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*ABSTRACT LINEAR DEPENDENCE**

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1. In the first edition (1930) of his *Moderne Algebra* [10] van der Waerden developed, in § 28, the theory of linear dependence over a field and, in § 61, the theory of algebraic dependence over a field. In § 33 of the second edition (1937) he lists four basic properties of the relation: "an element x depends on a set A of elements" and shows that these imply all he requires, both in the case of linear dependence and of algebraic dependence. He verifies his basic properties in both these cases. Thus the axiomatic theory of linear dependence seems to have originated between 1930 and 1937. Independently, in 1935, H. Whitney published the fundamental paper [11] in this field in which he investigates various approaches to the theory. In what follows I shall discuss (i) methods of defining LI ($=$ linear independence)-structures, (ii) some general theorems, (iii) the representation problem and other open questions.

2. S is a fixed set of cardinal $|S|$, finite or infinite. The letters x, y, z denote elements of S, and A, B, C subsets of S. The relation $A \subset\subset S$ means that $A \subset B$ and $|A| < \aleph_0$. The symbol $\{x_0, \ldots, x_{n-1}\}_{\neq}$, and similarly $\{x_0, \ldots, x_{n-1}\}_{<}$, denotes the set $\{x_0, \ldots, x_{n-1}\}$ and, at the same time, expresses the fact that $x_r \neq x_s$ or $x_r < x_s$ (for $r < s < n$) respectively. Unless the contrary is stated, all systems $x_0, x_1, \ldots$ are finite.

I shall define, in various equivalent ways, an LI structure on S.

(a) Van der Waerden: The basic relation is $x \mid (y_0, \ldots, y_{n-1})$ (x depends on $y_0, \ldots, y_{n-1}$), whose negation is denoted by $x \nmid (y_0, \ldots, y_{n-1})$. It satisfies the axioms:

(i) $y_\nu \mid (y_0, \ldots, y_{n-1})$ $(\nu < n)$.

(ii) *If* $n \geqslant 1$ *and* $x \mid (y_0, \ldots, y_{n-1})$ *and* $x \nmid (y_1, \ldots, y_{n-1})$, then $y_0 \mid (x, y_1, \ldots, y_{n-1})$.

(iii) *If* $x \mid (y_0, \ldots, y_{n-1})$ *and* $y_\nu \mid (z_0, \ldots, z_{m-1})$ *for* $\nu < n$, *then* $x \mid (z_0, \ldots, z_{m-1})$.

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(iv) *If* $x \mid (y_0, \ldots, y_{n-1})$, *then* $x \mid (y_{\alpha_0}, \ldots, y_{\alpha_{n-1}})$ *for every permutation* $\nu \to \alpha_\nu$.

(b) Whitney, slightly modified:

(b1) Basic notion: The set $I = I(S)$ the elements of which are finite systems $(x_0, \ldots, x_{n-1})$ or finite subsets of S. Axioms:

(i) $(x_0, \ldots, x_{n-1}) \in I \Leftrightarrow \{x_0, \ldots, x_{n-1}\}_{\neq} \in I$.

(ii) *If* $A \subset B \in I$, *then* $A \in I$.

(iii) $\emptyset \in I$.

(iv) *If* $A, B \in I$ *and* $|A| + 1 = |B|$, *then* $A \cup \{x\} \in I$ *for some* $x \in B - A$.

(b2) Basic notion: A function f ($= I$-function) such that $f(A), f(x_0, \ldots, x_{n-1}) \in \{0, 1\}$, defined for $A \subset\subset S$; $x_0, \ldots, x_{n-1} \in S$.

Axioms:

(i) $f(x, x) = 0$.

(ii) *If* $A = \{x_0, \ldots, x_{n-1}\}_{\neq}$, *then* $f(A) = f(x_0, \ldots, x_{n-1})$

(iii) $f(x_0, \ldots, x_{n-1}) \geqslant f(x_0, \ldots, x_{n-1}, x_n)$.

(iv) $f(x_0, \ldots, x_{n-1}) f(y_0, \ldots, y_n) \leqslant \sum_\nu f(x_0, \ldots, x_{n-1}, y_\nu)$.

(b3) Basic notion: The rank $\varrho(A)$, a non-negative integer, defined for $A \subset\subset S$.

Axioms:

(i) $\varrho(\emptyset) = 0$.

(ii) $\varrho(A) \leqslant \varrho(A \cup \{x\}) \leqslant \varrho(A) + 1$.

(iii) *If* $\varrho(A) = \varrho(A \cup \{x\}) = \varrho(A \cup \{y\})$, *then* $\varrho(A) = \varrho(A \cup \{x, y\})$.

(b4) Basic notion: The set $\beta(S)$ whose elements are subsets of S (the bases, i. e. maximal independent subsets). Here it is assumed that $|S| < \aleph_0$.

