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*GENERAL ALGEBRAIC DEPENDENCE STRUCTURES AND SOME APPLICATIONS**

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Here, we introduce a set-theoretical scheme of algebraic (i. e. of finite character) dependence, which is general enough to accomodate previously investigated concepts of dependence (or independence). An attempt to treat algebraic dependence axiomatically was made with the introduction of GA-dependence structures in [1]. The primitive notion was there the relation "an element depends on a set", i. e. a binary relation between the given set and its power-set. In the first part of the present paper we generalize some ideas of [1] and introduce the concept of an A-dependence structure in terms of a relation of the mentioned type, in terms of a closure operation, and in terms of a family of independent sets. Next, the translation of the notions into each other is described. Accordingly, in the second part aiming to derive the concept of rank (dimension) and to show the relations to some earlier results, two particular types of A-dependence structures are defined and studied in terms of independent sets. Finally, some applications of the theory are given in the last section.

Let S be a given set, $\mathfrak{P}S$ its power-set and $\mathscr{F}$ the subfamily of its all finite subsets. x and X denote always an element and a subset of S, respectively.

A subfamily $\mathscr{I}$ of $\mathfrak{P}S$ satisfying the conditions

(f) $$\forall F(F \subseteq X \wedge F \in \mathscr{F} \Rightarrow F \in \mathscr{I}) \Rightarrow x \in \mathscr{I}$$

and

(m) $$X_1 \subseteq X_2 \wedge X_2 \in \mathscr{I} \Rightarrow X_1 \in \mathscr{I}$$

is said to be an *A-independence net of S*.

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By a *relation ϱ on S* we understand a subset ϱ of the cartesian product $S \times \mathfrak{P}S$. For a relation ϱ on S, define the subfamily $\mathscr{I}_\varrho \subseteq \mathfrak{P}S$ of *ϱ-independent subsets* by

$$(\varrho \to \mathscr{I}) \qquad X \in \mathscr{I}_\varrho \Leftrightarrow \forall x (x \in X \Rightarrow [x, X \setminus (x)] \notin \varrho).$$

Two relations ϱ_1 and ϱ_2 on S are said to be *associated* or *similar* if

$$\mathscr{I}_{\varrho_1} = \mathscr{I}_{\varrho_2}$$

or

$$x \notin X \Rightarrow ([x, X] \in \varrho_1 \Leftrightarrow [x, X] \in \varrho_2),$$

respectively.

A relation ϱ on S satisfying the following three conditions:

$$(\mathrm{F}) \qquad [x, X] \in \varrho \to \exists F (F \subseteq X \wedge F \in \mathscr{F} \wedge [x, F] \in \varrho),$$

$$(\mathrm{M}) \qquad X_1 \subseteq X_2 \wedge [x, X_1] \in \varrho \to [x, X_2] \in \varrho,$$

and

$$(\mathrm{E}_r) \qquad x_1 \notin X \wedge X \in \mathscr{I}_\varrho \wedge [x_1, X] \notin \varrho \wedge [x_1, X \cup (x_2)] \in \varrho \to [x_2, X \cup (x_1)] \in \varrho,$$

is called an *A-dependence relation on S*. In this case, the corresponding subfamily $\mathscr{I}_\varrho$ is an A-independence net of S. The relation ϱ is said to be *proper* or *regular* if, moreover,

$$(\mathrm{I}) \qquad x \in X \Rightarrow [x, X] \in \varrho$$

or

$$(\mathrm{R}) \qquad x \notin X \wedge [x, X] \in \varrho \Rightarrow \exists I (I \subseteq X \wedge I \in \mathscr{I}_\varrho \wedge [x, I] \in \varrho)$$

is satisfied, respectively.

