there exists a transformation $g \in \mathscr{S}$ such that $g_1 = g_2 g$ or $g_2 = g_1 g$.

Let A_0 be a subset of A containing all fixed points of transformations from $\mathscr{S}$ that are not the identical transformation and satisfying the conditions $g(A_0) \subset A_0$ and $\{a: g(a) \in A_0\} \subset A_0$ for all $g \in \mathscr{S}$. If $\boldsymbol{A}$ is the class of all operations f defined as

$$f(x_1, x_2, \ldots, x_n) = g(x_j) \qquad (1 \leqslant j \leqslant n),$$

or

$$f(x_1, x_2, \ldots, x_n) = a,$$

where $g \in \mathscr{S}$ and $a \in A_0$, then the algebra $\mathfrak{A} = (A; \boldsymbol{A})$ is a v_*-algebra (see [14], Section 3). Of course, $\boldsymbol{A}^{(n)} = \boldsymbol{A}^{(n,1)}$ for all $n \geqslant 1$. This algebra is an analogue of the algebra (2.3).

From these examples it follows, in particular, that the class of v_*-algebras does not coincide with the class of v-algebras.

Representation theorem for v_*-algebras. *Let $\mathfrak{A}$ be a v_*-algebra.*

(i) *If $\boldsymbol{A}^{(0)} \neq \emptyset$ and $\boldsymbol{A}^{(3)} \neq \boldsymbol{A}^{(3,1)}$, then A is a unital left-module satisfying the cancellation law over an associative ring $\mathscr{R}$ with the unit element, without divisors of zero, such that for any pair of elements of $\mathscr{R}$ at least one element is left-divisible by the other one. Moreover, there exists a divisible submodule A_0 of A such that all algebraic operations are given by the formula*

$$f(x_1, \ldots, x_n) = \sum_{k=1}^{n} \lambda_k x_k + a,$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathscr{R}$ and $a \in A_0$.

(ii) *If $\boldsymbol{A}^{(0)} = \emptyset$ and $\boldsymbol{A}^{(3)} \neq \boldsymbol{A}^{(3,1)}$, then A is a unital left-module satisfying the cancellation law over an associative ring $\mathscr{R}$ with the unit element, without divisors of zero, and such that for any pair of elements of $\mathscr{R}$ at least one element is left-divisible by the other one. Moreover, there exists an admissible subset B of $\mathscr{R} \times A$ such that all algebraic operations are given by the formula*

$$f(x_1, x_2, \ldots, x_n) = \sum_{k=1}^{n} \lambda_k x_k + a,$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathscr{R}$, $a \in A$ and $\langle \sum_{k=1}^{n} \lambda_k, a \rangle \in B$.

(iii) *If* $A^{(3)} = A^{(3,1)}$, *then there is a semigroup* $\mathcal{S}$ *of one-to-one transformations of the set* A *into itself containing the identical transformation and satisfying the conditions*

(*) *each transformation that is not the identical transformation has at most one fixed point,*

(**) *if* $g_1 g_2 \in \mathcal{S}$ *and* $g_1(A) \cap g_2(A) \neq \emptyset$, *then at least one element of the pair* g_1, g_2 *is left-divisible by the other one. Moreover, there exists a subset* A_0 *of the set* A *containing all fixed points of transformations that are not the identical transformation and satisfying the conditions* $g(A_0) \subset A_0$ *and* $\{a: g(a) \in A_0\} \subset A_0$ *for all* $g \in \mathcal{S}$ *such that all algebraic operations are given by the formulas*

$$f(x_1, x_2, \ldots, x_n) = g(x_j) \qquad (1 \leqslant j \leqslant n)$$

and

$$f(x_1, x_2, \ldots, x_n) = a,$$

where $g \in \mathcal{S}$ *and* $a \in A_0$.

The proof of this representation theorem is given in [14].

As a consequence of the representation theorem for v_*-algebras we obtain the following theorem:

THEOREM 3.1. *Finite* v_**-algebras are* v*-algebras.*

Proof. Suppose that the algebra $(A; \boldsymbol{F})$ is a finite v_*-algebra. In the cases (i) and (ii) of the representation theorem for v_*-algebras the set A is a unital left-module satisfying the cancellation law over a ring $\mathcal{R}$ without divisors of zero. By the cancellation law for A the ring $\mathcal{R}$ is also finite and, consequently, is a field. Thus in the case (i) the algebra $(A; \boldsymbol{F})$ is a v-algebra. In the case (ii) the only admissible subsets of $\mathcal{R} \times A$ are the products $\{1\} \times A_0$, where A_0 is a linear subspace of A_0 (see [14], Theorem 2.2). Hence it follows that in the case (ii) the algebra $(A; \boldsymbol{F})$ is also a v-algebra. In the case (iii) the semigroup of one-to-one transformations of the finite set A is a permutation group and, consequently, the algebra $(A; \boldsymbol{F})$ is a v-algebra.

THEOREM 3.2. *Let* $(A; \boldsymbol{F})$ *be a* v_**-algebra containing at least two independent elements. If every set of independent elements can be extended to a basis, then* $(A; \boldsymbol{F})$ *is a* v*-algebra.*

Proof. The same reasoning as in the proof of the preceding theorem shows that in the cases (i) and (ii) of the representation theorem to prove our statement it suffices to prove that the ring $\mathcal{R}$ is a field.

Let a_1, a_2 be a pair of independent elements of A and λ an arbitrary element of $\mathcal{R}$ different from zero. Obviously, the operation $f(x, y) = \lambda x + (1-\lambda)y$ is algebraic and depends on the variable x.

First we shall prove that the elements $f(a_1, a_2)$ and a_2 are independent. Suppose the contrary. Since v_*-algebras are v_*^*-algebras, we infer that either $a_2 \in [f(a_1, a_2)]$ or $f(a_1, a_2) \in [a_2]$. Consequently, there is an algebraic unary operation g such that either $a_2 = g(f(a_1, a_2))$ or $f(a_1, a_2) = g(a_2)$. By the independence of a_1 and a_2 we get either $g(f(x, y)) = y$ or $f(x, y) = g(y)$ for all x and y from A. Since $f(x, x) = x$, we have $g(x) = x$ in both cases. Consequently, $f(x, y) = y$ which is impossible because $f(x, y)$ depends on the variable x. Thus the elements $f(a_1, a_2)$ and a_2 are independent.