Axioms:

(i) *If* $A \subset \neq B \in \beta(S)$, *then* $A \notin \beta(S)$.

(ii) *If* $A, B \in \beta(S)$ *and* $x \in A$, *then* $(A - \{x\}) \cup \{y\} \in \beta(S)$ *for some* $y \in B$.

(b5) Basic notion: The set $\gamma(S)$ whose elements are finite subsets of S (the circuits, i. e. minimal dependent subsets).

Axioms:

(i) *If* $A \subset \neq B \in \gamma(S)$, *then* $A \notin \gamma(S)$.

(ii) *If* $A, B \in \gamma(S)$; $x \in A \cap B$ *and* $y \in A - B$, *then there is* $C \in \gamma(S)$ *such that* $x \notin C$; $y \in C$.

3. It is possible to pass, by formulae obviously suggested by the terminology, from any one of the structures (a), (b1)-(b5) to any other, and the stated axioms are necessary and sufficient to ensure the logical equivalence of all these structures. Thus each of the six sets of axioms defines the same structure called an LI-structure. Let us extend and modify van der Waerden's notation in (a) by writing $A \mid B$ to express the fact that if $x \in A$ and $B = \{y_0, \ldots, y_{n-1}\}_{\neq}$, then $x \mid (y_0, \ldots, y_{n-1})$. Also, $A \equiv B$ means, by definition, that $A \mid B$ and $B \mid A$.

Lazarson [5] has made a thorough study of LI-structures. I shall now describe three ways in which, according to [5], a LI-structure is determined, provided that $|S| < \aleph_0$.

(α) The LI-structure is determined by the set

$$\pi(S) = \{(A, B) \colon A \equiv B\}$$

of pairs of equivalent sets.

(β) Define a closure operation $A \to A'$ by putting $A' = \{x \colon x \mid A\}$. In fact, $A \subset A' = A''$. A *subspace* is a set B such that $B' = B$. Then the LI-structure is determined by the set

$$\sigma(S) = \{A' \colon A \subset S\}$$

of all subspaces.

(γ) The set A is called *separable* if there is a partition $A = B \cup C$ such that $B, C \neq \emptyset$; $B \cap C = \emptyset$; $\varrho(A) = \varrho(B) + \varrho(C)$. Then the LI-structure is determined by the set

$$\varphi(S) = \{A \colon A \text{ separable}\}$$

of all separable sets provided, however, that all one-element sets are independent.

The problem of finding necessary and sufficient conditions on a set of pairs of subsets of S in order that it should equal $\pi(S)$ for some LI-structure is still open, and similarly in the cases (β) and (γ).

4. I now turn to some general theorems on LI-structures. We suppose that $I(S)$ is given and that the axioms under (b1) hold. If $|S| \geqslant \aleph_0$, then I adjoin to $I(S)$ all infinite A such that $B \subset\subset A$ implies $B \in I$, and then all transfinite sequences $(x_0, x_1, \ldots)$ such that $\{x_0, x_1, \ldots\}_{\neq} \in I$. Similarly, the relation $A \mid B$ is extended to infinite sets, and now $\beta(S)$ denotes the set of all maximal elements of the extended set $I(S)$. I write I for $I(S)$. The LI-structure is called a *natural* structure if S is a subset of a vector space over a division ring R, and the abstract linear independence coincides with linear independence over R.

I begin by stating the exchange theorem which is often named after E. Steinitz but which was already published earlier in Grassmann's *Ausdehnungslehre*.

THEOREM 1. *If* $(x_0, \ldots, x_{m-1}) \in I$ *and* $x_\mu \mid (y_0, \ldots, y_{n-1})$ *for* $\mu < m$, *then there are indices* $a_0 < \ldots < a_{m-1} < n$ *such that*

$$\big(\{y_0, \ldots, y_{n-1}\} - \{y_{a_0}, \ldots, y_{a_{m-1}}\}\big) \cup \{x_0, \ldots, x_{m-1}\} \equiv \{y_0, \ldots, y_{n-1}\}.$$

THEOREM 2. (1) *If* $A \in I$, *then* $A \subset B \in \beta(S)$ *for some* B.

(2) *If* $A \mid B$ *and* $B \mid C$, *then* $A \mid C$.

(3) *If* $A, B \in \beta(S)$, *then* $|A| = |B|$.

(4) *If* $A, B \in I$ *and* $|A| < |B|$, *then there is* $C \in I$ *such that* $A \subset C \subset A \cup B$ *and* $|C| = |B|$.

These results have been well known for some time.

The unique number $|A|$, for $A \in \beta(S)$, is called the *rank* of S and denoted by $\varrho(S)$. The simplest proof of (3) is one based on a method due to H. Löwig, quoted in [4], p. 240. The first proof seems to be that in [8].