Evidently, a relation ϱ on S can be fully described by the following mapping $\mathscr{D}_\varrho$ of S into $\mathfrak{P}\mathfrak{P}S$:

$$(\mathscr{D}_\varrho) \qquad X \in \mathscr{D}_\varrho(x) \Leftrightarrow [x, X] \in \varrho.$$

A relation ϱ is an A-dependence relation on S if and only if the mapping $\mathscr{D}_\varrho$ possesses the following properties:

$$(\mathrm{F}^{\mathscr{D}}) \qquad X \in \mathscr{D}_\varrho(x) \Rightarrow \exists F \big(F \subseteq X \wedge F \in \mathscr{F} \wedge F \in \mathscr{D}_\varrho(x)\big),$$

$$(\mathrm{M}^{\mathscr{D}}) \qquad X_1 \subseteq X_2 \wedge X_1 \in \mathscr{D}_\varrho(x) \Rightarrow X_2 \in \mathscr{D}_\varrho(x),$$

and

$$(\mathrm{E}_r^{\mathscr{D}}) \qquad x \notin X \wedge X \in \mathscr{I}_\varrho \wedge X \cup (x) \notin \mathscr{I}_\varrho \Rightarrow X \in \mathscr{D}_\varrho(x).$$

Consider also the mapping $\mathscr{D}_\varrho^R$ defined by

$$(\mathscr{D}_\varrho^R) \qquad X \in \mathscr{D}_\varrho^R(x) \Leftrightarrow \exists I (I \subseteq X \wedge x \notin I \wedge I \in \mathscr{I}_\varrho \wedge [x, I] \in \varrho).$$

When a relation ϱ on S satisfies (M), we deduce $\mathscr{D}_\varrho^R(x) \subseteq \mathscr{D}_\varrho(x)$ for every $x \in S$. Also, if ϱ_1 and ϱ_2 satisfy (M) and (E_r), then $\mathscr{D}_{\varrho_1}^R$ coincides with $\mathscr{D}^R_{\varrho_2}$ if and only if ϱ_1 and ϱ_2 are associated. A similar characterization for ϱ_1 and ϱ_2 to be similar can be given. Since, for an A-dependence relation ϱ on S,

$$x \notin X \wedge X \in \mathscr{I}_\varrho \Rightarrow ([x, X] \in \varrho \Leftrightarrow X \cup (x) \notin \mathscr{I}_\varrho),$$

the mapping $\mathscr{D}^R_\varrho$ is uniquely determined by the corresponding family of ϱ-independent sets $\mathscr{I}_\varrho$:

$$X \in \mathscr{D}^R_\varrho(x) \Leftrightarrow \exists I (I \subseteq X \wedge x \notin I \wedge I \in \mathscr{I}_\varrho \wedge I \cup (x) \notin \mathscr{I}_\varrho).$$

For a mapping C of $\mathfrak{P}S$ into $\mathfrak{P}S$, define the subfamily $\mathscr{I}_c \subseteq \mathfrak{P}S$ of *C-independent subsets* by

$$(C \to \mathscr{I}) \qquad I \in \mathscr{I}_c \Leftrightarrow \forall X (X \subseteq I \wedge I \subseteq C(X) \Rightarrow X = I).$$

If the four conditions

$$(\varphi) \qquad C(X) \subseteq \bigcup_{\substack{F \subseteq X \\ F \in \mathscr{F}}} C(F),$$

$$(\mu) \qquad X_1 \subseteq X_2 \Rightarrow C(X_1) \subseteq C(X_2),$$

$$(\varepsilon_r) \qquad X \in \mathscr{I}_c \wedge x_1 \in C(X \cup (x_2)) \setminus C(X) \Rightarrow x_2 \in C(X \cup (x_1)),$$

and

$$(\iota) \qquad X \subseteq C(X),$$

are fulfilled, then C is called the *A-dependence closure operation in S*. Then, $\mathscr{I}_c$ is an A-independence net of S. Also, for such an A-dependence closure operation

$$C(I) = I \cup \bigcup_{I \cup (x) \notin \mathscr{I}_c} I \cup (x)$$

holds for every $I \in \mathscr{I}_c$; hence, C is uniquely determined by $\mathscr{I}_c$ on $\mathscr{I}_c$.

The following theorem extends the previous results and describes the relation between any two of the following concepts of the *A-dependence structure* (S, ϱ), (S, C) and $(S, \mathscr{I})$, where ϱ, C and $\mathscr{I}$ are an A-dependence relation on S, an A-dependence closure operation in S and an A-independence net of S, respectively.