By the assumption the pair $f(a_1, a_2)$, a_2 can be extended to a basis. Consequently, there exists a system $b_1, b_2, \ldots, b_n$ of elements of A forming together with $f(a_1, a_2)$, a_2 a set of independent elements and an operation $h \in \boldsymbol{A}^{(n+2)}$ such that

$$a_1 = h(f(a_1, a_2), a_2, b_1, b_2, \ldots, b_n).$$

Hence we get the formula

$$f(a_1, a_2) = f(h(f(a_1, a_2), a_2, b_1, b_2, \ldots, b_n), a_2).$$

Since the elements $f(a_1, a_2), a_2, b_1, b_2, \ldots, b_n$ are independent, the last equation implies the equation

$$x = f(h(x, y, y, \ldots, y), y)$$

for all $x, y \in A$. Setting $h(x, y, y, \ldots, y) = \alpha x + \beta y + a$ and taking into account the formula $f(x, y) = \lambda x + (1-\lambda) y$, we have the equation

$$x = \lambda \alpha x + (1 + \lambda\beta - \lambda) y + \lambda a$$

for all $x, y \in A$. Hence, by virtue of the cancellation law in A, we get the equation $\lambda\alpha = 1$. Since the ring $\mathscr{R}$ has no divisors of zero, the last formula shows that each element λ of $\mathscr{R}$ different from zero is invertible. Thus $\mathscr{R}$ is a field which completes the proof in the cases (i) and (ii).

In the case (iii) to prove the theorem it suffices to prove that the semigroup $\mathscr{S}$ is a group. Let a be an independent element of A. From the definition of the set A_0 consisting of algebraic constants for every $g \in \mathscr{S}$ it follows that $g(a) \notin A_0$ and, consequently, is not an algebraic constant. Since v_*-algebras are v_*^*-algebras, we infer that $g(a)$ is an independent element. Thus it can be extended to a basis. Consequently, there exists a system $b_1, b_2, \ldots, b_n$ of elements of A forming together with $g(a)$ a set of independent elements and an operation $h \in \boldsymbol{A}^{(n+1)}$ such that $a = h(g(a), b_1, b_2, \ldots, b_n)$. In view of the independence of a the operation h is not constant. Since in the case (iii) all algebraic operations depend on at most one variable, there exists an element $g_0 \in \mathscr{S}$ such that either $a = g_0(b_j)$ for an index j $(1 \leqslant j \leqslant n)$ or $a = g_0(g(a))$. In the first case $g(a) \in [b_j]$, which contradicts the independence of $g(a)$ and b_j. In the last case we have, by the independence of a, the relation $g_0 g = 1$. Moreover, $g(a)$

$= g(g_0(g(a)))$ which, by the independence of $g(a)$, implies the equation $x = g(g_0(x))$. Consequently, $gg_0 = 1$. Thus each element of $\mathscr{S}$ is invertible and $\mathscr{S}$ is a group which completes the proof.

It should be noted that the assumption that the algebra contains at least two independent elements is essential. A simple counterexample can be constructed as follows. Let $\mathscr{R}_p$ be the ring defined in the example 3.2, and let B be the set of all pairs

$$\left\langle \frac{pn+1}{pm+1}, \frac{n-m}{pm+1} \right\rangle,$$

where n and m are arbitrary integers. Let A be the class of all operations f on $\mathscr{R}_p$ defined by the formula

$$f(x_1, x_2, \ldots, x_n) = \sum_{k=1}^{n} \lambda_k x_k + a,$$

where $a, \lambda_1, \lambda_2, \ldots, \lambda_n \in \mathscr{R}_p$ and $\langle \sum_{k=1}^{n} \lambda_k, a \rangle \in B$. The algebra $(\mathscr{R}_p; A)$ is a v_*-algebra in which every element is independent. Moreover, this algebra is generated by every element and consequently, each one-point set is a basis. Suppose that $(\mathscr{R}_p; A)$ is a v-algebra. Let λ be an element of $\mathscr{R}_p$ different from zero. Of course, the operation $\lambda x + (1-\lambda)y$ is algebraic and the equation $\lambda x + (1-\lambda)y = z$ depends on the variable x. Consequently, there is an operation $ay + \beta z + \gamma$ such that the last equation is equivalent to the equation $x = ay + \beta z + \gamma$. Hence it follows that the equation $\lambda(ay + \beta z + \gamma) + (1-\lambda)y = z$ holds for all y and z from $\mathscr{R}_p$. Consequently, $\lambda\beta = 1$ which shows that each element of $\mathscr{R}_p$ different from zero is invertible in $\mathscr{R}_p$. But the prime p is not invertible in $\mathscr{R}_p$. Thus the algebra $(\mathscr{R}_p; A)$ is not a v-algebra.

4. v^*-algebras. We know that each v-algebras is also a v^*-algebra. Now we shall give some other examples of v^*-algebras.

4.1. All algebras in which all elements are algebraic constants are v^*-algebras. Moreover, these algebras are the only zero-dimensional v^*-algebras.

4.2. Let $\mathscr{S}$ be a semigroup with a unit element such that each non-invertible element from $\mathscr{S}$ is a left zero-element. An n-ary operation f on $\mathscr{S}$ is said to be *$\mathscr{S}$-homogeneous* if for every system $a, a_1, a_2, \ldots, a_n$ of elements of $\mathscr{S}$ the equation

$$f(a_1 a, a_2 a, \ldots, a_n a) = f(a_1, a_2, \ldots, a_n)a$$

holds. It is evident that the composition of $\mathscr{S}$-homogeneous operations is $\mathscr{S}$-homogeneous. Moreover, the operations $f(x) = ax$ $(a \in \mathscr{S})$ are the only $\mathscr{S}$-homogeneous unary operations.

Let $\boldsymbol{F}$ be an arbitrary class of $\mathscr{S}$-homogeneous operations containing all $\mathscr{S}$-homogeneous unary operations. The algebra $(\mathscr{S}; \boldsymbol{F})$ will be called an *$\mathscr{S}$-homogeneous algebra over $\mathscr{S}$*. It is very easy to verify that the left zero-elements are the only algebraic constants. Moreover, each invertible element is independent and generates the whole algebra. Thus each $\mathscr{S}$-homogeneous algebra over $\mathscr{S}$ is a one-dimensional v^*-algebra.

4.3. A set $\mathscr{Q}$ containing at least two elements with a multiplication and a substraction operations is called a *quasifield* if it contains an element 0 such that $0a = a0 = 0$, $\mathscr{Q}\setminus\{0\}$ is a group with respect to the multiplication, and

(i) $$a-0 = a,$$

(ii) $$a(b-c) = ab-ac,$$

(iii) $$a-(a-c) = c,$$

(iv) $$a-(b-c) = (a-b)-(a-b)(b-a)^{-1}c \quad \text{if} \quad a \neq b.$$

In the last axiom $(b-a)^{-1}$ denotes the multiplicative inverse of $b-a$. The notion of a quasifield was introduced by G. Grätzer ([1], Section 2) and can be regarded as a generalization of the notion of a near-field, i. e. the notion of an algebraic system in which all laws of a division ring excepting the right-distributivity hold (see [2], Chapter 20, Section 4, where the dual notion is used, namely the right distributive law is postulated). In fact, if $a\ominus b$ is the substraction in a near-field $\mathscr{Q}$ and u is an arbitrary element of order two from the centre of $\mathscr{Q}$, then $\mathscr{Q}$ is a quasifield under the multiplication and the substraction defined by the formula $a-b = a\ominus b$ if $a \neq 0$ and $0-b = bu$ (see [1], Theorems 3 and 4). From results of Grätzer and M. Hall ([1], Section 2; [2], Chapter 20, Section 7) it follows that each finite quasifield can be obtained in such a way from a near-field.

