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GROUP THEORETICAL PROPERTIES AND FUNCTIONS*

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When investigating group theoretical properties (like commutativity or the property of being noetherian) one soon makes the following observations: firstly it is possible to derive from these properties certain characteristic subgroups and secondly these characteristic subgroups play a central part in the study of these properties. This second phenomenon becomes even more noticeable if one investigates not a single property but a whole class of properties. Then the class under consideration will have to be singled out in such a way that the derived characteristic subgroups are in some sense amenable to treatment.

Let us illustrate this general remark by a fairly well-known example: If $\mathfrak{e}$ is some group theoretical property, then the $\mathfrak{e}$-radical of a group G is the product $\mathfrak{e}'G$ of all the normal $\mathfrak{e}$-subgroups of G. The property $\mathfrak{e}$ has been termed a radical property whenever $\mathfrak{e}'G$ is likewise an $\mathfrak{e}$-group (Plotkin [10] et alii).

Now $\mathfrak{e}'$ constitutes an example of what we have in mind when talking of group theoretical functions. Such a function $\mathfrak{f}$ attaches to every group G a subgroup $\mathfrak{f}G$ subject to the single rule:

$$(\mathfrak{f}G)^\sigma = \mathfrak{f}(G^\sigma) \text{ for every isomorphism } \sigma \text{ of } G.$$

Then one may prove (Theorem 3.5): The group theoretical function $\mathfrak{f}$ has the form $\mathfrak{f} = \mathfrak{e}'$ for a radical property $\mathfrak{e}$ such that normal subgroups of $\mathfrak{e}$-groups are $\mathfrak{e}$-groups if, and only if,

$$\mathfrak{f}N = N \cap \mathfrak{f}G \text{ for every normal subgroup } N \text{ of } G.$$

It seems to the author that the characterization of the class of functions is more satisfactory than that of the class of properties under

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consideration; and thus it may be advantageous to consider properties and functions simultaneously, giving emphasis to that concept which is more appropriate to the problem attacked.

We have shown in the above example how to derive a function from a property. Naturally we need ways of deriving properties from functions. In our example the derived property $\mathfrak{f}'$ belonging to the function $\mathfrak{f}$ is defined by the rule:

The group G is an $\mathfrak{f}'$-group if, and only if, $G = \mathfrak{f}G$.

The prime objective of our investigation will be the discussion of several possible relations between group theoretical properties and functions. Apart from the "derivations" already mentioned we shall be concerned with the following operations:

If $\mathfrak{e}$ is a group theoretical property, then

$\mathfrak{e}^*G$ is the intersection of all normal subgroups N of G with G/N an $\mathfrak{e}$-group (§ 3);

$\mathfrak{h}_{\mathfrak{e}}G$ is the set of all elements g of G such that $\{g, E\}$ is contained in an $\mathfrak{e}$-group whenever E is an $\mathfrak{e}$-subgroup of G (§ 4);

${}^{co}\mathfrak{h}_{\mathfrak{e}}G$ is the intersection of all the normal subgroups N of G such that $G/(N \cap X)$ is an $\mathfrak{e}$-group whenever G/X is an $\mathfrak{e}$-group (§ 5).

If $\mathfrak{f}$ is a group theoretical function, then the property $\mathfrak{f}^*$ is defined by the rule:

G is an $\mathfrak{f}^*$-group if, and only if, $\mathfrak{f}G = 1$ (§ 3, 5).

It is necessary to make one aspect at the same time more general and more precise. Our discussion will not be effected in the universe of all groups, but in a class $\mathfrak{D}$ of groups which will be the domain of definition. Important classes which may be chosen as $\mathfrak{D}$ — and have been discussed in previous investigations — are the class of finite groups or the class of abelian groups and the like. The gain obtained this way may be illustrated by the following example.

A *variety* on $\mathfrak{D}$ is a class $\mathfrak{v}$ of groups in $\mathfrak{D}$, meeting the following requirements:

$\mathfrak{v}$ is inherited by subgroups and epimorphic images; and $\mathfrak{v}$ is residual.

If $\mathfrak{D}$ happens to be the class of all groups, then the varieties are just the classes of groups defined by identical relations. If, however, $\mathfrak{D}$ is the class of all finite groups, then the finite p-groups, for p a fixed prime, form likewise a variety on $\mathfrak{D}$. But the only class of groups defined by identical relations, which comprises all finite p-groups, is the class of all groups (see Baer [1], p. 205, Satz 5.1).

A last remark seems to be in order. Many of the ideas developped here — and also some related ideas — can be applied in the theory of rings and, more generally, in the theory of universal algebras. But the acid test for all such methods will always be found in the theory of groups.

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1. THE GENERAL SETTING

Our discussion will be conducted within the framework of a class $\mathfrak{D}$ of groups which will be subject to varying requirements and for which class $\mathfrak{D}$ the following classes will be typical examples:

the universal class of all groups;

the class of all finite groups;

the class of all noetherian groups (= groups by whose subgroups the maximum condition is satisfied);

the class of all artinian groups (= groups by whose subgroups the minimum condition is satisfied); and so on.

The examples indicate that these classes $\mathfrak{D}$ may be comparatively large and may meet many requirements. These will be stated as need arises; but we shall always require that 1 is a member of $\mathfrak{D}$ and that $\mathfrak{D}$ contains complete classes of isomorphic groups.

On $\mathfrak{D}$ we shall discuss firstly group theoretical properties. If $\mathfrak{e}$ is such a property, then a group in $\mathfrak{D}$ is either an $\mathfrak{e}$-group or it is not; and we shall always impose the following two requirements:

1 is an $\mathfrak{e}$-group and isomorphic images of $\mathfrak{e}$-groups are $\mathfrak{e}$-groups.

Thus group theoretical properties are essentially subclasses of $\mathfrak{D}$ meeting various appropriate requirements.

On $\mathfrak{D}$ we shall discuss secondly group theoretical functions. Such a function $\mathfrak{f}$ assigns to every group G in $\mathfrak{D}$ a uniquely determined subgroup $\mathfrak{f}G \subseteq G$; and we shall always require the validity of

$$(\mathfrak{f}G)^\sigma = \mathfrak{f}(G^\sigma) \text{ for every isomorphism } \sigma \text{ of } G.$$

This implies in particular that the $\mathfrak{f}$-subgroup $\mathfrak{f}G$ is always a characteristic subgroup of G.

2. RESIDUALITY

General requirement. $\mathfrak{D}$ contains with any group all its epimorphic images.

Notational reminder. If $\mathfrak{e}$ is a property, then a *co-$\mathfrak{e}$-subgroup* is a normal subgroup X of G with $\mathfrak{e}$-quotient group G/X.

$$\mathfrak{e}^*G = \text{intersection of all co-}\mathfrak{e}\text{-subgroups of } G.$$

The characterization of residual properties is effected by means of the following

THEOREM 2.1. *The following properties of the property* $\mathfrak{e}$ (*on* $\mathfrak{D}$) *are equivalent*:

(i) $\begin{cases} \text{(a) } \textit{Intersections of co-}\mathfrak{e}\textit{-subgroups are co-}\mathfrak{e}\textit{-subgroups.} \\ \text{(b) } \textit{Epimorphic images of } \mathfrak{e}\textit{-groups are } \mathfrak{e}\textit{-groups.} \end{cases}$

(ii) $\begin{cases} \text{(a) } \mathfrak{e}^*G \textit{ is for every } G \textit{ (in } \mathfrak{D}) \textit{ a co-}\mathfrak{e}\textit{-subgroup of } G. \\ \text{(b) } \textit{Epimorphic images of } \mathfrak{e}\textit{-groups are } \mathfrak{e}\textit{-groups.} \end{cases}$

(iii) *The normal subgroup N of G (in $\mathfrak{D}$) is a co-$\mathfrak{e}$-subgroup of G if, and only if,* $\mathfrak{e}^*G \subseteq N$.

(iv) $\begin{cases} \text{(a) } G \textit{ is an } \mathfrak{e}\textit{-group if, and only if, } \mathfrak{e}^*G = 1. \\ \text{(b) } (\mathfrak{e}^*G)^\sigma = \mathfrak{e}^*(G^\sigma) \textit{ for every epimorphism } \sigma \textit{ of } G. \end{cases}$

Properties $\mathfrak{e}$, meeting the equivalent requirements (i)-(iv), shall be termed *residual.*

Proof. (ii.a) is just a slightly weakened form of (i. a) so that (ii) is a consequence of (i). If (ii) is true, and if N is a normal subgroup of G with $\mathfrak{e}^*G \subseteq N$, then G/N is an epimorphic image of $G/\mathfrak{e}^*G$ so that N is a co-$\mathfrak{e}$-subgroup of G; and now it is clear that (iii) is a consequence of (ii). If (iii) is true, and if J is an intersection of co-$\mathfrak{e}$-subgroups of G, then $\mathfrak{e}^*G \subseteq J$ so that J is a co-$\mathfrak{e}$-subgroup of G, implying (i. a). If furthermore G is an $\mathfrak{e}$-group, then $\mathfrak{e}^*G = 1$ by definition; and it follows from (iii) that every normal subgroup of G is a co-$\mathfrak{e}$-subgroup of G and that consequently every epimorphic image of G is an $\mathfrak{e}$-group. Hence conditions (i)-(iii) are equivalent.

Assume next the validity of the equivalent conditions (i)-(iii). If G is an $\mathfrak{e}$-group, then $\mathfrak{e}^*G = 1$ by definition. If conversely $\mathfrak{e}^*G = 1$, then we deduce from (ii.a) that G is an $\mathfrak{e}$-group. Hence (iv.a) is true. If furthermore σ is an epimorphism of G upon H, then σ induces an epimorphism of $G/\mathfrak{e}^*G$ upon $H/(\mathfrak{e}^*G)^\sigma$ so that $H/(\mathfrak{e}^*G)^\sigma$ is by (ii) an $\mathfrak{e}$-group. Hence $\mathfrak{e}^*H \subseteq (\mathfrak{e}^*G)^\sigma$ by (iii). We form the inverse image $(\mathfrak{e}^*H)^{\sigma^{-1}}$ and note the isomorphism $G/(\mathfrak{e}^*H)^{\sigma^{-1}} \simeq H/\mathfrak{e}^*H$. Since the latter group is an $\mathfrak{e}$-group by (ii.a), we may deduce $\mathfrak{e}^*G \subseteq (\mathfrak{e}^*H)^{\sigma^{-1}}$ from (iii); and now we find that

$$(\mathfrak{e}^*G)^\sigma \subseteq [(\mathfrak{e}^*H)^{\sigma^{-1}}]^\sigma = \mathfrak{e}^*H \subseteq (\mathfrak{e}^*G)^\sigma,$$

showing the validity of (iv.b). Assume conversely the validity of (iv). If σ is the canonical epimorphism of G upon $G/\mathfrak{e}^*G$, then we deduce from (iv.b) that

$$\mathfrak{e}^*(G/\mathfrak{e}^*G) = \mathfrak{e}^*(G^\sigma) = (\mathfrak{e}^*G)^\sigma = \mathfrak{e}^*G/\mathfrak{e}^*G = 1;$$

and it follows from (iv.a) that $G/\mathfrak{e}^*G$ is an $\mathfrak{e}$-group, showing the validity of (ii.a). If furthermore λ is an epimorphism of the $\mathfrak{e}$-group G upon the group H, then we deduce $\mathfrak{e}^*G = 1$ from (iv.a) so that $\mathfrak{e}^*H = 1$ by (iv.b). It follows from (iv.a) again that H is an $\mathfrak{e}$-group, proving (ii.b). Thus we have deduced (ii) from (iv), showing the equivalence of (i) to (iv).