THEOREM. *To any A-dependence relation ϱ on S there corresponds a well-defined A-independence net $\mathscr{I}_\varrho$ of S. On the other hand, to any A-independence net of S there corresponds a set of (associated) A-dependence relations on S which form, under the natural operations of join and meet, a lattice* **L** *with infinite joins and* **O**. *The lattice* **L** *splits into convex sublattices of similar relations, the greatest element of each of these sublattices being the corresponding proper relation. The correspondence, in which every element of such sublattice is mapped into the corresponding greatest element, is a lattice-homomorphism of* **L** *onto the sublattice* $\mathbf{L}_p$ *of all proper relations* *with the ideal of all regular relations. Denoting by* $\mathbf{1}$, $\mathbf{O}_p$ *and* $\mathbf{O}$ *the greatest element of* $\mathbf{L}$, *the least element of* $\mathbf{L}_p$ *and* $\mathbf{L}$, *respectively, we have*

$$\mathscr{D}_{\mathbf{1}}(x) = \mathscr{D}^R(x) \cup (\mathfrak{P}S \setminus \mathscr{I}) \cup \mathscr{S}(x),$$
$$\mathscr{D}_{\mathbf{O}_p}(x) = \mathscr{D}^R(x) \cup \mathscr{S}(x),$$
$$\mathscr{D}_{\mathbf{O}}(x) = \mathscr{D}^R(x),$$

where $\mathscr{S}(x)$ *is the subfamily of all subsets* X *such that* $x \in X$.

As a consequence, for any A-independence net of S, *there is a uniquely determined proper regular A-dependence relation on* S.

To any A-dependence closure operation C *in* S *there corresponds a well-defined A-independence net* $\mathscr{I}_c$ *of* S. *On the other hand, to any A-independence net of* S *there corresponds a lattice of A-dependence closure operations in* S *which is isomorphic to the correspoding lattice* $\mathbf{L}_p$ *of all proper A-dependence relations. The least element of this lattice is the corresponding Schmidt's "mehrstufiger Austauschoperator"* (see [11]).

Now, let us consider a (fixed) A-dependence structure $(S, \mathscr{I})$ with the closure operation $C_{\mathscr{I}}$:

$$C_{\mathscr{I}}(I) = I \cup \bigcup_{I \cup (x) \notin \mathscr{I}} I \cup (x)$$

for every $I \in \mathscr{I}$ ($C_{\mathscr{I}}(I)$ is the greatest subset X of S such that $I \subseteq X$ and $I \subsetneqq X' \subseteq X \Rightarrow X' \notin \mathscr{I}$) (1). Define the subfamilies $\mathscr{C}$ and ${}^s\mathscr{C}$ of the *canonic* and *strongly canonic* subsets by

$$(\mathscr{C}) \quad I \in \mathscr{C} \Leftrightarrow I \in \mathscr{I} \wedge \forall X[X \in \mathscr{I} \wedge I \subseteq C(X) \wedge X \subseteq C(I) \Rightarrow C(I) \subset C(X)]$$

and

$$({}^s\mathscr{C}) \qquad I \in {}^s\mathscr{C} \Leftrightarrow I \in \mathscr{I} \wedge \forall X[X \in \mathscr{I} \wedge I \subseteq C(X) \Rightarrow C(I) \subseteq C(X)],$$

respectively. Clearly, ${}^s\mathscr{C} \subseteq \mathscr{C} \subseteq \mathscr{I}$; in general, ${}^s\mathscr{C} \neq \mathscr{C}$. Also, define the family $\mathscr{M}$ of all *maximal subsets* of S by

$$(\mathscr{M}) \qquad I \in \mathscr{M} \Leftrightarrow I \in \mathscr{I} \wedge C(I) = S,$$

and the family $\mathscr{B}$ of all *bases* of S by

$$(\mathscr{B}) \qquad \mathscr{B} = \mathscr{M} \cap \mathscr{C} = \mathscr{M} \cap {}^s\mathscr{C}.$$

Let us introduce, moreover, the relation $\varepsilon \subseteq \mathscr{I} \times \mathscr{I}$ defined by

$$(\varepsilon) \qquad [I_1, I_2] \in \varepsilon \Leftrightarrow I_1 \subseteq C(I_2) \wedge I_2 \subseteq C(I_1).$$

When restricted to $\mathscr{C} \times \mathscr{C}$, the corresponding partial relation $\varepsilon_{\mathscr{C}}$ is evidently an equivalence.