An n-ary operation f on a quasifield $\mathscr{Q}$ is said to be *$\mathscr{Q}$-homogeneous* if for every system $a, b, a_1, a_2, \ldots, a_n$ of elements of $\mathscr{Q}$ the equation

$$f(a-ba_1, a-ba_2, \ldots, a-ba_n) = a-bf(a_1, a_2, \ldots, a_n)$$

holds. It is evident that the composition of $\mathscr{Q}$-homogeneous operations is also $\mathscr{Q}$-homogeneous. One can prove that the trivial operations are the only $\mathscr{Q}$-homogeneous unary operations. Moreover, the operations $f(x, y) = x-(x-y)a$ $(a \in \mathscr{Q})$ are the only $\mathscr{Q}$-homogeneous binary operations.

Let $\boldsymbol{F}$ be an arbitrary class of $\mathscr{Q}$-homogeneous operations containing all $\mathscr{Q}$-homogeneous binary operations. The algebra $(\mathscr{Q}; \boldsymbol{F})$ will be called a *$\mathscr{Q}$-homogeneous algebra over $\mathscr{Q}$*. Since every pair of elements of $\mathscr{Q}$ is a basis of $(\mathscr{Q}; \boldsymbol{F})$ (see [1], Theorems 6 and 7), the algebra $(\mathscr{Q}; \boldsymbol{F})$ is a two-dimensional v^*-algebra. We note that all unary algebraic operations in $(\mathcal{Q}; \boldsymbol{F})$ are trivial.

4.4. Consider an algebra $(E; i, q^*)$, where E is a four-element set, the unary operation i is an involution without fixed points and the ternary symmetrical operation q^* is uniquely determined by the conditions $q^*\big(x, y, i(x)\big) = y$, $q^*(x, y, x) = x$. The algebra $(E; i, q^*)$ will be called *exceptional*. It should be noted that the exceptional algebra can be also defined in terms of Boolean operations. Namely, the set E can be treated as a four-element Boolean algebra, $i(x) = x'$, $q^*(x_1, x_2, x_3) = (x_1' \cap x_2' \cap x_3') \cup (x_1' \cap x_2 \cap x_3) \cup (x_1 \cap x_2' \cap x_3) \cup (x_1 \cap x_2 \cap x_3')$ if all elements x_1, x_2, x_3 are different and $q^*(x_1, x_2, x_3) = (x_1 \cap x_2 \cap x_3) \cup (x_1' \cap x_2 \cap x_3) \cup (x_1 \cap x_2' \cap x_3) \cup (x_1 \cap x_2 \cap x_3')$ in the opposite case.

It is every easy to prove that the involution i is the only non-trivial algebraic unary operation in the exceptional algebra. Moreover, there is no binary algebraic operation in $(E; i, q^*)$ depending on every variable. Hence it follows that the elements $a, b \in E$ $(a \neq b)$ are independent if and only if $a \neq i(b)$. Furthermore, since the involution i has no fixed points, the algebra $(E; i, q^*)$ is generated by every pair of independent elements. Consequently, the exceptional algebra is a two-dimensional v^*-algebra.

4.5. Let $\mathcal{G}$ be a group of permutations of a set A and let A_0 be a subset of A containing all fixed points of permutations from $\mathcal{G}$ that are not the identity and which is invariant under all permutations from $\mathcal{G}$. If $\boldsymbol{A}$ is the class of all operations f defined as

$$f(x_1, x_2, \ldots, x_n) = g(x_j) \qquad (1 \leqslant j \leqslant n)$$

or

$$f(x_1, x_2, \ldots, x_n) = a,$$

where $g \in \mathcal{G}$ and $a \in A_0$, then $\mathfrak{A} = (A; \boldsymbol{A})$ is a v^*-algebra. Moreover, $\dim \mathfrak{A} = t(A \setminus A_0)$, where $t(A \setminus A_0)$ denotes the cardinal number of transitive constituents of $A \setminus A_0$ with respect to the permutation group $\mathcal{G}$. This example is an analogue of the example (2.3) of a v-algebra. The only difference is that we do not restrict here the number of fixed points of permutations from $\mathcal{G}$.

From these examples of v^*-algebras it follows, in particular, that the class of v^*-algebras coincides neither with the class of v-algebras nor with the class of v_*-algebras.

REPRESENTATION THEOREM FOR v^*-ALGEBRAS. *Let $\mathfrak{A}$ be a v^*-algebra of dimension at least one. Then one of the following cases holds.*

(i) *$\mathfrak{A}$ is an $\mathcal{S}$-homogeneous algebra over a semigroup $\mathcal{S}$ with the unit element and such that each non-invertible element from $\mathcal{S}$ is a left zero-element of $\mathcal{S}$.*

(ii) $\mathfrak{A}$ *is a* $\mathcal{Q}$*-homogeneous algebra over a quasifield* $\mathcal{Q}$.

(iii) $\mathfrak{A}$ *is an exceptional algebra.*

(iv) *There exists a permutation group* $\mathcal{G}$ *of the set* A *and a subset* A_0 *of* A *containing all fixed points of permutations from* $\mathcal{G}$ *that are not the identity and which is invariant under all permutations from* $\mathcal{G}$ *such that all algebraic operations are given by the formulas*

$$f(x_1, x_2, \ldots, x_n) = g(x_j) \qquad (1 \leqslant j \leqslant n)$$

and

$$f(x_1, x_2, \ldots, x_n) = a,$$

where $g \in \mathcal{G}$ *and* $a \in A_0$.

(v) $\mathfrak{A}$ *is a v-algebra.*

For v^**-algebras of dimension at least three one of the cases* (iv) *and* (v) *holds.*

Proof. The last assertion was proved in [12]. If $\mathfrak{A}$ is a two-dimensional v^*-algebra without non-trivial unary algebraic operations, then every pair of its elements is a basis. Consequently, by Grätzer's theorem the set $\mathcal{Q}$ of elements of the algebra $\mathfrak{A}$ is a quasifield and $\mathfrak{A}$ is a $\mathcal{Q}$-homogeneous algebra over $\mathcal{Q}$ (see [1], Theorems 8 and 9). Further, we have proved in [13] that for two-dimensional v^*-algebras with non-trivial unary algebraic operations one of the cases (iii), (iv) and (v) holds. Consequently, to prove the theorem it suffices to prove that for one-dimensional v^*-algebras the case (i) holds.