THEOREM 3. *Let k be a positive integer. Then there is a partition* $S = S_0 \cup \ldots \cup S_{k-1}$ *such that* $S_0, \ldots, S_{k-1} \in I$ *if and only if, whenever* $A \subset\subset S$, *then* $|A| \leqslant k\varrho(A)$.

This result, in the special case of natural LI-structures, was first proved by Horn [2]. His argument can be adapted so as to yield the more general version of Theorem 3.

Hall [1] proved the following result. Let n be a positive integer and $A_0, \ldots, A_{n-1}$ be n sets. Then it is possible to select $x_\nu \in A_\nu$, for $\nu < n$, such that $x_\alpha \neq x_\beta$ for $\alpha < \beta < n$, if and only if $|A_{\nu_0} \cup \ldots \cup A_{\nu_{p-0}}| \geqslant p$ whenever $\nu_0 < \ldots < \nu_{p-1} < n$. The following theorem [7] is a generalisation.

THEOREM 4. *Let n be a positive integer and* $A_0, \ldots, A_{n-1} \subset S$. *Then it is possible to select* $x_\nu \in A_\nu$, *for* $\nu < n$, *such that* $(x_0, \ldots, x_{n-1}) \in I$, *if and only if* $\varrho(A_{\nu_0} \cup \ldots \cup A_{\nu_{p-1}}) \geqslant p$ *whenever* $\nu_0 < \ldots < \nu_{p-1} < n$.

It is shown in [7] that, vice versa, I-functions are the only functions with values in $\{0, 1\}$ which satisfy $f(x, x) = 0$ and for which the assertion of Theorem 4 holds.

In the next theorem n denotes any ordinal number; its cardinal is $|n|$.

THEOREM 5. *Let, for* $\nu < n$, $A_\nu \notin I$ *and* $\bigcup(\mu < \nu)A_\mu \cap A_\nu \in I$. *Then* $\varrho\big(\bigcup(\nu < n)A_\nu\big) + |n| \leqslant |\bigcup(\nu < n)A_\nu|$.

Originally this was only proved [9] and in a somewhat weaker form, for natural LI-structures, and the proof did not appear to be capable of an extension. But Ingleton [3] found an argument which establishes the more general proposition as stated in Theorem 5.

The next theorem is most easily stated by using the *obliteration operator* $\wedge$ whose effect consists in removing the symbol above which it is placed. Again, n denotes an ordinal number, and ω denotes the least infinite ordinal number. On the assumption that S is well ordered a certain base, the *minimal base*, is characterised by certain extremal properties [9].

THEOREM 6. *Let* S *be well ordered.*

(i) *There is exactly one ordinal* n *and one set* $\{b_0, \ldots, \hat{b}_n\} \in \beta^*(S)$ *such that* $(b_0, \ldots, \hat{b}_\nu, x) \notin I$ *for* $\nu < n$; $x < b_\nu$.

(ii) *If* $\nu < \min(n, \omega)$, *then* b_ν *is the least* x_ν *such that, for suitable* $x_0, \ldots, \hat{x}_\nu$, *we have* $\{x_0, \ldots, x_\nu\}_< \in I$.

This is false if $\nu = \omega < n$.

(iii) *If* $n < \omega$ *and* $\{x_0, \ldots, x_{n-1}\}_< \in I$, *then* $x_\nu \geqslant b_\nu$ *for* $\nu < n$.

(iv) $S - \{b_0, \ldots, b_n\}$ *is the set of all maximal elements of circuits, i. e. of elements of* $\gamma(S)$.

5. Representations. Let V be a vector space over a division ring R. A *representation* of an LI-structure is a one-one map $S \to V$ which takes the given LI-structure into a natural LI-structure over V. If such a representation exists we say that $I(S)$ is *representable over* R. It is convenient to call $|S|$ the *order* of every LI-structure over S.

Whitney formulated the problem of deciding whether every LI-structure of finite order is representable over some field, and he was particularly interested in the real and the complex number field. He constructed an LI-structure which is representable over the finite field $GF(2)$ but not over the reals, and he established necessary and sufficient conditions for representability over $GF(2)$. Lazarson [6] proved the following two theorems.

THEOREM 7. R *is a division ring with prime characteristic* p, *and* V *a vector space over* R. *Let* $x_0, \ldots, x_p$ *be* $p+1$ *linearly independent elements of* V, *and* $x = x_0 + \ldots + x_p$. *Then the natural LI-structure, of order* $2p+3$, *over the set*

$$\{x_0, \ldots, x_p, x - x_0, \ldots, x - x_p, x\} \tag{1}$$

is only representable over division rings of characteristic p.