PROPOSITION 2.2. *If the class $\mathfrak{D}$ contains with any group all its subgroups, then the following properties of the residual property $\mathfrak{e}$ are equivalent:*

(a) *Subgroups of $\mathfrak{e}$-groups are $\mathfrak{e}$-groups.*

(b) $\mathfrak{e}^*S \subseteq \mathfrak{e}^*G$ *for every subgroup S of a group G in $\mathfrak{D}$.*

Proof. Assume first the validity of (a) and consider a subgroup S of a group G in $\mathfrak{D}$. Then $S \cap \mathfrak{e}^*G$ is a normal subgroup of S such that

$$S/(S \cap \mathfrak{e}^*G) \simeq S\mathfrak{e}^*G/\mathfrak{e}^*G \subseteq G/\mathfrak{e}^*G.$$

The last of these groups is an $\mathfrak{e}$-group by Theorem 2.1(ii.a). Since $\mathfrak{e}$ is by (a) subgroup-inherited, $S \cap \mathfrak{e}^*G$ is a co-$\mathfrak{e}$-subgroup of S; and this implies by Theorem 2.1, (iii) that

$$\mathfrak{e}^*S \subseteq S \cap \mathfrak{e}^*G \subseteq \mathfrak{e}^*G,$$

showing that (b) is a consequence of (a).

That (a) is a consequence of (b), is immediately deduced from Theorem 2.1(iv.a).

REMARK 2.3. If $\mathfrak{D}$ happens to be the universal class of all groups, then subgroup-inherited residual properties are just the properties defined by identical relations; see, for instance, Baer [1], p. 183, Satz 1.1. But the property of nilpotency, defined on the class of all finite groups, is a subgroup-inherited residual property, though it is in no way related to properties defined by identical relations. This fact is contained in Baer [1], p. 205, Satz 5.1.

REMARK 2.4. There exists any number of residual properties which are not subgroup-inherited. A simple example is obtained as follows: Let $\mathfrak{D}$ be the class of all finite groups and denote by $\mathfrak{e}$ the class of all finite direct products of non-abelian simple groups. Then $\mathfrak{e}$ is residual, though not subgroup-inherited.

The group theoretical function $\mathfrak{f}$ on $\mathfrak{D}$ is termed *residual* if $(\mathfrak{f}G)^\sigma = \mathfrak{f}(G^\sigma)$ for every group G in $\mathfrak{D}$ and every epimorphism σ of G.

If $\mathfrak{f}$ is a function on $\mathfrak{D}$, then we derive from $\mathfrak{f}$ a group theoretical property $\mathfrak{f}^*$ by means of the following rule:

The group G in $\mathfrak{D}$ is an $\mathfrak{f}^*$-group if, and only if, $\mathfrak{f}G = 1$.

These two definitions are clearly suggested by Theorem 2.1(iv). This relation is made explicit by the following

THEOREM 2.5. *The following properties of the group-theoretical*

property $\mathfrak{e}$ are equivalent:	*function $\mathfrak{f}$ are equivalent:*
(1′) $\mathfrak{e}$ *is residual.*	(1″) $\mathfrak{f}$ *is residual.*
(2′) $\mathfrak{e}^*$ *is residual and* $\mathfrak{e} = \mathfrak{e}^{**}$.	(2″) $\mathfrak{f}^*$ *is residual and* $\mathfrak{f} = \mathfrak{f}^{**}$.

Proof. If firstly $\mathfrak{e}$ is residual, then we deduce from Theorem 2.1 (iv.b) that $\mathfrak{e}^*$ is residual. Furthermore a group G is by definition an $\mathfrak{e}^{**}$-group if, and only if, $\mathfrak{e}^*G = 1$; and it is a consequence of Theorem 2.1 (iv.a) that $\mathfrak{e}^*G = 1$ is necessary and sufficient for G to be an $\mathfrak{e}$-group. Hence G is an $\mathfrak{e}$-group if, and only if, G is an $\mathfrak{e}^{**}$-group, showing $\mathfrak{e} = \mathfrak{e}^{**}$. Thus (2′) is a consequence of (1′).

Assume secondly the residuality of $\mathfrak{f}$. If G is an $\mathfrak{f}^*$-group and σ an epimorphism of G upon H, then $\mathfrak{f}H = (\mathfrak{f}G)^\sigma = 1$ by residuality, showing that H is an $\mathfrak{f}^*$-group. Hence $\mathfrak{f}^*$ is epimorphism-inherited. If λ is the canonical epimorphism of G upon $G/\mathfrak{f}G = L$, then $\mathfrak{f}L = (\mathfrak{f}G)^\lambda = 1$ so that L is an $\mathfrak{f}^*$-group. If next N is a normal subgroup of G with $\mathfrak{f}G \subseteq N$, then G/N is an $\mathfrak{f}^*$-group as an epimorphic image of the $\mathfrak{f}^*$-group $G/\mathfrak{f}G$. If next N is a co-$\mathfrak{f}^*$-subgroup of G, then we get by residuality and the definition of $\mathfrak{f}^*$ that

$$1 = \mathfrak{f}(G/N) = N \cdot \mathfrak{f}G/N;$$

and this implies $\mathfrak{f}G \subseteq N$. Thus we see that the normal subgroup N of G is a co-$\mathfrak{f}^*$-subgroup of G if, and only if, $\mathfrak{f}G \subseteq N$. Since $\mathfrak{f}^{**}G$ is the intersection of all the co-$\mathfrak{f}^*$-subgroups of G, this implies $\mathfrak{f}G \subseteq \mathfrak{f}^{**}G$. But $\mathfrak{f}G$ is itself a co-$\mathfrak{f}^*$-subgroup of G so that $\mathfrak{f}^{**}G \subseteq \mathfrak{f}G$; and we have shown that

$$\mathfrak{f} = \mathfrak{f}^{**}$$

and that $\mathfrak{f}^{**}G = \mathfrak{f}G$ is a co-$\mathfrak{f}^*$-subgroup of G. Thus we have shown that $\mathfrak{f}^*$ meets requirement (ii) of Theorem 2.1; and hence we have deduced (2″) from (1″).

If next $\mathfrak{e}$ is a property meeting requirement (2′), then it follows from the implication of (2″) by (1″) that $\mathfrak{e} = \mathfrak{e}^{**}$ is residual; and from the implication of (2′) by (1′) we deduce likewise that (2″) implies (1″). This completes the proof.

REMARK 2.6. If $\mathfrak{D}$ is the class of all finite groups and $\mathfrak{e}$ the property of being a cyclic group, then $\mathfrak{e}^*G = G'$ is for every finite group G the commutator subgroup of G. Naturally $\mathfrak{e}^*$ is a residual function, but $\mathfrak{e}^{**}$ is the property of being abelian so that $\mathfrak{e} \neq \mathfrak{e}^{**}$. This shows that the second part of (2′) is not a consequence of the first part. Consequently, it is indispensable.

REMARK 2.7. Denote by $\mathfrak{D}$ some admissible class of groups. We define the function $\mathfrak{f}$ by the rule

$$\mathfrak{f}G = \begin{cases} 1 & \text{if } G \text{ is abelian,} \\ G & \text{if } G \text{ is not abelian.} \end{cases}$$

Then $\mathfrak{f}^*$ is the property of being an abelian group in $\mathfrak{D}$ and $\mathfrak{f}^{**}$ is the commutator subgroup. It is clear that most choices of $\mathfrak{D}$ lead to a situation with $\mathfrak{f} \neq \mathfrak{f}^{**}$, proving that the second part of (2″) is not a consequence of its first part and that therefore the second part of (2″) is indispensable.

REMARK 2.8. If $\mathfrak{f}$ is a residual function and $\mathfrak{D}$ is subgroup-inherited, then we deduce from Proposition 2.2 that the derived property $\mathfrak{f}^*$ is subgroup-inherited if, and only if,

$$\mathfrak{f}A \subseteq \mathfrak{f}B \quad \text{for} \quad A \subseteq B.$$

Such functions we have termed "functors" elsewhere (Baer [2], p. 179), since they meet the requirement

$$(\mathfrak{f}G)^\lambda \subseteq \mathfrak{f}H \text{ for every homomorphism } \lambda \text{ of } G \text{ into } H.$$

COROLLARY 2.9. *The following properties of the group-theoretical function $\mathfrak{f}$ are equivalent:*

(i) $\mathfrak{f}$ *is residual.*

(ii) (a) $\mathfrak{f}(G/\mathfrak{f}G) = 1$.
(b) $\mathfrak{f}^*$ *is epimorphism-inherited.*
(c) $\mathfrak{f} = \mathfrak{f}^{**}$.

(iii) (a) $\mathfrak{f}(G^\sigma) = 1$ *if, and only if,* $(\mathfrak{f}G)^\sigma = 1$ *for σ an epimorphism of G.*
(b) $\mathfrak{f} = \mathfrak{f}^{**}$.

Proof. It is clear that (iii.a) is a consequence of the residuality of $\mathfrak{f}$; and (iii.b) may be deduced from (i) by means of Theorem 2.5. Conditions (ii.a) and (ii.b) are clearly contained in (iii.a) so that (ii) is a consequence of (iii). If finally (ii) is satisfied by $\mathfrak{f}$, then we deduce from (a) and (c) that $\mathfrak{f}^{**}G$ is a co-$\mathfrak{f}^*$-subgroup of G. Hence it follows from (b) and Theorem 2.1(ii) that $\mathfrak{f}^*$ is a residual property. Thus Theorem 2.5(2″), is satisfied by $\mathfrak{f}$, showing the residuality of $\mathfrak{f}$.

