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NOTE ON NORMAL SEQUENCES OF CHAIN COMPLEXES

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In [5] Simson and Tyc proved an E. H. Brown representation theorem for chain complexes using *normal sequences* of chain complexes, i.e. short exact sequences of chain complexes which split in each dimension. A normal sequence consists of a normal monomorphism, followed by a normal epimorphism. We show that normal monomorphisms are cofibrations in the sense of [1] in the category of chain complexes. Dually, normal epimorphisms are fibrations. Thus categorical homotopy theory as in [1]-[3] applies to a part of [5].

1. Categorical homotopy. Let (I, j_0, j_1, q) be a *homotopy system* in a category $\mathscr{C}$ (see [1], (1.1)), that is $I\colon \mathscr{C} \to \mathscr{C}$ is a functor (*cylinder functor*), and $j_0, j_1\colon 1_{\mathscr{C}} \to I$ and $q\colon I \to 1_{\mathscr{C}}$ are natural transformations such that $qj_0 = qj_1 = 1$.

A morphism $f\colon X \to Y$ of $\mathscr{C}$ is a *cofibration* if

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle j_0^X}\downarrow & & \downarrow{\scriptstyle j_0^Y} \\ IX & \xrightarrow[If]{} & IY \end{array}$$

is a weak pushout in $\mathscr{C}$.

Let

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle j_0^X}\downarrow & & \downarrow{\scriptstyle k} \\ IX & \xrightarrow[l]{} & Z_f \end{array}$$

be a pushout in $\mathscr{C}$ (Z_f is called the *mapping cylinder* of f), and let $f'\colon Z_f \to IY$ be induced by j_0^Y and If. Then it is easy to show that f is a cofibration if and only if f' is a section, that is there exists a morphism $r\colon IY \to Z_f$ with $rf' = 1_{Z_f}$.

2. Homotopy of chain complexes. Let $d\mathscr{A}$ be the category of chain complexes over an Abelian category $\mathscr{A}$ (see [4], VI.8). Then a homotopy system in $d\mathscr{A}$ is defined as follows.

Let X be a chain complex, and $f\colon X \to Y$ a chain morphism. Then

$$(IX)_n = X_n \oplus X_n \oplus X_{n-1}, \qquad (If)_n = f_n \oplus f_n \oplus f_{n-1}.$$

The boundary d^{IX} of IX is given by the matrix

$$d_n^{IX} = \begin{pmatrix} d_n^X & 0 & 1_{X_{n-1}} \\ 0 & d_n^X & -1_{X_{n-1}} \\ 0 & 0 & -d_{n-1}^X \end{pmatrix}.$$

j_0^X, j_1^X, q^X are given by the matrices

$$\begin{pmatrix} 1_{X_n} \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1_{X_n} \\ 0 \end{pmatrix}, \quad (1_{X_n} \quad 1_{X_n} \quad 0),$$

respectively. For the mapping cylinder Z_f of f we have

$$(Z_f)_n = Y_n \oplus X_n \oplus X_{n-1}$$

and

$$d_n^{Z_f} = \begin{pmatrix} d_n^Y & 0 & f_{n-1} \\ 0 & d_n^X & -1_{X_{n-1}} \\ 0 & 0 & -d_{n-1}^X \end{pmatrix}.$$

The morphisms k, l, f' are represented by

$$k_n = \begin{pmatrix} 1_{Y_n} \\ 0 \\ 0 \end{pmatrix}, \quad l_n = f_n \oplus 1_{X_n} \oplus 1_{X_{n-1}}, \quad f_n' = 1_{Y_n} \oplus f_n \oplus f_{n-1}.$$

3. Normal morphisms.

Definition (see [5], 1). Let $f\colon X \to Y$ be a morphism of $d\mathscr{A}$.

(1) f is a *normal monomorphism* if $f_n\colon X_n \to Y_n$ is a section in $\mathscr{A}$ for each $n \in \mathbf{Z}$.

(1*) f is a *normal epimorphism* if $f_n\colon X_n \to Y_n$ is a retraction in $\mathscr{A}$ for each $n \in \mathbf{Z}$.

PROPOSITION 1. *A morphism $f\colon X \to Y$ of $d\mathscr{A}$ is a normal monomorphism if and only if it is a cofibration in $d\mathscr{A}$.*

Proof. Let f be a cofibration. Then we have a chain morphism $r\colon IY \to Z_f$ such that $rf' = 1_{Z_f}$. For $n \in \mathbf{Z}$, r_n is given by a (3×3)-matrix

$(r_{ij}^{(n)})$ of morphisms of $\mathscr{A}$. It is easy to check

$$r_{22}^{(n)} f_n = 1_{X_n}.$$

Thus f_n is a section in $\mathscr{A}$.

Conversely, let f be a normal monomorphism. Write $X' = \operatorname{coker} f$. Then, as in [5], 1, we can assume without loss of generality that

$$Y_n = X_n \oplus X'_n \quad \text{and} \quad f_n = \begin{pmatrix} 1_{X_n} \\ 0 \end{pmatrix}.$$

Then the boundary d_n^Y of Y is of the form

$$d_n^Y = \begin{pmatrix} d_n^X & \Theta_n \\ 0 & d_n^{X'} \end{pmatrix},$$

where $\Theta_n \colon X'_n \to X_{n-1}$ is a suitable morphism of $\mathscr{A}$. Since $d_{n-1}^Y d_n^Y = 0$, we have

$$d_{n-1}^X \Theta_n = -\Theta_{n-1} d_n^{X'}.$$

Now we are in a position to define $r\colon IY \to Z_f$ by the matrix

$$r_n = \begin{pmatrix} 1_{Y_n} & \begin{pmatrix} 0 & 0 \\ 0 & 1_{X'_n} \end{pmatrix} & 0 \\ 0 & (1_{X_n} \quad 0) & 0 \\ 0 & (0 \quad -\Theta_n) & (1_{X_{n-1}} \quad 0) \end{pmatrix}.$$

It is left to the reader to verify that r is a chain morphism such that $rf' = 1_{Z_f}$.

A suitable dualization (see [1], 7) of Proposition 1 yields

PROPOSITION 1*. *A morphism of $d\mathscr{A}$ is a normal epimorphism if and only if it is a fibration* ([1], (1.7)) *in $d\mathscr{A}$.*

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