Let $\mathfrak{A}$ be a one-dimensional v^*-algebra and $\mathcal{S}$ its set of elements. Let e be an independent element of $\mathcal{S}$. Since the one-point set $\{e\}$ is a basis, to every element $a \in \mathcal{S}$ there corresponds one and only one unary algebraic operation h_a such that $a = h_a(e)$. Of course, the operation h_e is trivial. We define a multiplication in $\mathcal{S}$ by the formula $ab = h_a\big(h_b(e)\big)$. It is very easy to verify that the set $\mathcal{S}$ is a semigroup under this multiplication and the element e is a unit element. Moreover, each algebraic constant is a left zero-element. If a is not an algebraic constant and, consequently, is an independent element, then there is a unary algebraic operation g such that $g(a) = e$. Put $a^{-1} = g(e)$. Of course, $a^{-1}a = g\big(h_a(e)\big) = g(a) = e$. Moreover, $h_a\big(g(a)\big) = h_a(e) = a$, whence, by the independence of a, the formula $h_a\big(g(e)\big) = e$ follows. Consequently, $aa^{-1} = e$ which shows that each non-constant element of the algebra is invertible.

Further, for any n-ary algebraic operation we have he equation

$$f(a_1, a_2, \ldots, a_n) = f\big(h_{a_1}(e), h_{a_2}(e), \ldots, h_{a_n}(e)\big)$$

which implies the $\mathcal{S}$-homogeneity condition

$$\begin{aligned} f(a_1, a_2, \ldots, a_n)b &= f\big(h_{a_1}(h_b(e)), h_{a_2}(h_b(e)), \ldots, h_{a_n}(h\ (e))\big) \\ &= f(a_1 b, a_2 b, \ldots, a_n b). \end{aligned}$$

Consequently, $\mathfrak{A}$ is an $\mathscr{S}$-homogeneous algebra over $\mathscr{S}$. The representation theorem for v^*-algebras is thus proved.

It is evident that each set of independent elements in a v^*-algebra can be extended to a basis. Consequently, from Theorem 3.2 we obtain the following theorem:

THEOREM 4.1. *If* $\mathfrak{A}$ *is a* v_**-algebra and a* v^**-algebra simultaneously and* $\dim\mathfrak{A} \geqslant 2$, *then it is a* v*-algebra.*

We note that the algebra $(\mathscr{R}_p; \boldsymbol{A})$, being an example for the essentiality of the assumption that the algebras in Theorem 3.2 contain at least two independent elements, is also v^*-algebra. Consequently, the assumption $\dim\mathfrak{A} \geqslant 2$ in the last Theorem is essential too.

Finally we note that *each two-element algebra with at least one constant algebraic operation is a* v^**-algebra.* Of course, it suffices to prove this statement for algebras over the set $T = \{0, 1\}$ regarded as a group under the addition mod 2. Taking into account that the operations $0, 1, x$ and $x+1$ are the only unary operations in T, we infer that each self-dependent element in an algebra over T is an algebraic constant. Further, if an algebra over T contains a constant algebraic operation and a is an independent element of T, then the remaining element b of T is an algebraic constant and, consequently, $b \in [a]$. Thus each algebra over T with a constant algebraic operation is a v^*-algebra.

We have seen in Section 2 that there are exactly ten v-algebras over the set T. E. L. Post has proved in [10] (Chapter 23) that there are denumerably many different algebras over the set T with at least one constant algebraic operation. Consequently, there are denumerably many v^*- algebras over the set T. The trivial algebra over T is of course a two-dimensional v^*-algebra. The following three algebras are the only non-trivial two-dimensional v^*-algebras on the set T (see [6]):

$$\mathfrak{P}_* = (T; x+y+z), \qquad \mathfrak{P}^* = (T; xy+yz+xz),$$
$$\mathfrak{P} = (T; x+y+z, xy+yz+xz).$$

The addition and the multiplication are here taken mod 2. Further, from Post's results [10] it follows that the following seven algebras are the only zero-dimensional v^*-algebras on T:

$$\mathfrak{N}_1 = (T; x+y+1), \qquad \mathfrak{N}_2 = (T; x+1, 0), \qquad \mathfrak{N}_3 = (T; 0, 1),$$
$$\mathfrak{N}_4 = (T; 0, 1, xy), \qquad \mathfrak{N}_5 = (T; x+1, xy),$$
$$\mathfrak{N}_6 = (T; 0, 1, xy+x+y), \qquad \mathfrak{N}_7 = (T; 0, 1, xy+yz+xz).$$

All remaining v^*-algebras on T are one-dimensional.

5. v_*^*-algebras. All v_*-algebras and v^* algebras are v_*^*-algebras. Now we shall give some other examples of v_*^*-algebras.

5.1. Let N be the set of all non-negative integers. We define two unary operations f_1 and f_2 in the Cartesian product $N\times N$ by the formulas

$$f_1(\langle p,q\rangle) = \langle p+1,0\rangle,\qquad f_2(\langle p,q\rangle) = \langle p,q+1\rangle.$$

Since $f_1(f_2(x)) = f_1(x)$, we infer that each unary algebraic operation in the algebra $(N\times N; f_1, f_2)$ is of the form $f(x) = f_2^n(f_1^m(x))$ $(n, m = 0, 1, \ldots)$, where $f_1^0(x) = f_2^0(x) = x$ and $f_j^{n+j}(x) = f_j(f_j^n(x))$ $(j = 1, 2;$ $n = 0, 1, \ldots)$. Hence, by a simple reasoning for all $a\in N\times N$ we get the inequality $f(a)\neq g(a)$ whenever $f\neq g$. Thus each element of $N\times N$ is independent. Moreover, we have the equations

$$f_2^s(f_1^{r-p}(\langle p,q\rangle)) = \langle r,s\rangle \quad \text{if} \quad r>p$$

and

$$f_2^{s-q}(\langle p,q\rangle) = \langle r,s\rangle \quad \text{if} \quad r=p \text{ and } s>q.$$

Consequently, for any pair of elements of $N\times N$ one element is generated by the other one. Thus $(N\times N; f_1, f_2)$ is a v_*^*-algebra.