3. CO-RESIDUALITY

GENERAL REQUIREMENT. $\mathfrak{D}$ contains with any group all its normal subgroups.

NOTATIONAL REMINDER. If $\mathfrak{e}$ is a property, then $\mathfrak{e}'G$ is the product of all normal $\mathfrak{e}$-subgroups of the group G in $\mathfrak{D}$.

The characterization of co-residual properties is effected by means of the following

THEOREM 3.1. *The following properties of the property $\mathfrak{e}$ (on $\mathfrak{D}$) are equivalent:*

(i) (a) *Products of normal $\mathfrak{e}$-subgroups are $\mathfrak{e}$-groups.*
(b) *Normal subgroups of $\mathfrak{e}$-groups are $\mathfrak{e}$-groups.*

(ii) (a) *$\mathfrak{e}'G$ is (for every G in $\mathfrak{D}$) an $\mathfrak{e}$-group.*
(b) *Normal subgroups of $\mathfrak{e}$-groups are $\mathfrak{e}$-groups.*

(iii) *The normal subgroup N of G is an $\mathfrak{e}$-group if, and only if, $N \subseteq \mathfrak{e}'G$.*

(iv) $\begin{cases} \text{(a)} & G \textit{ is an } \mathfrak{e}\textit{-group if, and only if, } G = \mathfrak{e}'G. \\ \text{(b)} & \mathfrak{e}'N = N \cap \mathfrak{e}'G \textit{ for every normal subgroup } N \textit{ of } G. \end{cases}$

Properties $\mathfrak{e}$, meeting the equivalent requirements (i)-(iv), shall be termed *co-residual.*

Proof. It is clear that (ii) is just a slightly weakened form of (i). It is a consequence of (ii.a) that $\mathfrak{e}'G$ is an $\mathfrak{e}$-group. If N is a normal subgroup of G with $N \subseteq \mathfrak{e}'G$, then N is a normal subgroup of the $\mathfrak{e}$-group $\mathfrak{e}'G$; and it is a consequence of (ii.b) that N is an $\mathfrak{e}$-group. Now it is clear that (iii) is a consequence of (ii).

Assume next the validity of (iii). Since $\mathfrak{e}'G$ is then an $\mathfrak{e}$-group, it is easy to verify (iv.a). If N is a normal subgroup of G, then $\mathfrak{e}'N$ is a characteristic $\mathfrak{e}$-subgroup of N and hence a normal subgroup of G so that $\mathfrak{e}'N \subseteq \mathfrak{e}'G$. Next we note that $N \cap \mathfrak{e}'G$ is a normal subgroup of $\mathfrak{e}'G$ and as such it is an $\mathfrak{e}$-group by (iii). It is furthermore a normal subgroup of N. Thus it follows that

$$\mathfrak{e}'N \subseteq N \cap \mathfrak{e}'G \subseteq \mathfrak{e}'N,$$

showing that (iv.b) is a consequence of (iii) too.

Assume finally the validity of (iv). If the group G is a product of normal $\mathfrak{e}$-subgroups, then $G = \mathfrak{e}'G$ is an $\mathfrak{e}$-group by (iv.a), proving the validity of (i. a). If furthermore N is a normal subgroup of the $\mathfrak{e}$-group G, then

$$\mathfrak{e}'N = N \cap \mathfrak{e}'G = N \cap G = N$$

by (iv); and it is a consequence of (iv.a) that N is an $\mathfrak{e}$-group too, showing that (i) is a consequence of (iv) and that therefore (i)-(iv) are equivalent.

COROLLARY 3.2. *If $\mathfrak{e}$ is co-residual, then $\mathfrak{e}'G$ contains every subnormal $\mathfrak{e}$-subgroup of G.*

Proof. If S is a subnormal $\mathfrak{e}$-subgroup of G, then there exist, by definition of subnormality, finitely many subgroups $S(i)$ such that

$$S = S(0),\ S(i) \text{ is a normal subgroup of } S(i+1),\ S(n) = G.$$

Naturally $S = \mathfrak{e}'S = \mathfrak{e}'S(0)$. Since $\mathfrak{e}'S(i)$ is a characteristic $\mathfrak{e}$-subgroup of the normal subgroup $S(i)$ of $S(i+1)$, it is a normal $\mathfrak{e}$-subgroup of $S(i+1)$ so that $\mathfrak{e}'S(i) \subseteq \mathfrak{e}'S(i+1)$. It follows that

$$S = \mathfrak{e}'S(0) \subseteq \ldots \subseteq \mathfrak{e}'S(i) \subseteq \mathfrak{e}'S(i+1) \subseteq \ldots \subseteq \mathfrak{e}'S(n) = \mathfrak{e}'G,$$

as we wanted to show.

REMARK 3.3. A subgroup A of G is termed *accessible*, if there exist subgroups A_σ of G such that

$$A = A_0, \quad A_\beta = G,$$

$$A_\sigma \text{ is a normal subgroup of } A_{\sigma+1},$$

$$A_\lambda = \bigcup_{\sigma<\lambda} A_\sigma \text{ for } \lambda \text{ a limit ordinal.}$$

It is impossible to substitute in Corollary 3.2 for the word "subnormal" the word "accessible" as may be seen from the following example.

Let $\mathfrak{D}$ be the universal class of all groups and let $\mathfrak{e}$ be the property defined by the rule:

The group G is an $\mathfrak{e}$-group if, and only if, every element in G generates a subnormal subgroup of G.

It is clear that subgroups and epimorphic images of $\mathfrak{e}$-groups are $\mathfrak{e}$-groups.

If G is any group, then $\mathfrak{e}'G$ is generated by its cyclic subnormal subgroups. But the set of elements in a group, generating cyclic subnormal subgroups, is always a characteristic subgroup; cf. Baer [6], in particular p. 418, Satz 2. Consequently $\mathfrak{e}'G$ is always an $\mathfrak{e}$-group. Application of Theorem 3.1, (ii) shows that $\mathfrak{e}$ is co-residual.

Let A be an infinite abelian p-group with $A = A^p$. The group G arises from A by adjoining an element g, subject to the relations:

$$a^g = a^{1+p} \text{ for every } a \text{ in } A.$$

Then $A = \mathfrak{c}A$ is the centralizer of A and G/A is an infinite cyclic group. It is furthermore clear that $A \subseteq \mathfrak{e}'G$. If the element x in G does not belong to A, then A is generated by all the $a^{-1}x^{-1}ax$ for a in A. This implies $\{x^G\} = A\{x\}$. Consequently $\{x\}$ is not subnormal. Hence $A = \mathfrak{e}'G$.

The totality of elements a in A with $a^{p^n} = 1$ is the n-th term $\mathfrak{z}_nG$ of the ascending central chain of G. It follows that $A = \mathfrak{z}_\omega G$ and $G = \mathfrak{z}_{\omega+1}G$. If U is any subgroup of G, then the chain of subgroups $U\mathfrak{z}_\sigma G$ for $0 \leqslant \sigma \leqslant \omega+1$ shows that U is an accessible subgroup of G. In particular, every cyclic subgroup of G is accessible. It follows that $\mathfrak{e}'G = A \subset G$ does not contain every accessible $\mathfrak{e}$-subgroup of G.

COROLLARY 3.4. *Suppose that $\mathfrak{e}$ is a co-residual property.*

(A) *If $\mathfrak{D}$ contains the epimorphic images of all groups in $\mathfrak{D}$, then epimorphic images of $\mathfrak{e}$-groups are $\mathfrak{e}$-groups if, and only if, $(\mathfrak{e}'G)^\sigma \subseteq \mathfrak{e}'(G^\sigma)$ for every epimorphism σ of a group G.*

(B) *If $\mathfrak{D}$ contains with any group all its subgroups, then subgroups of $\mathfrak{e}$-groups are $\mathfrak{e}$-groups if, and only if,*

$$S \cap \mathfrak{e}'G \subseteq \mathfrak{e}'S \text{ for every subgroup } S \subseteq G.$$

Proof. If $\mathfrak{e}$ is epimorphism-inherited and if σ is an epimorphism of G upon H, then every normal $\mathfrak{e}$-subgroup of G is mapped by σ upon a normal $\mathfrak{e}$-subgroup of H so that $(\mathfrak{e}'G)^\sigma \subseteq \mathfrak{e}'H$. If conversely this condition is satisfied by $\mathfrak{e}$, then we deduce from co-residuality (Theorem 3.1(iii)) that $\mathfrak{e}$ is epimorphism-inherited.

If $\mathfrak{e}$ is subgroup-inherited, and if $S \subseteq G$, then $S \frown \mathfrak{e}'G$ is a normal $\mathfrak{e}$-subgroup of S (since $\mathfrak{e}'G$ is by Theorem 3.1(ii.a) a normal $\mathfrak{e}$-subgroup of G), so that $S \frown \mathfrak{e}'G \subseteq \mathfrak{e}'S$. If conversely this condition is satisfied by $\mathfrak{e}$, then we deduce from Theorem 3.1, (iv.a) that $\mathfrak{e}$ is subgroup-inherited.

The group-theoretical function $\mathfrak{f}$ on $\mathfrak{D}$ is termed *co-residual*, if

$$\mathfrak{f}N = N \frown \mathfrak{f}G \text{ for every normal subgroup } N \text{ of } G.$$

If $\mathfrak{f}$ is any function, then we derive from $\mathfrak{f}$ a property $\mathfrak{f}'$ by means of the following rule:

The group G in $\mathfrak{D}$ is an $\mathfrak{f}'$-group if, and only if, $G = \mathfrak{f}G$.

THEOREM 3.5. *The following properties of the group-theoretical property $\mathfrak{e}$ are equivalent:*

(1′) $\mathfrak{e}$ *is co-residual.*

(2′) $\mathfrak{e}'$ *is co-residual and* $\mathfrak{e} = \mathfrak{e}''$.

The following properties of the group-theoretical function $\mathfrak{f}$ are equivalent:

(1″) $\mathfrak{f}$ *is co-residual.*

(2″) $\mathfrak{f}'$ *is co-residual and* $\mathfrak{f} = \mathfrak{f}''$.

Proof. If firstly $\mathfrak{e}$ is co-residual, then we deduce from Theorem 3.1 (iv.b) the co-residuality of $\mathfrak{e}'$. Furthermore a group G is by definition an $\mathfrak{e}''$-group if, and only if, $\mathfrak{e}'G = G$; and this latter property is by Theorem 3.1(iv.a) equivalent with the fact that G is an $\mathfrak{e}$-group. Hence G is an $\mathfrak{e}$-group if, and only if, G is an $\mathfrak{e}''$-group. Consequently $\mathfrak{e} = \mathfrak{e}''$; and we have deduced (2′) from (1′).