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(1) Since there is no danger of confusion, we simply write $C(I)$, $\mathscr{C}$, ${}^s\mathscr{C}$, $\mathscr{M}$ instead of $C_{\mathscr{I}}(I)$, $\mathscr{C}_{\mathscr{I}}$, ${}^s\mathscr{C}_{\mathscr{I}}$, $\mathscr{M}_{\mathscr{I}}$ etc.

For any $I_1 \in \mathscr{I}$ and $I_2 \in \mathscr{I}$ with $I_1 \subseteq C(I_2)$, there exists $I_0 \subseteq I_2 \setminus I_1$ such that $I_1 \cup I_0 \in \mathscr{I}$ and $[I_1 \cup I_0, I_2] \in \varepsilon$. From here the first part of the following lemma, which can be interpreted as a generalization of the Steinitz's Exchange Theorem, follows readily:

LEMMA. $I_1 \in \mathscr{I} \wedge I_2 \in \mathscr{I} \wedge [I_1, I_2] \in \varepsilon \Rightarrow$

(i) $\forall x [x \in I_1 \setminus I_2 \Rightarrow \exists I_0 (\emptyset \neq I_0 \subseteq I_2 \setminus I_1 \wedge [(I_1 \setminus (x)) \cup I_0, I_2] \in \varepsilon)]$;

(ii) $\operatorname{card}(I_1 \setminus I_2) \leqslant \operatorname{card}(I_2 \setminus I_1)$.

Hence, we get immediately the following theorem and corollary:

THEOREM. $I_1 \in \mathscr{I} \wedge I_2 \in \mathscr{C} \wedge I_1 \subseteq C(I_2) \Rightarrow \operatorname{card}(I_1) \leqslant \operatorname{card}(I_2)$.

COROLLARY. $I_1 \in \mathscr{M} \wedge I_2 \in \mathscr{B} \wedge I_3 \in \mathscr{B} \Rightarrow \operatorname{card}(I_1) \leqslant \operatorname{card}(I_2) = \operatorname{card}(I_3)$.

As a consequence, we are able to define the *rank* of any A-dependence structure whose family of bases is non-empty as the common cardinality of its bases. In accordance with [1] and the "classical" theory of linear independence we define *GA-dependence structures* and *LA-dependence structures* in the following way: $(S, \mathscr{I})$ is said to be a GA-dependence structure if there exists a subfamily $\mathscr{C}^*$ of $\mathscr{C}$ such that $\mathscr{C}^* \cap \mathscr{B} = \mathscr{C}^* \cap \mathscr{M} \neq \emptyset$ and

$$I_1 \subseteq I_2 \wedge I_2 \in \mathscr{C}^* \Rightarrow I_1 \in \mathscr{C}^*.$$

If, moreover, $\mathscr{B} = \mathscr{M}$, then $(S, \mathscr{I})$ is called a *LA-dependence structure.* In fact, in view of the following theorem, the mere assumption $\mathscr{B} = \mathscr{M}$ implies that $(S, \mathscr{I})$ is a LA-dependence structure.

THEOREM [4]. $\mathscr{B} = \mathscr{M} \Rightarrow \mathscr{C} = {}^s\mathscr{C} = \mathscr{I}$.

Let us point out that the A-dependence structure $(S, \mathscr{I}_\varrho)$, where ϱ is an A-dependence relation on S satisfying

$$(\mathrm{T}_r) \qquad x_0 \notin X_0 \wedge X_0 \in \mathscr{I}_\varrho \wedge X \in \mathscr{I}_\varrho \wedge [x_0, X] \in \varrho \wedge \forall x (x \in X \Rightarrow [x, X_0] \in \varrho) \Rightarrow \Rightarrow [x_0, X_0] \in \varrho,$$

is a LA-dependence structure. In particular, the A-dependence structure $(S, \mathscr{I}_\delta)$ corresponding to a linear dependence relation δ ([12] and a series of papers generalizing the theory to infinite case), i. e. a proper regular A-dependence relation δ on S satisfying (T_r), is a LA-dependence structure. On the other hand, to any LA-dependence structure $(S, \mathscr{I})$ there exists a lattice of A-dependence relations with (T_r) and a uniquely determined linear dependence (in the correspondence described in our first theorem). There is a similar correspondence between GA-dependence structures and GA-dependence relations of [1].