Of course, each algebraic operation in $(N\times N; f_1, f_2)$ essentially depends on one variable. Since $f_1(f_2(x)) = f_1(x)$, the left-cancellation law does not hold in the semigroup with respect to the composition of all algebraic unary operations in $(N\times N; f_1, f_2)$. Consequently, the algebra in question is a counterexample to a theorem formulated in [9] (Theorem IV) that the left-cancellation law holds in the semigroup of non-constant unary algebraic operations provided the algebra is a v_*^*-algebra containing at least one independent element and all algebraic operations depend on at most one variable (see [9], Correction).

5.2. Let $\mathscr{S}$ be a semigroup with a unit element satisfying the conditions

(*) for each pair of elements of $\mathscr{S}$ at least one element is right-divisible by the other one,

(**) if $a, b, c\in\mathscr{S}$, $a\neq b$ and $ac = bc$, then c is a left zero-element.

Let $(\mathscr{S}; \boldsymbol{F})$ be an $\mathscr{S}$-homogeneous algebra over $\mathscr{S}$, i. e. the class $\boldsymbol{F}$ of fundamental operations consists of $\mathscr{S}$-homogeneous operations and contains all unary $\mathscr{S}$-homogeneous operations $f(x) = ax$ $(a\in\mathscr{S})$. It is very easy to verify that the left zero-elements are the only algebraic constants. Moreover, each element which is not a left zero-element is independent and for each pair of elements of $\mathscr{S}$ at least one element is generated by the other one. Furthermore, the unit element is a basis. Thus $(\mathscr{S}; \boldsymbol{F})$ is a one-dimensional v_*^*-algebra. This example is an analogue of the example 4.2. of v^*-algebras.

The product $N\times N$ of the set N of all non-negative integers is a semigroup with the unit element satisfying conditions (*) and (**) with respect to the multiplication

$$\langle p, q\rangle\langle r, s\rangle = \begin{cases} \langle p+r, s\rangle & \text{if} \quad q = 0, \\ \langle p, q+s\rangle & \text{if} \quad q > 0. \end{cases}$$

If $\boldsymbol{F}$ is the set od all unary $N\times N$-homogeneous operations, then the algebra $(N\times N; \boldsymbol{F})$ and the algebra $(N\times N; f_1, f_2)$ considered in 5.1 are identical.

Now we shall prove that *each one-dimensional v_*^*-algebra in which every pair of elements is dependent is an $\mathscr{S}$-homogeneous algebra over a semigroup $\mathscr{S}$ with the unit element and satisfying the conditions* (*) *and* (**).

Let $\mathfrak{A}$ be a one-dimensional v_*^*-algebra in which every pair of elements is dependent. By $\mathscr{S}$ we shall denote the set of elements of the algebra $\mathfrak{A}$. Let $\{e\}$ be a one-point basis of $\mathfrak{A}$. To every element a of $\mathscr{S}$ there corresponds one and only one unary algebraic operation h_a such that $a = h_a(e)$. We define a multiplication in $\mathscr{S}$ by the formula $ab = h_a(h_b(e))$. It is evident that the set $\mathscr{S}$ is a semigroup with respect to this multiplication and the element e is the unit element. Let $a, b\in\mathscr{S}$. Since every pair of elements of $\mathscr{S}$ is dependent, there exists an algebraic unary operation g such that $a = g(b)$ or $b = g(a)$. Setting $d = g(e)$, we have $db = g(h_b(e)) = g(b) = a$ or $da = g(h_a(e)) = g(a) = b$ respectively. Consequently, condition (*) is satisfied. If $a \neq b$ and $ac = bc$, then $h_a \neq h_b$ and $h_a(c) = h_b(c)$ which implies that the element c is an algebraic constant. Consequently, the operation h_c is constant and for any element $u\in\mathscr{S}$ we have the equation $cu = h_c(h_u(e)) = c$. Thus the element c is a left zero-element of $\mathscr{S}$ which completes the proof of the condition (**). Further, for any n-ary algebraic operation f we have the equation

$$f(a_1, a_2, \dots, a_n) = f(h_{a_1}(e), h_{a_2}(e), \dots, h_{a_n}(e))$$

which implies the $\mathscr{S}$-homogeneity condition

$$\begin{aligned} f(a_1, a_2, \dots, a_n)b &= f\big(h_{a_1}(h_b(e)), h_{a_2}(h_b(e)), \dots, h_{a_n}(h_b(e))\big) \\ &= f(a_1 b, a_2 b, \dots, a_n b). \end{aligned}$$

Consequently, $\mathfrak{A}$ is an $\mathscr{S}$-homogeneous algebra over $\mathscr{S}$.

5.3. Let w be a one-to-one mapping of all ordered m-tuples $(m \geqslant 2)$ $k_1, k_2, \dots, k_m$ of different non-negative integers into the set N of non-negative integers satisfying the condition

$$w(k_1, k_2, \dots, k_m) > k_s \quad (s = 1, 2, \dots, m). \tag{5.1}$$

For instance, as a mapping w we can take the mapping

$$w(k_1, k_2, \dots, k_m) = p_1^{k_1} p_2^{k_2} \dots p_m^{k_m},$$

where $p_1, p_2, \ldots, p_m$ are primes. We extend the mapping w over all m-tuples of non-negative integers by setting $w(k_1, k_2, \ldots, k_m) = k_1$ in all remaining cases.

We shall prove that the algebra $(N; w)$ is a v_*^*-algebra. First we shall prove an auxiliary Lemma.

LEMMA. *Let $a_1, a_2, \ldots, a_n$ be a system of elements of N. If f is an n-ary algebraic operation in $(N; w)$ satisfying the condition*

$$f(a_1, a_2, \ldots, a_n) \notin [a_2, a_3, \ldots, a_n], \tag{5.2}$$

then the inequality $f(a_1, a_2, \ldots, a_n) \geqslant a_1$ holds.

Proof. The class $A^{(n)}$ of all n-ary algebraic operations in $(N; w)$ is the union $A^{(n)} = \bigcup_{k=0}^{\infty} A_k^{(n)}$, where the classes $A_k^{(n)}$ are defined recursively as follows

$$A_0^{(n)} = \{e_1^{(n)}, e_2^{(n)}, \ldots, e_n^{(n)}\},$$

$$A_{k+1}^{(n)} = A_k^{(n)} \cup \{w(f_1, f_2, \ldots, f_m): f_j \in A_k^{(n)},\ j = 1, 2, \ldots, m\} \qquad (k = 0, 1, \ldots)$$