Assume secondly the validity of (1″). If X is a normal $\mathfrak{f}'$-subgroup of G, then

$$X = \mathfrak{f}X = X \frown \mathfrak{f}G \subseteq \mathfrak{f}G.$$

This implies in particular $G = \mathfrak{f}G$ in case G is the product of normal $\mathfrak{f}'$-subgroups so that products of normal $\mathfrak{f}'$-subgroups are $\mathfrak{f}'$-groups. If furthermore N is a normal subgroup of the $\mathfrak{f}'$-group G, then we have

$$\mathfrak{f}N = N \frown \mathfrak{f}G = N \frown G = N$$

so that normal subgroups of $\mathfrak{f}'$-groups are $\mathfrak{f}'$-groups. Hence $\mathfrak{f}'$ is a co-residual property by Theorem 3.1(i). Since $\mathfrak{f}G$ is a normal subgroup of G, we have

$$\mathfrak{f}(\mathfrak{f}G) = \mathfrak{f}G \frown \mathfrak{f}G = \mathfrak{f}G$$

so that $\mathfrak{f}G$ is a normal $\mathfrak{f}'$-subgroup of G; and this implies

$$\mathfrak{f}G \subseteq \mathfrak{f}''G.$$

Since $\mathfrak{f}'$ has been shown to be co-residual, $\mathfrak{f}''G$ is always a normal $\mathfrak{f}'$-subgroup of G. Hence $\mathfrak{f}''G \subseteq \mathfrak{f}G$, as has been shown before; and this shows $\mathfrak{f}G = \mathfrak{f}''G$ for every G. Thus (2″) is a consequence of (1″).

That (2′) implies (1′), is a consequence of the fact that (1″) implies (2″); and that (2″) implies (1″), may be deduced from the fact that (1′) implies (2′).

4. PRE-LOCALITY AND LOCALITY

GENERAL REQUIREMENT. $\mathfrak{D}$ contains with any group all its subgroups.

THE $\mathfrak{e}$-HYPERCENTER $\mathfrak{h}_{\mathfrak{e}}G$ of the group G is the *set* of all the elements g in G meeting the requirement:

$\{g, E\}$ is, for every $\mathfrak{e}$-subgroup E of G, part of some $\mathfrak{e}$-subgroup of G.

Subsets of $\mathfrak{h}_{\mathfrak{e}}G$ will be termed *$\mathfrak{e}$-hypercentral.*

Clearly $G = \mathfrak{h}_{\mathfrak{e}}G$ if G is an $\mathfrak{e}$-group. The falsity of the converse may be seen from any number of easily constructed examples; see Example 4.3 below.

The wide range of this construction is best seen from the following two instances:

A. Let $\mathfrak{e}$ be commutativity. Then $\mathfrak{h}_{\mathfrak{e}} = \mathfrak{z}$ is just the center.

B. Let $\mathfrak{D}$ be the class of all finite groups and let $\mathfrak{e}$ be nilpotency. Then $\mathfrak{h}_{\mathfrak{e}} = \mathfrak{h}$ is just the hypercenter; for a proof see Baer [3], p. 42, Theorem 3.

LEMMA 4.1. *The $\mathfrak{e}$-hypercenter is always a characteristic subgroup.*

Proof. Clearly 1 is an $\mathfrak{e}$-hypercentral element of the group G. If a and b are $\mathfrak{e}$-hypercentral elements of G, and if E is an $\mathfrak{e}$-subgroup of G, then there exists an $\mathfrak{e}$-subgroup A of G with

$$\{a, E\} \subseteq A.$$

Likewise there exists an $\mathfrak{e}$-subgroup B of G with

$$\{b, A\} \subseteq B.$$

Hence

$$\{ab^{-1}, E\} \subseteq \{a, b, E\} \subseteq \{b, A\} \subseteq B,$$

proving the $\mathfrak{e}$-hypercentrality of ab^{-1}. Hence $\mathfrak{h}_{\mathfrak{e}}G$ is a, necessarily characteristic, subgroup of G.

LEMMA 4.2. *If $\mathfrak{e}$ is subgroup-inherited, then*

(a) *the element g in G is $\mathfrak{e}$-hypercentral if, and only if, $\{g, E\}$ is an $\mathfrak{e}$-group for every $\mathfrak{e}$-subgroup E of G.*

(b) *$U \cap \mathfrak{h}_{\mathfrak{e}}G \subseteq \mathfrak{h}_{\mathfrak{e}}U$ for every subgroup U of G.*

(c) *If F is a finitely generated subgroup of $\mathfrak{h}_{\mathfrak{e}}G$ and if E is an $\mathfrak{e}$-subgroup of G, then $\{F, E\}$ is an $\mathfrak{e}$-group.*

Proof. If g is an element in $\mathfrak{h}_{\mathfrak{e}}G$ and E is an $\mathfrak{e}$-subgroup of G, then there exists an $\mathfrak{e}$-subgroup S of G with $\{g, E\} \subseteq S$. But $\mathfrak{e}$ is subgroup-inherited and so $\{g, E\}$ is an $\mathfrak{e}$-group. This proves (a); and (b) is a fairly immediate consequence of (a).

If $F = \{g_1, \ldots, g_n\}$ is a finitely generated subgroup of $\mathfrak{h}_{\mathfrak{e}}G$, then we let

$$F_0 = 1, \qquad F_i = \{g_1, \ldots, g_i\}, \qquad F_n = F.$$

If E is an $\mathfrak{e}$-subgroup of G, then $\{E, F_0\} = E$ is an $\mathfrak{e}$-subgroup of G; and thus we may make the inductive hypothesis that $i < n$ and $\{E, F_i\}$ is an $\mathfrak{e}$-group. Now

$$\{E, F_{i+1}\} = \{\{E, F_i\}, g_{i+1}\}$$

is an $\mathfrak{e}$-group by (a), since g_{i+1} is $\mathfrak{e}$-hypercentral. Hence it follows (by complete induction) that $\{E, F\} = \{E, F_n\}$ is an $\mathfrak{e}$-group, proving (c).

EXAMPLE 4.3. If $\mathfrak{D}$ is the universal class of all groups and $\mathfrak{e} = \mathfrak{fg}$ is the property of being finitely generated, then

$$(*) \qquad \mathfrak{h}_{\mathfrak{e}}G = G \text{ for every group } G$$

and $\mathfrak{e}$ is not subgroup-inherited. Hence property (b) of Lemma 4.2 is not sufficient for subgroup-inheritance; and G need not be an $\mathfrak{e}$-group if $G = \mathfrak{h}_{\mathfrak{e}}G$ (cf. however Proposition 4.4). Note furthermore that many properties $\mathfrak{e}$ meet requirement $(*)$. Hence $\mathfrak{e}$ is, in general, in no sense determined by the nature of $\mathfrak{h}_{\mathfrak{e}}$. But see Theorem 4.6(a) for a particular instance where $\mathfrak{e}$ is determined by $\mathfrak{h}_{\mathfrak{e}}$.

PRE-LOCALITY: THE FIRST STAGE. This will be described by the following

PROPOSITION 4.4. *The following properties of the property $\mathfrak{e}$ are equivalent:*

(i) $\begin{cases} \text{(a) } \mathfrak{e} \textit{ is subgroup-inherited.} \\ \text{(b) } G \textit{ is an } \mathfrak{e}\textit{-group if, and only if, } G = \mathfrak{h}_{\mathfrak{e}}G. \end{cases}$

(ii) $\begin{cases} \text{(a) } U \cap \mathfrak{h}_{\mathfrak{e}}G \subseteq \mathfrak{h}_{\mathfrak{e}}U \textit{ for } U \subseteq G. \\ \text{(b) } G \textit{ is an } \mathfrak{e}\textit{-group if, and only if, } G = \mathfrak{h}_{\mathfrak{e}}G. \end{cases}$

(iii) $\begin{cases} \text{(a) } U \cap \mathfrak{h}_{\mathfrak{e}}G \subseteq \mathfrak{h}_{\mathfrak{e}}U \textit{ for } U \subseteq G. \\ \text{(b) } \mathfrak{h}_{\mathfrak{e}}G \textit{ is always an } \mathfrak{e}\textit{-group.} \end{cases}$

A property $\mathfrak{e}$ meeting the equivalent requirements (i)-(iii) will be termed *first stage pre-local.*

Proof. It is a consequence of Lemma 4.2(b) that (i) implies (ii).

If (ii) is true, then we apply (ii.a) on $U = \mathfrak{h}_{\mathfrak{e}}G$ to obtain

$$\mathfrak{h}_{\mathfrak{e}}G = \mathfrak{h}_{\mathfrak{e}}G \cap \mathfrak{h}_{\mathfrak{e}}G \subseteq \mathfrak{h}_{\mathfrak{e}}[\mathfrak{h}_{\mathfrak{e}}G] \subseteq \mathfrak{h}_{\mathfrak{e}}G$$

so that $\mathfrak{h}_{\mathfrak{e}}G = \mathfrak{h}_{\mathfrak{e}}[\mathfrak{h}_{\mathfrak{e}}G]$ is an $\mathfrak{e}$-group by (ii.b). Hence (ii) implies (iii).

Assume finally the validity of (iii). If U is a subgroup of the $\mathfrak{e}$-group G, then

$$U = U \cap G = U \cap \mathfrak{h}_{\mathfrak{e}}G \subseteq \mathfrak{h}_{\mathfrak{e}}U \subseteq U$$

so that $U = \mathfrak{h}_{\mathfrak{e}}U$ is an $\mathfrak{e}$-group. Hence $\mathfrak{e}$ is subgroup-inherited; and the validity of (i.b) is contained in (iii.b).

We term the function $\mathfrak{f}$ on $\mathfrak{D}$ *first stage pre-local*, if

$$U \cap \mathfrak{f}G \subseteq \mathfrak{f}U \quad \text{for} \quad U \subseteq G;$$

and we recall that the group X is an $\mathfrak{f}'$-group if, and only if, $X = \mathfrak{f}X$.

LEMMA 4.5. *If the function $\mathfrak{f}$ on $\mathfrak{D}$ is first stage pre-local, then $\mathfrak{f}'$ is subgroup-inherited and $\mathfrak{f}G$ is always an $\mathfrak{f}'$-group.*

Proof. If U is a subgroup of the $\mathfrak{f}'$-group G, then

$$U = U \cap G = U \cap \mathfrak{f}G \subseteq \mathfrak{f}U \subseteq U$$

so that $U = \mathfrak{f}U$ is likewise an $\mathfrak{f}'$-group. Likewise

$$\mathfrak{f}G = \mathfrak{f}G \cap \mathfrak{f}G \subseteq \mathfrak{f}(\mathfrak{f}G) \subseteq \mathfrak{f}G$$

so that $\mathfrak{f}G = \mathfrak{f}(\mathfrak{f}G)$ is an $\mathfrak{f}'$-group.