The LA-dependence structures can be characterized in various ways. Some of such characterizations (cf. [4], [6], [9], [10] and [13]) are included in the following theorem the proof of which is — on the basis of our theory — quite simple.

THEOREM. *For a given A-dependence structure $(S, \mathscr{I})$, the following conditions are equivalent:*

$(\mathscr{F}\mathscr{C})$ $\quad \mathscr{I} \cap \mathscr{F} \subseteq \mathscr{C}$;

$(\mathscr{C})$ $\quad \mathscr{I} = \mathscr{C}$;

$(\mathscr{F}^s\mathscr{C})$ $\quad \mathscr{I} \cap \mathscr{F} \subseteq {}^s\mathscr{C}$;

$({}^s\mathscr{C})$ $\quad \mathscr{I} = {}^s\mathscr{C}$;

$(\mathscr{F}\mathscr{N})$ $\quad I \in \mathscr{I} \cap \mathscr{F} \wedge I \cup (x_1) \notin \mathscr{I} \wedge I \cup (x_2) \notin \mathscr{I} \wedge x_1 \neq x_2 \wedge x_3 \in I \Rightarrow$
$\Rightarrow \big(I \setminus (x_3)\big) \cup (x_1) \cup (x_2) \notin \mathscr{I}$;

$(\mathscr{N})$ $\quad I \in \mathscr{I} \wedge I \cup (x_1) \notin \mathscr{I} \wedge I \cup (x_2) \notin \mathscr{I} \wedge x_1 \neq x_2 \wedge x_3 \in I \Rightarrow \big(I \setminus (x_3)\big) \cup$
$\cup (x_1) \cup (x_2) \notin \mathscr{I}$;

$(\mathscr{F}\mathscr{W}\mathscr{R})$ $\quad I_1 \in \mathscr{I} \cap \mathscr{F} \wedge I_2 \in \mathscr{I} \cap \mathscr{F} \wedge \operatorname{card}(I_1) < \operatorname{card}(I_2) \Rightarrow$
$\Rightarrow \exists x\big(x \in I_2 \setminus I_1 \wedge I_1 \cup (x) \in \mathscr{I}\big)$;

$(\mathscr{W}\mathscr{R})$ $\quad I_1 \in \mathscr{I} \wedge I_2 \in \mathscr{I} \wedge \operatorname{card}(I_1) < \operatorname{card}(I_2) \Rightarrow \exists x\big(x \in I_2 \setminus I_1 \wedge I_1 \cup (x) \in \mathscr{I}\big)$;

$(\mathscr{F}\mathscr{E})$ $\quad I_1 \in \mathscr{I} \cap \mathscr{F} \wedge I_2 \in \mathscr{I} \cap \mathscr{F} \wedge [I_1, I_2] \in \varepsilon \wedge x_1 \in I_1 \setminus I_2 \Rightarrow$
$\Rightarrow \exists x_2\big(x_2 \in I_2 \setminus I_1 \wedge (I_1 \setminus (x_1)) \cup (x_2) \in \mathscr{I}\big)$ (2);

$(\mathscr{E})$ $\quad I_1 \in \mathscr{I} \wedge I_2 \in \mathscr{I} \wedge [I_1, I_2] \in \varepsilon \wedge x_1 \in I_1 \setminus I_2 \Rightarrow \exists x_2\big(x_2 \in I_2 \setminus I_1 \wedge$
$(I_1 \setminus (x_1)) \cup (x_2) \in \mathscr{I}\big)$ (2);

$(\mathscr{M})$ $\quad I_1 \in \mathscr{M} \wedge I_2 \in \mathscr{M} \wedge x_1 \in I_1 \Rightarrow \exists x_2\big(x_2 \in I_2 \wedge (I_1 \setminus (x_1)) \cup (x_2) \in \mathscr{M}\big)$;