(see [5], p. 47). Let f be an operation from $A_k^{(n)}$ satisfying condition (5.2). We shall prove the Lemma by induction with respect to k. If $k = 0$, then, in view of the assumption (5.2), $f = e_1^{(n)}$ and, consequently, $f(a_1, a_2, \ldots, a_n) = a_1$. Suppose now that the Lemma is true for all operations from $A_k^{(n)}$ satisfying condition (5.2). Let f be an operation from $A_{k+1}^{(n)}$ satisfying condition (5.2). There exist then operations $f_1, f_2, \ldots, f_m$ from $A_k^{(n)}$ such that

$$\begin{aligned} f(x_1, x_2, \ldots, x_n) \\ = w\big(f_1(x_1, x_2, \ldots, x_n), f_2(x_1, x_2, \ldots, x_n), \ldots, f_m(x_1, x_2, \ldots, x_n)\big). \end{aligned} \tag{5.3}$$

Suppose first that all elements $f_1(a_1, a_2, \ldots, a_n), f_2(a_1, a_2, \ldots, a_n), \ldots \ldots, f_m(a_1, a_2, \ldots, a_n)$ are different. Then, by the condition (5.1), we have the inequality

$$f(a_1, a_2, \ldots, a_n) > f_j(a_1, a_2, \ldots, a_n) \qquad (j = 1, 2, \ldots, m). \tag{5.4}$$

Furhter, from the condition (5.2) and from (5.3) it follows that $f_r(a_1, a_2, \ldots, a_n) \notin [a_2, a_3, \ldots, a_n]$ for at least one index r. Since $f_r \in A_k^{(n)}$, we have, by inductive assumption, the inequality $f_r(a_1, a_2, \ldots, a_n) \geqslant a_1$ which, in view of (5.4), implies the inequality $f(a_1, a_2, \ldots, a_n) > a_1$.

Suppose now that at least two elements among $f_1(a_1, a_2, \ldots, a_n)$, $f_2(a_1, a_2, \ldots, a_n), \ldots, f_m(a_1, a_2, \ldots, a_n)$ are equal. Then, by (5.3) and the definition of the fundamental operation w, we have the equation

$$f(a_1, a_2, \ldots, a_n) = f_1(a_1, a_2, \ldots, a_n).$$

Moreover, $f_1(a_1, a_2, \ldots, a_n) \notin [a_2, a_3, \ldots, a_n]$. Since $f_1 \in A_k^{(n)}$, we have, by the inductive assumption, the inequality $f_1(a_1, a_2, \ldots, a_n) \geqslant a_1$ and, consequently, the inequality $f(a_1, a_2, \ldots, a_n) \geqslant a_1$ which completes the proof.

Now we shall prove that the algebra $(N; w)$ is a v_*^*-algebra. It is evident that all $(m-1)$-ary algebraic operations in the algebra $(N; w)$ are trivial. Hence it follows that each $(m-1)$-element set is independent. Suppose that $n \geqslant m$ and the elements $a_1, a_2, \ldots, a_n$ are dependent. Consequently, there are two different n-ary algebraic operations f and g such that

$$f(a_1, a_2, \ldots, a_n) = g(a_1, a_2, \ldots, a_n). \tag{5.5}$$

Suppose that $f, g \in \boldsymbol{A}_k^{(n)}$. We shall prove by induction with respect to k that there exists an index j $(1 \leqslant j \leqslant n)$ for which $a_j \in [a_1, a_2, \ldots, a_{j-1}, a_{j+1}, \ldots, a_n]$.

If $k = 0$, then $f = e_i^{(n)}$ and $g = e_j^{(n)}$, where the indices i and j are different because of the inequality $f \neq g$. Hence and from (5.5) it follows that $a_i = a_j$ and, consequently, $a_j \in [a_1, a_2, \ldots, a_{j-1}, a_{j+1}, \ldots, a_n]$.

Suppose now that our statement is true for operations f, g from $\boldsymbol{A}_k^{(n)}$. Let $f, g \in \boldsymbol{A}_{k+1}^{(n)}$. Of course, without loss of generality we may assume that $f \notin \boldsymbol{A}_k^{(n)}$. Consequently, there are operations $f_1, f_2, \ldots, f_m$ belonging to $\boldsymbol{A}_k^{(n)}$ such that

$$\begin{aligned} f(x_1, x_2, \ldots, x_n) & \\ = w\big(f_1(x_1, x_2, \ldots, x_n), & f_2(x_1, x_2, \ldots, x_n), \ldots, f_m(x_1, x_2, \ldots, x_n)\big). \end{aligned} \tag{5.6}$$

Since $f \notin \boldsymbol{A}_k^{(n)}$, all the operations are different. If for a pair i, j of different indices the equation

$$f_i(a_1, a_2, \ldots, a_n) = f_j(a_1, a_2, \ldots, a_n)$$

holds, then, by inductive assumption, one element among $a_1, a_2, \ldots, a_n$ is generated by the other ones. Consequently, we may assume that all the elements $f_1(a_1, a_2, \ldots, a_n), f_2(a_1, a_2, \ldots, a_n), \ldots, f_m(a_1, a_2, \ldots, a_n)$ are different. Thus, in virtue of (5.1) and (5.6), we have the inequality

$$f(a_1, a_2, \ldots, a_n) > f_i(a_1, a_2, \ldots, a_n) \quad (i = 1, 2, \ldots, m). \tag{5.7}$$

First consider the case $g \in \boldsymbol{A}_0^{(n)}$. Without loss of generality we may assume that $g = e_1^{(n)}$. Thus, by (5,5),

$$f(a_1, a_2, \ldots, a_n) = a_1. \tag{5.8}$$

Suppose that $f(a_1, a_2, \ldots, a_n) \notin [a_2, a_3, \ldots, a_n]$. Then $f_j(a_1, a_2, \ldots, a_n) \notin [a_2, a_3, \ldots, a_n]$ for an index j $(1 \leqslant j \leqslant m)$ and, consequently, by the Lemma, $f_j(a_1, a_2, \ldots, a_n) \geqslant a_1$. Hence and from (5.7) we obtain the inequality $f(a_1, a_2, \ldots, a_n) > a_1$ which contradicts (5.8). Thus $f(a_1, a_2, \ldots, a_n) \in [a_2, a_3, \ldots, a_n]$ and, consequently, by (5.8), $a_1 \in [a_2, a_3, \ldots, a_n]$.