THEOREM 4.6. (a) *The property $\mathfrak{e}$ is first stage pre-local if, and only if, the function $\mathfrak{h}_{\mathfrak{e}}$ is first stage pre-local and $\mathfrak{e} = (\mathfrak{h}_{\mathfrak{e}})'$.*

(b) *The function $\mathfrak{f}$ on $\mathfrak{D}$ with $\mathfrak{f} = \mathfrak{h}_{\mathfrak{f}'}$ is first stage pre-local if, and only if, the property $\mathfrak{f}'$ is first stage pre-local.*

Proof. If $\mathfrak{e}$ is first stage pre-local, then we conclude the first stage pre-locality of $\mathfrak{h}_{\mathfrak{e}}$ from Proposition 4.4 (ii.a); and from (ii.b) we deduce the sequence of equivalences:

$$G \text{ is an } \mathfrak{e}\text{-group}; \ G = \mathfrak{h}_{\mathfrak{e}}G; \ G \text{ is an } (\mathfrak{h}_{\mathfrak{e}})'\text{-group}.$$

Hence $\mathfrak{e} = (\mathfrak{h}_{\mathfrak{e}})'$. If conversely $\mathfrak{h}_{\mathfrak{e}}$ is first stage pre-local and $\mathfrak{e} = (\mathfrak{h}_{\mathfrak{e}})'$, then condition (ii) of Proposition 4.4 is satisfied by $\mathfrak{e}$ so that $\mathfrak{e}$ is first stage pre-local.

If next $\mathfrak{f}$ is a function on $\mathfrak{D}$ with $\mathfrak{f} = \mathfrak{h}_{\mathfrak{f}'}$ then we let $\mathfrak{e} = \mathfrak{f}'$ so that $\mathfrak{f} = \mathfrak{h}_{\mathfrak{e}}$ and $\mathfrak{e} = \mathfrak{f}' = (\mathfrak{h}_{\mathfrak{e}})'$. Application of (a) shows that $\mathfrak{e}$ is first stage pre-local if, and only if, $\mathfrak{h}_{\mathfrak{e}}$ is first stage pre-local and $\mathfrak{e} = (\mathfrak{h}_{\mathfrak{e}})'$. But the last condition is satisfied by hypothesis so that $\mathfrak{f}' = \mathfrak{e}$ is first stage pre-local if, and only if, $\mathfrak{h}_{\mathfrak{e}} = \mathfrak{h}_{\mathfrak{f}'} = \mathfrak{f}$ is first stage pre-local.

Remark 4.7. Let $\mathfrak{D}$ be any class of groups containing some non-abelian groups whose center is not 1; and let

$$\mathfrak{f}G = \begin{cases} G, & \text{if } G \text{ is abelian,} \\ 1, & \text{if } G \text{ is non-abelian.} \end{cases}$$

Then $\mathfrak{f}$ is first stage pre-local. Next a group is an $\mathfrak{f}'$-group if, and only if, it is abelian; and consequently

$$\mathfrak{h}_{\mathfrak{f}'} = \mathfrak{z} \ (= \text{center}).$$

It follows that a group G is an $\mathfrak{f}'$-group if, and only if, $G = \mathfrak{h}_{\mathfrak{f}'}G$. Thus $\mathfrak{f}'$ is first stage pre-local. But $\mathfrak{f} \neq \mathfrak{h}_{\mathfrak{f}'}$ showing the impossibility of deriving the general hypothesis $\mathfrak{f} = \mathfrak{h}_{\mathfrak{f}'}$ in Theorem 4.6(b) from its other parts.

As a second example consider the class $\mathfrak{D}$ of all abelian groups (or that of all abelian torsion groups) and let

$$\mathfrak{f}G = \begin{cases} G, & \text{if } G \text{ is cyclic,} \\ 1, & \text{if } G \text{ is not cyclic.} \end{cases}$$

Then $\mathfrak{f}$ is first stage pre-local; and a group is an $\mathfrak{f}'$-group if, and only if, it is cyclic. If G is a group of Prüfer's type p^∞, then every pair of elements in G generates a cyclic subgroup so that $G = \mathfrak{h}_{\mathfrak{f}'}G$. But G is not cyclic and hence not an $\mathfrak{f}'$-group showing that $\mathfrak{f}'$ is not first stage pre-local. This shows the impossibility of omitting the general hypothesis $\mathfrak{f} = \mathfrak{h}_{\mathfrak{f}'}$ in Theorem 4.6(b).

Pre-locality: the second stage. This will be described by the following

Proposition 4.8. *The following properties of the property $\mathfrak{e}$ are equivalent*:

(i) (a) *$\mathfrak{e}$ is subgroup-inherited.*
(b) *The subset S of G is part of $\mathfrak{h}_{\mathfrak{e}}G$ if, and only if, $\{S, E\}$ is an $\mathfrak{e}$-group for every $\mathfrak{e}$-subgroup E of G.*

(ii) (a) *$U \cap \mathfrak{h}_{\mathfrak{e}}G \subseteq \mathfrak{h}_{\mathfrak{e}}U$ for $U \subseteq G$.*
(b) *$\mathfrak{h}_{\mathfrak{e}}[U \cdot \mathfrak{h}_{\mathfrak{e}}G] = \mathfrak{h}_{\mathfrak{e}}U \cdot \mathfrak{h}_{\mathfrak{e}}G$ for $U \subseteq G$.*
(c) *$\mathfrak{h}_{\mathfrak{e}}G$ is an $\mathfrak{e}$-group.*

(iii) (a) *$\mathfrak{e}$ is subgroup-inherited.*
(b) *$E \cdot \mathfrak{h}_{\mathfrak{e}}G$ is an $\mathfrak{e}$-group for every $\mathfrak{e}$-subgroup E of G.*

Properties meeting these equivalent requirements (i)-(iii) shall be termed *second stage pre-local.*

NOTES. If we let $S = G$ or $\mathfrak{h}_e G$ and $E = 1$ in (i. b) or just $E = 1$ in (iii.b), then we see that (i) as well as (iii) implies condition (i) of Proposition 4.4. Likewise we see that (ii) implies condition (iii) of Proposition 4.4. Consequently each of the conditions (i)-(iii) implies that $\mathfrak{e}$ is first stage-pre-local.

Proof. Assume first the validity of (i). Then $\mathfrak{e}$ is first stage pre-local (as has been noted just now) and (ii.a) and (ii.c) follow from Proposition 4.4. Consider next any subgroup U of G and let $V = U \cdot \mathfrak{h}_e G$. Then we deduce

$$\text{(1)} \qquad \mathfrak{h}_e G = V \cap \mathfrak{h}_e G \subseteq \mathfrak{h}_e V$$

and

$$U \cap \mathfrak{h}_e V \subseteq \mathfrak{h}_e U$$

from (ii.a). Application of Dedekind's modular law shows that

$$\text{(2)} \qquad \mathfrak{h}_e V = \mathfrak{h}_e V \cap V = \mathfrak{h}_e V \cap U \cdot \mathfrak{h}_e G = \mathfrak{h}_e G[U \cap \mathfrak{h}_e V] \subseteq \mathfrak{h}_e G \cdot \mathfrak{h}_e U.$$

Consider now an $\mathfrak{e}$-subgroup E of V. Apply (i.b) with $S = \mathfrak{h}_e G$ to see that $E \cdot \mathfrak{h}_e G = F$ is an $\mathfrak{e}$-subgroup of V. Because of

$$\mathfrak{h}_e G \subseteq F \subseteq V = U \cdot \mathfrak{h}_e G$$

we may apply Dedekind's modular law to show

$$F = [U \cap F]\mathfrak{h}_e G.$$

From (i.a) we deduce that $U \cap F$ is an $\mathfrak{e}$-subgroup of U. Apply (i.b) to show that $(U \cap F) \cdot \mathfrak{h}_e U$ is an $\mathfrak{e}$-subgroup of U; and a second application of (i.b) shows that

$$\{E, \mathfrak{h}_e U\} \subseteq \{F, \mathfrak{h}_e U\} = [(U \cap F)\mathfrak{h}_e U]\mathfrak{h}_e G$$

is an $\mathfrak{e}$-subgroup of G (and V). Since $\{E, \mathfrak{h}_e U\}$ is an $\mathfrak{e}$-subgroup of V for every $\mathfrak{e}$-subgroup E of V, another application of (i.b) shows

$$\mathfrak{h}_e U \subseteq \mathfrak{h}_e V.$$

Combine this with (1) and (2) to show that

$$\mathfrak{h}_e U \cdot \mathfrak{h}_e G \subseteq \mathfrak{h}_e V \subseteq \mathfrak{h}_e U \cdot \mathfrak{h}_e G,$$

proving the validity of (ii.b).

Assume next the validity of (ii). Then $\mathfrak{e}$ is first stage pre-local (as noted before) and $\mathfrak{e}$ is subgroup-inherited by Proposition 4.4. If next E is an $\mathfrak{e}$-subgroup of G, then $E = \mathfrak{h}_e E$ (as before); and we deduce from (ii.b) that

$$\mathfrak{h}_e[E \cdot \mathfrak{h}_e G] = \mathfrak{h}_e E \cdot \mathfrak{h}_e G = E \cdot \mathfrak{h}_e G.$$

Hence $E \cdot \mathfrak{h}_e G$ is by (ii.c) an $\mathfrak{e}$-group; and thus we have deduced (iii) from (ii).

Assume next the validity of (iii). If S is a subset of $\mathfrak{h}_e G$ and E is an $\mathfrak{e}$-subgroup of G, then

$$\{S, E\} \subseteq E \cdot \mathfrak{h}_e G.$$

The latter group is an $\mathfrak{e}$-group by (iii.b) and thus $\{S, E\}$ is an $\mathfrak{e}$-group by (iii.a). If conversely S is a subset of G such that $\{S, E\}$ is an $\mathfrak{e}$-subgroup of G for every n-subgroup E of G, then $\{s, E\}$ is for every s in S an $\mathfrak{e}$-group (by (iii.a)) so that every s in S belongs to $\mathfrak{h}_e G$. Hence (i.b) is a consequence of (iii), proving that (iii) implies (i).