$(\mathscr{M}_g)$ $\quad I_1 \in \mathscr{M} \wedge I_2 \in \mathscr{M} \wedge X_1 \subseteq I_1 \Rightarrow$
$\Rightarrow \exists X_2\big(X_2 \subseteq I_2 \wedge \operatorname{card}(X_1) = \operatorname{card}(X_2) \wedge (I_1 \setminus X_1) \cup X_2 \in \mathscr{M}\big)$;

$(\mathscr{M}^*)$ $\quad I_1 \in \mathscr{M} \wedge I_2 \in \mathscr{M} \wedge x_1 \in I_1 \setminus I_2 \Rightarrow \exists x_2\big(x_2 \in I_2 \setminus I_1 \wedge (I_1 \setminus (x_1)) \cup (x_2) \in \mathscr{M}\big)$;

$(\mathscr{M}^*_g)$ $\quad I_1 \in \mathscr{M} \wedge I_2 \in \mathscr{M} \wedge X_1 \subseteq I_1 \setminus I_2 \Rightarrow$
$\Rightarrow \exists X_2\big(X_2 \subseteq I_2 \setminus I_1 \wedge \operatorname{card}(X_1) = \operatorname{card}(X_2) \wedge (I_1 \setminus X_1) \cup X_2 \in \mathscr{M}\big)$;

$(\mathscr{M}^{**})$ $\quad I_1 \in \mathscr{M} \wedge I_2 \in \mathscr{M} \wedge x_1 \in I_1 \setminus I_2 \Rightarrow \exists x_2\big(x_2 \in I_2 \setminus I_1 \wedge (x_1) \cup (I_2 \setminus (x_2)) \in \mathscr{M}\big)$;

$(\mathscr{M}^{**}_g)$ $\quad I_1 \in \mathscr{M} \wedge I_2 \in \mathscr{M} \wedge X_1 \subseteq I_1 \setminus I_2 \Rightarrow$
$\Rightarrow \exists X_2\big(X_2 \subseteq I_2 \setminus I_1 \wedge \operatorname{card}(X_1) = \operatorname{card}(X_2) \wedge X_1 \cup (I_2 \setminus X_2) \in \mathscr{M}\big)$.

As to the applications, our theory covers besides the classical concepts and theories of dependence in algebra and geometry also the concepts of dependence in abelian groups (discussed in [1] and [2]) and in lattices, as well as the Marczewski's concept of independence introduced in [8]. It can be applied also to the study of linear combinations — e. g. in abelian groups — bringing some results on generating systems. Also, on the basis of our theory, the rank of a (non-commutative) group G

(2) In fact, $[(I_1 \setminus (x_1) \cup (x_2), I_1] \in \varepsilon$.

can be defined (cf. [7]) provided (i) there is a subset K in G such that $\{K\}_N = G$, and (ii) for every $g \in K$, $\{g\}_N$ does not contain an infinite direct product of non-trivial normal subgroups of G (here $\{K\}_N$ and $\{g\}_N$ denote the normal subgrups of G generated by K and g, respectively). Perhaps the following two results deserve to be mentioned here, too; they represent a part of an application of our theory to modules (see [3]) which generalizes the results of Kertész [7] and Fuchs [5]. Let R be an (associative) ring with identity and M a (left) module over R. A subset X of M is said to be *independent* if every relation

$$\sum_{i=1}^{n} \lambda_i x_i = 0 \quad \text{with} \quad \lambda_i \in R \text{ and } x_i \in X$$

implies $\lambda_i x_i = 0$ for each $i = 1, 2, \ldots, n$. The family of all independent subsets of M is an A-independent net and thus every module over an arbitrary ring R is in this way an A-dependence structure. A ring has the property that every module over R is a GA-dependence structure if and only if, for every proper left ideal L of R, there exists $\varrho \notin L$ such that the quotient ideal $L\colon \varrho$ is irreducible. Finally, a ring R has the property that every module over R is a LA-dependence structure (and thus, any two its maximal independent subsets have the same cardinality) if and only if every left ideal L of R is irreducible, i. e. if and only if the family of all left ideals of R is a chain.