Now consider the case $g \notin A_0^{(n)}$. Since, by the assumption, $g \in A_{k+1}^{(n)}$, there exists an index r $(1 \leqslant r \leqslant k+1)$ such that $g \in A_r^{(n)} \setminus A_{r-1}^{(n)}$. Consequently,

$$(5.9) \quad g(x_1, x_2, \ldots, x_n) \\ = w\big(g_1(x_1, x_2, \ldots, x_n), g_2(x_1, x_2, \ldots, x_n), \ldots, g_m(x_1, x_2, \ldots, x_n)\big),$$

where the operations $g_1, g_2, \ldots, g_m$ are different and belong to $A_{r-1}^{(n)}$. If for a pair i, j of different indices the equation

$$g_i(a_1, a_2, \ldots, a_n) = g_j(a_1, a_2, \ldots, a_n)$$

holds, then, by inductive assumption, one element among $a_1, a_2, \ldots, a_n$ is generated by the other ones. Consequently, we may assume that the elements $g_1(a_1, a_2, \ldots, a_n), g_2(a_1, a_2, \ldots, a_n), \ldots, g_m(a_1, a_2, \ldots, a_n)$ are different, Since, by (5.5), (5.6) and (5.9),

$$w\big(f_1(a_1, a_2, \ldots, a_n), f_2(a_1, a_2, \ldots, a_n), \ldots, f_m(a_1, a_2, \ldots, a_n)\big) \\ = w\big(g_1(a_1, a_2, \ldots, a_n), g_2(a_1, a_2, \ldots, a_n), \ldots, g_m(a_1, a_2, \ldots, a_n)\big)$$

and the mapping w is one-to-one on m-tuples of different elements, we have the equations

$$(5.10) \quad f_j(a_1, a_2, \ldots, a_n) = g_j(a_1, a_2, \ldots, a_n) \quad (j = 1, 2, \ldots, m).$$

Further, the inequality $f \neq g$ yields, by (5.6) and (5.9), the inequality $f_i \neq g_i$ for an index i. Since both operations f_i and g_i belong to $A_k^{(n)}$, we infer, in view of (5.10) and the inductive assumption, that one element among $a_1, a_2, \ldots, a_n$ is generated by the other ones which completes the proof that $(N; w)$ is a v_*^*-algebra.

From the representation theorems for v-algebras, v_*-algebras and v^*-algebras it follows that for any algebra belonging to one of these three classes, containing at least one independent element, and satisfying the condition $A^{(3)} = A^{(3,1)}$ every algebraic operation essentially depends on at most one variable. The algebra $(N; w)$ shows that for v_*^*-algebras the situation is quite different. Namely, each element of N is independent and all $(m-1)$-ary algebraic operations are trivial but the fundamental m-ary operation w depends on every variable.

5.4. Consider an *absolutely free algebra* $(A; \boldsymbol{F})$ with an arbitrary generating set and an arbitrary set $\boldsymbol{F}$ of fundamental non-constant free operations (see [3], p. 157). The set A is the set of all words. We note that the algebra $(A; \boldsymbol{F})$ has te following properties:

(i) if $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_m \in A$, f, g are n-ary and m-ary operations from $\boldsymbol{F}$ and $f(a_1, a_2, \ldots, a_n) = g(b_1, b_2, \ldots, b_m)$, then $n = m$, $f = g$ and $a_j = b_j$ $(j = 1, 2, \ldots, n)$,

(ii) if h is any n-ary algebraic operation depending on the first variable and different from $e_1^{(n)}$, then for any system $a_1, a_2, \ldots, a_n \epsilon A$ the inequality $h(a_1, a_2, \ldots, a_n) \neq a_1$ holds.

We shall prove that absolutely free algebras are v_*^*-algebras. This example of v_*^*-algebras was suggested to me by K. H. Diener.

We shall use in the proof the decomposition of the class $\boldsymbol{A}^{(n)}$ of n-ary algebraic operations into the classes $\boldsymbol{A}_k^{(n)}$ defined recursively as follows:

$$\boldsymbol{A}_0^{(n)} = \{e_1^{(n)}, e_2^{(n)}, \ldots, e_n^{(n)}\},$$

$$\boldsymbol{A}_{k+1}^{(n)} = \boldsymbol{A}_k^{(n)} \cup \{f(f_1, f_2, \ldots, f_m): f \epsilon \boldsymbol{F}, f_j \epsilon \boldsymbol{A}_k^{(n)},\ j = 1, 2, \ldots, m\}$$

(see [5], p. 47).

First we shall prove that every element in the absolutely free algebra $(A; \boldsymbol{F})$ is independent. Suppose the contrary. Given a dependent element $a \epsilon A$, we denote by r the smallest index for which there are different algebraic operations f and g belonging to $\boldsymbol{A}_r^{(1)}$ such that $f(a) = g(a)$. Of course, $r \geqslant 1$ and, consequently, there are operations $f_0, g_0 \epsilon \boldsymbol{F}$ and operations $f_1, f_2, \ldots, f_n, g_1, g_2, \ldots, g_m \epsilon \boldsymbol{A}_{r-1}^{(1)}$ such that

$$f(x) = f_0\big(f_1(x), f_2(x), \ldots, f_n(x)\big) \quad \text{and} \quad g(x) = g_0\big(g_1(x), g_2(x), \ldots, g_m(x)\big).$$

From the equation

$$f_0\big(f_1(a), f_2(a), \ldots, f_n(a)\big) = g_0\big(g_1(a), g_2(a), \ldots, g_m(a)\big)$$

and the property (i) it follows that $n = m$, $f_0 = g_0$ and $f_j(a) = g_j(a)$ $(j = 1, 2, \ldots, n)$. Since $f \neq g$, we have the inequality $f_i \neq g_i$ for at least one index i. But $f_i, g_i \epsilon \boldsymbol{A}_{r-1}^{(1)}$, which contradicts the definition of the number r. Thus every element of A is independent.

Let $a_1, a_2, \ldots, a_n$ $(n \geqslant 2)$ be a system of dependent elements. There are then two different n-ary algebraic operations f and g such that

$$f(a_1, a_2, \ldots, a_n) = g(a_1, a_2, \ldots, a_n). \tag{5.11}$$

Suppose that $f, g \epsilon \boldsymbol{A}_k^{(n)}$. We shall prove by induction with respect to k that there exists an index j $(1 \leqslant j \leqslant n)$ for which $a_j \epsilon [a_1, a_2, \ldots, a_{j-1}, a_{j+1}, \ldots, a_n]$.

If $k = 0$, then $f = e_i^{(n)}$ and $g = e_j^{(n)}$, where the indices i and j are different because the inequality $f \neq g$. Hence and from (5.11) it follows that $a_i \neq a_j$ and, consequently, $a_j \epsilon [a_1, a_2, \ldots, a_{j-1}, a_{j+1}, \ldots, a_n]$.

Suppose that our statement is true for operations $f, g \epsilon \boldsymbol{A}_k^{(n)}$. Let $f, g \epsilon \boldsymbol{A}_{k+1}^{(n)}$. Without loss of generality we may assume that $f \notin \boldsymbol{A}_k^{(n)}$. Consequently, the operation f is non-trivial. Suppose that $g \epsilon \boldsymbol{A}_k^{(n)}$. Without loss of generality we may assume that $g = e_1^{(n)}$. Thus, by (5.11), $f(a_1, a_2, \ldots, a_n) = a_1$ which, by the property (ii), implies the independence of the operation f on the first variable. Consequently, $a_1 = f(a_2, a_2, a_3, \ldots, a_n) \in [a_2, a_3, \ldots, a_n]$.