REMARK 4.9. Let $\mathfrak{D}$ be any class of groups which contains also non-countable groups and denote by $\mathfrak{e}$ the property of being countable. Then $\mathfrak{h}_e G = G$ for every group G. Thus (ii.a) and (ii.b) are satisfied. But in general $\mathfrak{h}_e G$ is not going to be countable so that (ii.c) is not a consequence of (ii.a) and (ii.b).

COROLLARY 4.10. *Suppose that the property $\mathfrak{e}$ is second stage pre-local.*

(1) *The normal subgroup N of G is a part of $\mathfrak{h}_e G$ if, and only if, EN is an $\mathfrak{e}$-group for every $\mathfrak{e}$-subgroup E of G.*

(2) *$\mathfrak{h}_e G$ is the set of all the elements g in G such that $E\{g^G\}$ is an $\mathfrak{e}$-group for every $\mathfrak{e}$-subgroup E of G.*

(3) *$\mathfrak{h}_e G$ is the product of all the normal subgroups X of G such that EX is an $\mathfrak{e}$-group for every $\mathfrak{e}$-subgroup E of G.*

(1) is an obvious special case of condition (i.b) of Proposition 4.8. If we note that the element g belongs to the characteristic subgroup $\mathfrak{h}_e G$ if, and only if, $\{g^G\} \subseteq \mathfrak{h}_e G$, then we see that (2) is a consequence of (1). Property (3) finally is easily derived from (1) and (2).

It is worth noting that (1) is a special case of Proposition 4.8(i.b) and that Proposition 4.8(iii.b) is a special case of (1).

The function $\mathfrak{f}$ on $\mathfrak{D}$ will be termed *second stage pre-local*, if

(a) $$U \cap \mathfrak{f}G \subseteq \mathfrak{f}U \quad \text{for} \quad U \subseteq G$$

and

(b) $$\mathfrak{f}[U \cdot \mathfrak{f}G] = \mathfrak{f}U \cdot \mathfrak{f}G \quad \text{for} \quad U \subseteq G.$$

It is clear that such a function is likewise first stage pre-local.

COROLLARY 4.11. *The property $\mathfrak{e}$ is second stage pre-local if, and only if, the function $\mathfrak{h}_e$ is second stage pre-local and $\mathfrak{e} = (\mathfrak{h}_e)'$.*

Proof. If $\mathfrak{e}$ is second stage pre-local, then we deduce from Proposition 4.8, (ii) that $\mathfrak{h}_e$ is likewise second stage pre-local. Furthermore G is an $\mathfrak{e}$-group if, and only if, $G = \mathfrak{h}_e G$; and this is equivalent with $\mathfrak{e} = (\mathfrak{h}_e)'$. If conversely $\mathfrak{h}_e$ is second stage pre-local, then conditions (ii.a) and (ii.b) of Proposition 4.8 are satisfied. If furthermore $\mathfrak{e} = (\mathfrak{h}_{\mathfrak{e}})'$, then application of Proposition 4.4 shows that $\mathfrak{e}$ is first stage pre-local and that therefore $\mathfrak{h}_{\mathfrak{e}}G$ is always an $\mathfrak{e}$-group. Hence Proposition 4.8(ii.c) holds true and $\mathfrak{e}$ is second stage pre-local.

LEMMA 4.12. (1) *If the function $\mathfrak{f}$ on $\mathfrak{D}$ is second stage pre-local, then $\mathfrak{f} \subseteq \mathfrak{h}_{\mathfrak{f}'}$.*

(2) *The function $\mathfrak{f}$ on $\mathfrak{D}$ is second stage pre-local if, and only if,*

(a) $U \cap \mathfrak{f}G \subseteq \mathfrak{f}U$ *for* $U \subseteq G$

and

(b) $\mathfrak{f}U \subseteq \mathfrak{f}[U \cdot \mathfrak{f}G]$ *for* $U \subseteq G$.

Proof. Assume that $\mathfrak{f}$ is second stage pre-local. If g belongs to $\mathfrak{f}G$ and E is an $\mathfrak{f}'$-subgroup of G, then $E = \mathfrak{f}E$ and hence

$$\{g, E\} = \{g, \mathfrak{f}E\} \subseteq \mathfrak{f}E \cdot \mathfrak{f}G = \mathfrak{f}[E \cdot \mathfrak{f}G]$$

so that $\{g, E\}$ is, by Lemma 4.5, part of an $\mathfrak{f}'$-subgroup of G. Hence g belongs to $\mathfrak{h}_{\mathfrak{f}'}G$, proving (1).

It is clear that second stage pre-local functions meet requirements (2.a) and (2.b). Assume conversely that the function $\mathfrak{f}$ on $\mathfrak{D}$ meets these requirements (a) and (b). If $U \subseteq G$, then by (a)

$$\mathfrak{f}G = \mathfrak{f}G \cap [U \cdot \mathfrak{f}G] \subseteq \mathfrak{f}[U \cdot \mathfrak{f}G] \tag{$*$}$$

so that by (b)

$$\mathfrak{f}U \cdot \mathfrak{f}G \subseteq \mathfrak{f}[U \cdot \mathfrak{f}G]. \tag{$**$}$$

Applying (a) again we find that

$$U \cap \mathfrak{f}[U \cdot \mathfrak{f}G] \subseteq \mathfrak{f}U.$$

Application of ($*$) and Dedekind's modular law shows that

$$\mathfrak{f}[U \cdot \mathfrak{f}G] = \mathfrak{f}G[U \cap \mathfrak{f}(U \cdot \mathfrak{f}G)] \subseteq \mathfrak{f}G \cdot \mathfrak{f}U.$$

Combination with ($**$) gives

$$\mathfrak{f}[U \cdot \mathfrak{f}G] = \mathfrak{f}U \cdot \mathfrak{f}G,$$

proving the validity of (2).

PROPOSITION 4.13. *The following properties of the function $\mathfrak{f}$ on $\mathfrak{D}$ are equivalent:*

(i) *$\mathfrak{f}$ is second stage pre-local and $\mathfrak{h}_{\mathfrak{f}'} \subseteq \mathfrak{f}$.*

(ii) *The property $\mathfrak{f}'$ is second stage pre-local and $\mathfrak{h}_{\mathfrak{f}'} = \mathfrak{f}$.*

(iii) (a) *The element g in G belongs to $\mathfrak{f}G$ if, and only if, g belongs to $\mathfrak{f}\{g, E\}$ for every $\mathfrak{f}'$-subgroup E of G.*
(b) $\mathfrak{f}U \subseteq \mathfrak{f}[U \cdot \mathfrak{f}G]$ *for* $U \subseteq G$.

Proof. If (i) is true, then we derive from Lemma 4.5 and Lemma 4.12(1) that

$$\mathfrak{f}G \text{ is always an } \mathfrak{f}'\text{-group and } \mathfrak{f} \subseteq \mathfrak{h}_{\mathfrak{f}'}.$$

It follows that $\mathfrak{f} = \mathfrak{h}_{\mathfrak{f}'}$. Hence $\mathfrak{e} = \mathfrak{f}'$ meets requirement (ii) of Proposition 4.8 and is consequently second stage pre-local. Thus (ii) is a consequence of (i).

Assume next the validity of (ii). According to Proposition 4.8, (i.b) the element g in G belongs to $\mathfrak{h}_{\mathfrak{f}'}G$ if, and only if, $\{g, E\}$ is an $\mathfrak{f}'$-group for every $\mathfrak{f}'$-subgroup E of G.

But if E is an $\mathfrak{f}'$-group, then

$$\mathfrak{h}_{\mathfrak{f}'}[E\cdot\mathfrak{h}_{\mathfrak{f}'}\{g, E\}] = E\cdot\mathfrak{h}_{\mathfrak{f}'}\{g, E\}$$

according to Proposition 4.8, (ii.b). If $\{g, E\}$ is an $\mathfrak{f}'$-group, then g belongs to $\{g, E\} = \mathfrak{f}\{g, E\}$. If conversely g belongs to $\mathfrak{f}\{g, E\}$, then

$$\{g, E\} = E\cdot\mathfrak{f}\{g, E\} = E\cdot\mathfrak{h}_{\mathfrak{f}'}\{g, E\} = \mathfrak{h}_{\mathfrak{f}'}[E\cdot\mathfrak{h}_{\mathfrak{f}'}\{g, E\}]$$

is an $\mathfrak{f}'$-group according to Proposition 4.8(ii.c); and thus we have shown that g belongs to $\mathfrak{f}G = \mathfrak{h}_{\mathfrak{f}'}G$ if, and only if, g belongs to $\mathfrak{f}\{g, E\}$ for every $\mathfrak{f}'$-subgroup E of G. Hence (iii.a) is a consequence of (ii); and it is a consequence of Corollary 4.11 that (ii) implies (iii.b).

Assume the validity of (iii). If $U \subseteq G$ and g belongs to $U \cap \mathfrak{f}G$, then g belongs to $\mathfrak{f}\{g, E\}$ for every $\mathfrak{f}'$-subgroup E of U so that g belongs to $\mathfrak{f}U$. Hence

$$U \cap \mathfrak{f}G \subseteq \mathfrak{f}U \quad \text{for} \quad U \subseteq G.$$

From (iii.b) and Lemma 4.12(2) we deduce now that $\mathfrak{f}$ is second stage pre-local.

Consider next an element g in $\mathfrak{h}_{\mathfrak{f}'}G$. If E is an $\mathfrak{f}'$-subgroup of G, then $\{g, E\}$ is an $\mathfrak{f}'$-subgroup of G too (Corollary 4.10(2)). Thus g belongs to $\{g, E\} = \mathfrak{f}\{g, E\}$; and it follows from (iii.a) that g belongs to $\mathfrak{f}G$. Thus we have shown that

$$\mathfrak{h}_{\mathfrak{f}'}G \subseteq \mathfrak{f}G;$$

and we have derived (i) from (iii).

Construction 4.14. If $\mathfrak{e}$ is a subgroup-inherited property on $\mathfrak{D}$, then we define a function $\mathfrak{e}^+$ on $\mathfrak{D}$ by the rule:

$$\mathfrak{e}^+G = \begin{cases} G & \text{if } G \text{ is an } \mathfrak{e}\text{-group}, \\ 1 & \text{if } G \text{ is not an } \mathfrak{e}\text{-group}. \end{cases}$$

Consider a subgroup U of G.

Case 1. G is not an $\mathfrak{e}$-group.