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APPENDIX

In her lecture, Miss E. Szodoray (this volume, p. 360-361) has presented an attempt to give an axiomatization of an (infinite) LA-dependence structure in terms of bases (i. e. maximal independent sets). Here we offer the following solution of this problem (to be published under the title *Axiomatic treatment of bases in arbitrary sets* in Czechoslovak Mathematical Journal):

Let S be a given set. A non-empty family $\mathscr{M}$ of subsets of S is called an A-*independence covering* of S if $\mathscr{M}$ possesses the following properties:

($\mathscr{M}_1$) No proper subset of an element of $\mathscr{M}$ belongs to $\mathscr{M}$.

($\mathscr{M}_2$) A set which is not contained in any element of $\mathscr{M}$ possesses a finite subset with the same property.

The concepts of an A-dependence structure $(S, \mathscr{I})$, where $\mathscr{I}$ is an A-independence net of S, and the concept of an A-dependence structure $(\mathscr{S}, \mathscr{M})$, where $\mathscr{M}$ is an A-independence covering of S, are equivalent, the equivalence being established by the mappings

$$(\mathscr{I} \to \mathscr{M}) \qquad M \in \mathscr{M}_{\mathscr{I}} \Leftrightarrow M \in \mathscr{I} \wedge \forall X (X \supsetneqq M \Rightarrow X \notin \mathscr{I})$$

and

$$(\mathscr{M} \to \mathscr{I}) \qquad I \in \mathscr{I}_{\mathscr{M}} \Leftrightarrow \exists M (M \supseteq I \wedge M \in \mathscr{M}).$$

The A-dependence structure $(S, \mathscr{M})$ is said to be a LA-*dependence structure* if $(S, \mathscr{I}_{\mathscr{M}})$ is an LA-dependence structure.

Now, let $(S, \mathscr{I})$ be an LA-dependence structure. Then, moreover, $\mathscr{M}_{\mathscr{I}}$ satisfies ($\mathscr{M}$), ($\mathscr{M}^*$), ($\mathscr{M}^{**}$) and ($\mathscr{M}_g$), $\mathscr{M}_g^*$), ($\mathscr{M}_g^{**}$) of the preceding paper where, instead of $\mathscr{M}$, $\mathscr{M}_{\mathscr{I}}$ should be read. On the other hand, the following theorem can be proved:

Let $\mathscr{M} \neq \emptyset$ be a family of subsets of S satisfying the properties of one of the following groups:

(i) ($\mathscr{M}_1$), ($\mathscr{M}_2$), ($\mathscr{M}$);

(ii) ($\mathscr{M}_1$), ($\mathscr{M}_2$), ($\mathscr{M}_g$);

(iii) ($\mathscr{M}_2$), ($\mathscr{M}^*$);

(iv) ($\mathscr{M}_2$), ($\mathscr{M}_g^*$);

(v) ($\mathscr{M}_2$), ($\mathscr{M}^{**}$);

(vi) ($\mathscr{M}_2$), ($\mathscr{M}_g^{**}$).

Then $(S, \mathcal{M})$ *is an LA-dependence structure (and, hence,* $\mathcal{M}$ *satisfies all the other properties).*

The following two results (see my paper *The rôle of the "finite character property" in the theory od dependence*, Commentationes Mathematicae Universitatis Carolinae 5, 1 (1965), p. 97-104) relate to one of the problems mentioned by Prof. R. Rado in his lecture (this volume, p. 257-264):

Let a relation ϱ on S satisfy (I), (E_r), (T_r) and (M). If a finite maximal ϱ-independent set exists, then all maximal ϱ-independent sets are finite and have the same number of elements. On the other hand, a family $(\mathfrak{A}_\gamma)_{\gamma\in\Gamma}$ of infinite cardinal numbers being given, there is a set S and a relation ϱ on S satisfying (I), (E) and (T) ((E) and (T) are the properties (E_r) and (T_r), respectively, without the restriction $X\in\mathscr{I}_\varrho$ and $X_0\in\mathscr{I}_\varrho$) such that a family $(M_\gamma)_{\gamma\in\Gamma}$ of maximal ϱ-independent sets exists with

$$\mathrm{card}(M_\gamma) = \mathfrak{A}_\gamma \quad \text{for each} \quad \gamma\in\Gamma.$$

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