THEOREM 8. *There is an LI-structure of order* 16 *which is not representable over any division ring.*

To deduce Theorem 8 from Theorem 7 we need only apply Theorem 7 to the primes 2 and 3 and consider the LI-structure which is, in the obvious sense, the direct sum of natural LI-structures on two disjoint sets (1) corresponding to the two primes. The order of the sum-structure is

$$|S| = (2p_1 + 3) + (2p_2 + 3) = 16.$$

Also, $\varrho(S) = (p_1 + 1) + (p_2 + 1) = 7$. If we replace 2 and 3 by any two large successive primes we find, in the same way, a non-representable structure on a set S such that

$$\varrho(S) > (\tfrac{1}{2} - \varepsilon)|S|, \tag{2}$$

where ε is any preassigned positive number.

Using the existence of finite projective planes in which (i) Desargues' theorem fails, or (ii) Pappus' theorem fails, and defining abstract linear dependence of triples of points by means of collinearity, Ingleton [3] has proved the following result:

THEOREM 9. (i) *There is an LI-structure of order* 10 *which is not representable over any division ring.*

(ii) *There is an LI-structure of order* 9 *which is not representable over any field.*

In contrast to Lazarson's construction, Ingleton's examples of non-representable LI-structures are of bounded rank, in fact of rank 2, and his method cannot lead to any higher rank since in every finite projective space of dimension greater than 2 the theorems of Desargues and Pappus hold.

To formulate the next theorem most easily we introduce, for finitely many sets A_ν, the notation

$$\sum^{(2)} A_\nu = \{x : |\{\nu : x \in A_\nu\}| \in \{1, 3, 5, \ldots\}\}.$$

The theorem that follows contains positive results on representability some of which are taken from [9] whilst others are well known. Here $|S| < \aleph_0$, and we are given an LI-structure on S.

THEOREM 10. (i) *If* $\varrho(S) \leqslant 2$ *or* $\varrho(S) \geqslant |S|-1$, *then the structure is representable over every field.*

(ii) *If* $|S| \leqslant 6$, *then the structure is representable over the rational field, and there is a structure of order* 7 *which is only representable over division rings with characteristic* 2.

(iii) *A necessary and sufficient condition for representability over* $GF(2)$ *is: Whenever* $A_0, \ldots, A_{n-1} \in \gamma(S)$ *and* $\sum^{(2)} A_\nu \neq \emptyset$, *then* $\sum^{(2)} A_\nu \notin I$.

(iv) *If the structure is representable over a field* F, *then it is also representable over some simple algebraic extension of the prime field of* F; *if, in addition,* F *is of characteristic zero, then there is* $c > 0$ *such that, for every prime* $p \geqslant c$ *and every integer* $k \geqslant c$, *the structure is representable over* $GF(p^k)$. *Furthermore, under the same conditions, it is representable over* $GF(p)$ *for infinitely many primes* p.

6. Open questions. (a) It seems that of the axioms (i)-(iv) of (b2), the condition (i) causes most of the difficulties. Is there an algebraic structure, in some sense related to that of a subset of a vector space, which in a natural way gives rise to a function f satisfying (ii)-(iv) of (b2) and not necessarily (i)? (**P 530**).

(b) Ingleton [3] put forward the conjectures that every LI-structure of order k is representable over some division ring if $k \leqslant 9$, and over some field if $k \leqslant 8$. These conjectures are still open. If true, they are best possible.

(c) In the existing theory of LI-structures linear independence is a property of finite character. It would be of interest to construct a non-trivial theory which does not present this restriction. (**P 531**)

(d) It would be of interest to investigate whether the constant $\frac{1}{2}$ is best possible in § 5, (2). (**P 532**)

(e) It is well known that for every natural LI-structure, and therefore for every representable LI-structure, of finite order, we have

$$\varrho(A \cup B)+\varrho(A' \cap B') = \varrho(A)+\varrho(B) \tag{3}$$

for $A, B \subset S$, where $A \to A'$ is the closure defined in § 3 (ii). The method of proof shows that for non-representable LI-structures we have, for $A, B \subset S$,

$$\varrho(A \cup B)+\varrho(A' \cap B') \leqslant \varrho(A)+\varrho(B).$$

From here a simple consideration leads to the following result:

THEOREM 11. *Let $|S|$ be arbitrary, and let there be given a representable LI-structures on S. Then there is an extension of this structure into a set T, where $S \subset T$ and*

$$|T| = \max(|S|, \aleph_0),$$

such that (3) *holds for $A, B \subset\subset T$.*

It would be of interest to decide whether in this theorem representability is a necessary hypothesis. This might well be a first step towards a solution of the central problem in this field, that of finding an intrinsic representability condition for LI-structures. (**P 533**)

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