Then $\mathfrak{e}^+G = 1$ and we have

$$U \cap \mathfrak{e}^+G = 1 \subseteq \mathfrak{e}^+U,$$
$$\mathfrak{e}^+[U\cdot\mathfrak{e}^+G] = \mathfrak{e}^+U = \mathfrak{e}^+U\cdot\mathfrak{e}^+G.$$

Case 2. G is an $\mathfrak{e}$-group.
Then U is likewise an $\mathfrak{e}$-group so that $G = \mathfrak{e}^+G$, $U = \mathfrak{e}^+U$ and

$$U \cap \mathfrak{e}^+G = U \cap G = U = \mathfrak{e}^+U,$$
$$\mathfrak{e}^+[U\cdot\mathfrak{e}^+G] = \mathfrak{e}^+[U\cdot G] = \mathfrak{e}^+G = G = \mathfrak{e}^+U\cdot\mathfrak{e}^+G.$$

Thus we have shown that $\mathfrak{e}^+$ *is second stage pre-local.*
It is obvious that

$$\mathfrak{e} = (\mathfrak{e}^+)'.$$

There exist, however, many properties $\mathfrak{e}$ (see, for instance, Remark 4.6) such that

$$1 \subset \mathfrak{h}_{\mathfrak{e}}G \subset G \qquad \text{for some } G,$$

implying

$$\mathfrak{e}^+ \neq \mathfrak{h}_{(\mathfrak{e}^+)'}$$

and this shows the indispensability of the second half of condition (i) of Proposition 4.13.

Remark 4.15. Suppose that $\mathfrak{D}$ is the class of all finite groups and denote by $\mathfrak{f}G$ for every finite group G its Fitting subgroup. This is at the same time the product of all nilpotent normal subgroups of G and the most comprehensive nilpotent normal subgroup of G. Then

(1) $\mathfrak{f}'$ is just nilpotency.

(2) $U \cap \mathfrak{f}G \subseteq \mathfrak{f}U$ for $U \subseteq G$;

since $U \cap \mathfrak{f}G$ is a nilpotent normal subgroup of U.

If g belongs to $\mathfrak{f}G$ and E is a subgroup of G, then we deduce from (2) that g belongs to $\{g, E\} \cap \mathfrak{f}G \subseteq \mathfrak{f}\{g, E\}$. Hence we have:

(3) If g belongs to $\mathfrak{f}G$ and $E \subseteq G$, then g belongs to $\mathfrak{f}\{g, E\}$.

Assume next that g belongs to $\mathfrak{f}\{g, x\}$ for every x in G. Then g is a so-called Engel-element of $\{g, x\}$ for every x and hence g is an Engel-element of G, proving that g belongs to the Fitting subgroup $\mathfrak{f}G$ of G; cf. Baer [4], p. 257, Satz L′. Thus we have shown:

(4) If g belongs to $\mathfrak{f}\{g, x\}$ for every x in G, then g belongs to $\mathfrak{f}G$.

Properties (3) and (4) together imply:

(5) Condition (iii.a) of Proposition 4.13 is satisfied by $\mathfrak{f}$.

If p and q are primes such that $p \equiv 1 \bmod q$ and if G is the essentially uniquely determined non-abelian group of order pq, then $\mathfrak{f}G$ is its subgroup of order p. If Q is any subgroup of order q, then $G = Q \cdot \mathfrak{f}G$ and

$$\mathfrak{f}[Q \cdot \mathfrak{f}G] = \mathfrak{f}G \subset Q \cdot \mathfrak{f}G = \mathfrak{f}Q \cdot \mathfrak{f}G = G.$$

Thus condition (iii.b) of Proposition 4.13 is not satisfied by $\mathfrak{f}$, showing its indispensability. Combining this last remark with (2) we see that

(6) $\mathfrak{f}$ is first stage, but not second stage pre-local.

If g belongs to $\mathfrak{h}_{\mathfrak{f}'}G$, then $\{g, x\}$ is nilpotent for every x in G; and it is well known that this property characterizes the elements in the hypercenter $\mathfrak{h}G$ of G; see Baer [3], p. 42, Theorem 3. The converse holds too; see Baer [3], p. 42, Theorem 3. Consequently

(7) $$\mathfrak{h}_{\mathfrak{f}'} = \mathfrak{h} = \text{Hypercenter}.$$

Since the hypercenter is nilpotent, but is, in general, not equal to the Fitting subgroup, we may say:

(8) $$\mathfrak{h}_{\mathfrak{f}'} \subset \mathfrak{f}.$$

This shows that in Lemma 4.12(1) it does not suffice to assume that $\mathfrak{f}$ is first stage pre-local.

PRE-LOCALITY: THE THIRD STAGE. The property $\mathfrak{e}$ will be termed *third stage pre-local*, if it meets the following two requirements:

(a) *$\mathfrak{e}$ is subgroup-inherited.*

(b) *Every $\mathfrak{e}$-subgroup of G is part of a maximal $\mathfrak{e}$-subgroup of G.*

If, for instance, all groups in $\mathfrak{D}$ are noetherian, then a property is third stage pre-local if, and only if, it is subgroup-inherited.

PROPOSITION 4.16. *If $\mathfrak{e}$ is third stage pre-local, then*

(a) *$\mathfrak{h}_{\mathfrak{e}}G$ is for every group G in $\mathfrak{D}$ the intersection of all maximal $\mathfrak{e}$-subgroups of G*

and

(b) *$\mathfrak{e}$ is second stage pre-local.*

Proof. Let G^* be for every G in $\mathfrak{D}$ the intersection of all maximal $\mathfrak{e}$-subgroups of G. If S is a subset of G^* and E is an $\mathfrak{e}$-subgroup of G, then there exists a maximal $\mathfrak{e}$-subgroup M of G which contains E. From $G^* \subseteq M$ we deduce $\{S, E\} \subseteq M$ and $\{S, E\}$ is an $\mathfrak{e}$-group, since M is an $\mathfrak{e}$-group and $\mathfrak{e}$ is subgroup-inherited. Thus we have shown:

(1) If S is a subset of G^* and E is an $\mathfrak{e}$-subgroup of G, then $\{S, E\}$ is an $\mathfrak{e}$-group.

If we apply (1) in particular upon the one-element-subsets of G^*, then we deduce from the definition of the $\mathfrak{e}$-hypercenter that

(2) $$G^* \subseteq \mathfrak{h}_{\mathfrak{e}}G.$$

If g is an element in $\mathfrak{h}_{\mathfrak{e}}G$ and M is a maximal $\mathfrak{e}$-subgroup of G, then $\{g, M\}$ is an $\mathfrak{e}$-group by $\mathfrak{e}$-hypercentrality; and we deduce $M = \{g, M\}$ from the maximality of M. Thus g belongs to M. Consequently $\mathfrak{e}$-hypercentral elements belong to every maximal $\mathfrak{e}$-subgroup of G; and we have shown:

$$\mathfrak{h}_{\mathfrak{e}}G \subseteq G^*. \tag{3}$$

From (2) and (3) we deduce $G^* = \mathfrak{h}_{\mathfrak{e}}G$; and this is just our property (a).

If we combine (a) and (1), then we obtain property (i.b) of Proposition 4.8; and since $\mathfrak{e}$ is subgroup-inherited, we have shown that $\mathfrak{e}$ is second stage pre-local.

THE FOURTH STAGE: LOCALITY. This will be described by the following

PROPOSITION 4.17. *The following properties of the property $\mathfrak{e}$ are equivalent:*

(i) *G is an $\mathfrak{e}$-group if, and only if, every finitely generated subgroup of G is an $\mathfrak{e}$-group.*

(ii)
- (a) *$\mathfrak{e}$ is subgroup-inherited.*
- (b) *G is an $\mathfrak{e}$-group, if there exists a set of $\mathfrak{e}$-subgroups of G containing $\{X, Y\}$ with X and Y and covering G.*

(iii)
- (a) *$\mathfrak{e}$ is third stage pre-local.*
- (b) *If every finite subset of G is part of an $\mathfrak{e}$-subgroup of G, then there exists at most one maximal $\mathfrak{e}$-subgroup of G.*

(iv)
- (a) *$U \cap \mathfrak{h}_{\mathfrak{e}}G \subseteq \mathfrak{h}_{\mathfrak{e}}U$ for $U \subseteq G$.*
- (b) *$\mathfrak{h}_{\mathfrak{e}}G$ is always an $\mathfrak{e}$-group.*
- (c) *If the element g in G belongs to $\mathfrak{h}_{\mathfrak{e}}\{g, E\}$ for every finitely generated $\mathfrak{e}$-subgroup E of G, then g belongs to $\mathfrak{h}_{\mathfrak{e}}G$.*

(v)
- (a) *The element g in G belongs to $\mathfrak{h}_{\mathfrak{e}}G$ if, and only if, g belongs to $\mathfrak{h}_{\mathfrak{e}}\{g, F\}$ for every finitely generated subgroup F of G.*
- (b) *$\mathfrak{e} = (\mathfrak{h}_{\mathfrak{e}})'$.*

A property $\mathfrak{e}$, meeting the equivalent requirements (i)-(v), will be termed a *local* property.

Proof. Assume first the validity of (i). If U is a subgroup of the $\mathfrak{e}$-group G, then every finitely generated subgroup of U is — as a finitely generated subgroup of the $\mathfrak{e}$-group G — an $\mathfrak{e}$-group; and so U is an $\mathfrak{e}$-group: $\mathfrak{e}$ is subgroup-inherited. Assume furthermore the existence of a set θ of $\mathfrak{e}$-subgroups of G which covers G and which contains $\{X, Y\}$ with X and Y. Then every finite subset of G is part of a subgroup in θ, as may be seen by complete induction. Since $\mathfrak{e}$ has been shown to be subgroup-inherited, finitely generated subgroups of G are $\mathfrak{e}$-groups; and so G itself is an $\mathfrak{e}$-group. Hence (ii) is a consequence of (i).

Assume next the validity of (ii). Consider an $\mathfrak{e}$-subgroup E of G and denote by $\mathfrak{E}$ the set of all $\mathfrak{e}$-subgroups of G containing E. Then E itself belongs to $\mathfrak{E}$. Consider a tower θ of subgroups of G which is part of $\mathfrak{E}$. If X belongs to θ, then X is an $\mathfrak{e}$-group containing E; and if X and Y belong to θ, then $X \subseteq Y$ or $Y \subseteq X$. The join J of the subgroups in θ is consequently a subgroup of G; and θ covers J and contains with X and Y also $\{X, Y\}$ $(= X$ or $Y)$. Application of (ii.b) shows that J is an $\mathfrak{e}$-group. But $E \subseteq J \subseteq G$ so that J belongs to $\mathfrak{E}$. Consequently we may apply the Maximum Principle of Set Theory on $\mathfrak{E}$. Hence there exists a maximal element in $\mathfrak{E}$ and this is a maximal $\mathfrak{e}$-subgroup of G which contains E. Thus we have shown that $\mathfrak{e}$ is third stage pre-local. Assume next that every finite subset of G is contained in an $\mathfrak{e}$-subgroup of G. Then every finitely generated subgroup of G is an $\mathfrak{e}$-group (subgroup-inheritance); and an immediate application of (ii.b) shows that G is an $\mathfrak{e}$-group and as such G is its one and only one maximal $\mathfrak{e}$-subgroup. Hence (iii.b) is true too; and we have derived (iii) from (ii).