Suppose now that $g \notin A_0^{(n)}$. Thus the operations f and g can be written in the form

$$f(x_1, x_2, \ldots, x_n) = f_0\big(f_1(x_1, x_2, \ldots, x_n), f_2(x_1, x_2, \ldots, x_n), \ldots, f_p(x_1, x_2, \ldots, x_n)\big),$$

$$g(x_1, x_2, \ldots, x_n) = g_0\big(g_1(x_1, x_2, \ldots, x_n), g_2(x_1, x_2, \ldots, x_n), \ldots, g_q(x_1, x_2, \ldots, x_n)\big),$$

where $f_0, g_0 \in \boldsymbol{F}$ and $f_1, f_2, \ldots, f_p, g_1, g_2, \ldots, g_q \in A_k^{(n)}$. By (5.11) we have the equation

$$f_0\big(f_1(a_1, a_2, \ldots, a_n), f_2(a_1, a_2, \ldots, a_n), \ldots, f_p(a_1, a_2, \ldots, a_n)\big) = g_0\big(g_1(a_1, a_2, \ldots, a_n), g_2(a_1, a_2, \ldots, a_n), \ldots, g_q(a_1, a_2, \ldots, a_n)\big)$$

which, in virtue of the property (i), implies the equations $p = q$, $f_0 = g_0$ and $f_j(a_1, a_2, \ldots, a_n) = g_j(a_1, a_2, \ldots, a_n)$ $(j = 1, 2, \ldots, p)$. Further, the inequality $f \neq g$ yields the inequality $f_i \neq g_i$ for an index i. Since both operations f_i and g_i belong to $A_k^{(n)}$, we infer, by the inductive assumption, that one of the elements $a_1, a_2, \ldots, a_n$ is generated by the others which completes the proof. Thus each absolutely free algebra is a v_*^*-algebra.

All these examples of v_*^*-algebras have extremely different algebraic structure. Thus the representation problem, which is still open, seems to be rather difficult (**P 529**).

We note that some conditions imposed on a v_*^*-algebra imply that it must be a v^*-algebra. Namely, the following two theorems hold.

THEOREM 5.1. *Each finite v_*^*-algebra is a v^*-algebra.*

This result was proved in [9] (Theorem III).

THEOREM 5.2. *If $\mathfrak{A}$ is a v_*^*-algebra in which every set of independent elements can be extended to a basis, then $\mathfrak{A}$ is a v^*-algebra.*

This assertion was proved under an additional assumption that $\mathfrak{A}$ possess a finite basis ([9], Theorem (ii), p. 124). Theorem 5.2 gives an affirmative answer to a question raised by Narkiewicz in [9] (p. 124).

Proof. By the definition of v_*^*-algebras each self-dependent element in $\mathfrak{A}$ is an algebraic constant. Suppose now that the elements $a_1, a_2, \ldots, a_n$ $(n \geqslant 1)$ are independent and the elements $a_1, a_2, \ldots, a_n, a_{n+1}$ are dependent. To prove the theorem it suffices to prove the relation $a_{n+1} \in [a_1, a_2, \ldots, a_n]$. By the definition of v_*^*-algebras there exists an index j $(1 \leqslant j \leqslant n+1)$ such that $a_j \in [a_1, a_2, \ldots, a_{j-1}, a_{j+1}, \ldots, a_{n+1}]$. If $j = n+1$, then the theorem is proved. Consider the case $j \leqslant n$. Without loss of generality we may assume that $j = 1$, i. e. $a_1 \in [a_2, a_3, \ldots, a_n, a_{n+1}]$. Consequently, there is an n-ary algebraic operation f for which the equation

$$a_1 = f(a_2, a_3, \ldots, a_n, a_{n+1}) \tag{5.12}$$

holds. The set $\{a_1, a_2, \ldots, a_n\}$ of independent elements can be extended to a basis. Consequently, there exists a system $b_1, b_2, \ldots, b_m$ forming together with $a_1, a_2, \ldots, a_n$ a set of independent elements and an $(n+m)$-ary algebraic operation g such that

$$a_{n+1} = g(a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_m). \tag{5.13}$$

Setting

$$c = g(a_1, a_2, \ldots, a_n, a_n, a_n, \ldots, a_n), \tag{5.14}$$

we have the relation

$$c \in [a_1, a_2, \ldots, a_n]. \tag{5.15}$$

From (5.12) and (5.13) we get the equation

$$a_1 = f\big(a_2, a_3, \ldots, a_n, g(a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_m)\big)$$

which, in view of the independence of $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_m$ implies the equation

$$a_1 = f\big(a_2, a_3, \ldots, a_n, g(a_1, a_2, \ldots, a_n, a_n, a_n, \ldots, a_n)\big).$$

Thus, by (5.14)

$$a_1 = f(a_2, a_3, \ldots, a_n, c) \tag{5.16}$$

and, consequently, by (5.15),

$$[a_1, a_2, \ldots, a_n] = [c, a_2, a_3, \ldots, a_n]. \tag{5.17}$$

From the definition of v_*^*-algebras it follows that the set $\{c, a_2, a_3, \ldots, a_n\}$ contains a subset E such that $[E] = [c, a_2, a_3, \ldots, a_n]$ and E is either empty or consists of independent elements. Since, by (5.17), $[E] = [a_1, a_2, \ldots, a_n]$ and the elements $a_1, a_2, \ldots, a_n$ are independent, we infer that the set E is non-void and, consequently, is a basis of $[a_1, a_2, \ldots, a_n]$. It is known that all bases of a v_*^*-algebra have the same cardinal number (see [9], Theorem I). Thus the basis E consists of n elements and, consequently, $E = \{c, a_2, a_3, \ldots, a_n\}$. Hence, in particular, the elements $c, a_2, a_3, \ldots, a_n$ are independent. From (5.14) and (5.16) we obtain the equation

$$c = g\big(f(a_2, a_3, \ldots, a_n, c)a_2, a_3, \ldots, a_n, a_n, a_n, \ldots, a_n\big)$$

which, by the independence of $c, a_2, a_3, \ldots, a_n$, implies the equation

$$a_{n+1} = g\big(f(a_2, a_3, \ldots, a_n, a_{n+1}), a_2, a_3, \ldots, a_n, a_n, a_n, \ldots, a_n\big).$$

Hence and from (5.12) and (5.14) we get the equation

$$a_{n+1} = g(a_1, a_2, \ldots, a_n, a_n, a_n, \ldots, a_n) = c.$$

Thus, by (5.15), $a_{n+1} \in [a_1, a_2, \ldots, a_n]$ which completes the proof.

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