Assume next the validity of (iii). If G is an $\mathfrak{e}$-group, then so is every subgroup of G. In particular every finitely generated subgroup of G is an $\mathfrak{e}$-group. Assume conversely that every finitely generated subgroup of G is an $\mathfrak{e}$-group. Then we deduce from (iii.a) and (iii.b) together that there exists one and only one maximal $\mathfrak{e}$-subgroup M of G. If g is an element of G, then $\{g\}$ is an $\mathfrak{e}$-group and as such $\{g\}$ is part of a maximal $\mathfrak{e}$-subgroup V of G (by (iii.a)). Since M is the only maximal $\mathfrak{e}$-subgroup of G, we have $M = V$ so that g belongs to M and $G = M$ is an $\mathfrak{e}$-group. We have derived (i) from (iii) and shown the equivalence of (i)-(iii).

Assume next the validity of the equivalent properties (i)-(iii). Then $\mathfrak{e}$ is, by (iii.a), third stage pre-local; and it follows from Proposition 4.16 that $\mathfrak{e}$ *is second stage pre-local.*

Application of Proposition 4.8 (i) and (ii) shows now the validity of (iv.a, b) and of the subgroup-inheritance of $\mathfrak{e}$.

Consider an element g which belongs to $\mathfrak{h}_{\mathfrak{e}}\{g, E\}$ for every finitely generated $\mathfrak{e}$-subgroup E of G. Then $\{g, E\} = E \cdot \mathfrak{h}_{\mathfrak{e}}\{g, E\}$ for every finitely generated $\mathfrak{e}$-subgroup E of G. But $\mathfrak{e}$ is second stage pre-local. Apply Proposition 4.8 (iii.b) to see that

(*) $\{g, E\}$ is an $\mathfrak{e}$-group for every finitely generated $\mathfrak{e}$-subgroup E of G.

Consider an $\mathfrak{e}$-subgroup S of G. If F is a finitely generated subgroup of $\{g, S\}$, then there exists a finitely generated subgroup E of S with $F \subseteq \{g, E\}$. Since $\mathfrak{e}$ is subgroup-inherited, E is a finitely generated $\mathfrak{e}$-subgroup of G. From (*) we conclude that $\{g, E\}$ is an $\mathfrak{e}$-group. Hence F is an $\mathfrak{e}$-group (subgroup-inheritance). Now we may apply (i) to see that

$\{g, S\}$ is an $\mathfrak{e}$-group. Hence $\{g, S\}$ is an $\mathfrak{e}$-group for every $\mathfrak{e}$-subgroup S of G.

Application of Proposition 4.8 (i.b) shows that g belongs to $\mathfrak{h}_\mathfrak{e}G$. Hence we have derived (iv.c) too, showing that (iv) is a consequence of the equivalent properties (i)-(iii).

Assume next the validity of (iv). If firstly g belongs to $\mathfrak{h}_\mathfrak{e}G$, then g belongs, by (iv.a), to

$$\{g, S\} \cap \mathfrak{h}_\mathfrak{e}G \subseteq \mathfrak{h}_\mathfrak{e}\{g, S\} \quad \text{for} \quad S \subseteq G.$$

If secondly g belongs to $\mathfrak{h}_\mathfrak{e}\{g, F\}$ for every finitely generated subgroup F of G, then g belongs in particular to $\mathfrak{h}_\mathfrak{e}\{g, E\}$ for every finitely generated $\mathfrak{e}$-subgroup E of G; and g belongs to $\mathfrak{h}_\mathfrak{e}G$ by (iv.c). Thus (v.a) is true.

If G is an $\mathfrak{e}$-group, then $G = \mathfrak{h}_\mathfrak{e}G$ (by definition of $\mathfrak{h}_\mathfrak{e}$). If conversely $G = \mathfrak{h}_\mathfrak{e}G$, then G is an $\mathfrak{e}$-group by (iv.b). Hence G is an $\mathfrak{e}$-group if, and only if, $G = \mathfrak{h}_\mathfrak{e}G$; and this is equivalent to $\mathfrak{e} = (\mathfrak{h}_\mathfrak{e})'$, proving the validity of (v.b).

Assume finally the validity of (v). If S is a subgroup of the $\mathfrak{e}$-group G, then $G = \mathfrak{h}_\mathfrak{e}G$ by (v.b) (or the definition of $\mathfrak{h}_\mathfrak{e}$). If s is an element in S and F is a finitely generated subgroup of $G = \mathfrak{h}_\mathfrak{e}G$, then s belongs to $\mathfrak{h}_\mathfrak{e}\{s, F\}$ by (v.a); and a second application of (v.a) shows that $S = \mathfrak{h}_\mathfrak{e}S$. Apply (v.b) to see that S is an $\mathfrak{e}$-group. Hence $\mathfrak{e}$ *is subgroup-inherited.*

Consider next a group G all of whose finitely generated subgroups are $\mathfrak{e}$-groups. If g is an element of G, and F is a finitely generated subgroup of G, then $\{g, F\}$ is an $\mathfrak{e}$-group as a finitely generated subgroup of G. Consequently g belongs to $\{g, F\} = \mathfrak{h}_\mathfrak{e}\{g, F\}$ by (v.b) (or the definition of $\mathfrak{h}_\mathfrak{e}$). Apply (v.a) to see that g belongs to $\mathfrak{h}_\mathfrak{e}G$. Hence $G = \mathfrak{h}_\mathfrak{e}G$ is an $\mathfrak{e}$-group by (v.b); and thus we have derived (i) from (v) and completed the proof of the equivalence of (i)-(v).

REMARK 4.18. Let $\mathfrak{D}$ be the class of all abelian groups and let $\mathfrak{e}$ denote the property of being a finitely generated abelian group. Then

$$\mathfrak{h}_\mathfrak{e}A = A \quad \text{for every } A \text{ in } \mathfrak{D}.$$

Clearly $\mathfrak{e}$ meets requirements (iv.a, c) and (v.a) of Proposition 4.17. But there exist abelian groups which are not finitely generated. Hence $\mathfrak{e}$ is not a local property, showing the indispensability of conditions (iv.b) and (v.b) of Proposition 4.17.

The function $\mathfrak{f}$ on $\mathfrak{D}$ is termed a *local function,* if it meets the following requirement:

The element g in G belongs to $\mathfrak{f}G$ if, and only if, g belongs to $\mathfrak{f}\{g, F\}$ for every finitely generated subgroup F of G.

Then it is nothing but a restatement of the equivalence of conditions (i) and (v) of Proposition 4.17, if we say:

The property $\mathfrak{e}$ *on* $\mathfrak{D}$ *is local if, and only if,* $\mathfrak{h}_{\mathfrak{e}}$ *is local and* $\mathfrak{e} = (\mathfrak{h}_{\mathfrak{e}})'$.

This may serve as a justification of our definition of local functions.

LEMMA 4.19. *Suppose that* $\mathfrak{f}$ *is a local function.*

(a) $\mathfrak{f}$ *is first stage pre-local.*

(b) $\mathfrak{f}$ *is second stage pre-local if, and only if,* $\mathfrak{f}U \subseteq \mathfrak{f}[U\cdot\mathfrak{f}G]$ *for* $U \subseteq G$.

Proof. If $U \subseteq G$ and if g belongs to $U \cap \mathfrak{f}G$, then g belongs to $\mathfrak{f}\{g, F\}$ for every finitely generated subgroup F of (G and) U. Hence g belongs to $\mathfrak{f}U$ so that

$$U \cap \mathfrak{f}G \subseteq \mathfrak{f}U \quad \text{for} \quad U \subseteq G,$$

proving (a); (b) is a fairly immediate consequence of (a) and Lemma 4.12 (2).

PROPOSITION 4.20. *The following properties of the function* $\mathfrak{f}$ *on* $\mathfrak{D}$ *are equivalent:*

(i) $\mathfrak{f}'$ *is local and* $\mathfrak{f} = \mathfrak{h}_{\mathfrak{f}'}$.

(ii)
- (a) $\mathfrak{f}$ *is local.*
- (b) $\mathfrak{f}U \subseteq \mathfrak{f}[U\cdot\mathfrak{f}G]$ *for* $U \subseteq G$.
- (c) $\mathfrak{h}_{\mathfrak{f}'} \subseteq \mathfrak{f}$.

(iii)
- (a) $\mathfrak{f}$ *is second stage pre-local.*
- (b) *If the element* g *in* G *belongs to* $\mathfrak{f}\{g, \mathfrak{f}S\}$ *for every subgroup* S *of* G *with finitely generated* $\mathfrak{f}S$, *then* g *belongs to* $\mathfrak{f}G$.

Proof. If (i) is true, and if we let $\mathfrak{e} = \mathfrak{f}'$, then $\mathfrak{e}$ is local and $\mathfrak{f} = \mathfrak{h}_{\mathfrak{e}}$ so that

$$\mathfrak{e} = \mathfrak{f}' = (\mathfrak{h}_{\mathfrak{e}})'.$$

Application of Proposition 4.17 (v.a) shows that $\mathfrak{h}_{\mathfrak{e}} = \mathfrak{f}$ is local and that $\mathfrak{e}$ is third stage — and by Proposition 4.16 (b) — second stage pre-local. Application of Proposition 4.8 (ii.b) shows then that

$$\mathfrak{f}U = \mathfrak{h}_{\mathfrak{e}}U \subseteq \mathfrak{h}_{\mathfrak{e}}[U\cdot\mathfrak{h}_{\mathfrak{e}}G] = \mathfrak{f}[U\cdot\mathfrak{f}G]$$

for $U \subseteq G$. Hence (ii) is a consequence of (i).

Assume next the validity of (ii). Then Lemma 4.19 (b) shows that $\mathfrak{f}$ is second stage pre-local; and we deduce $\mathfrak{f} \subseteq \mathfrak{h}_{\mathfrak{f}'}$ from Lemma 4.12 (1). Thus $\mathfrak{f} = \mathfrak{h}_{\mathfrak{f}'}$; and letting $\mathfrak{e} = \mathfrak{f}'$ we have: $\mathfrak{h}_{\mathfrak{e}} = \mathfrak{f}$ is local and $(\mathfrak{h}_{\mathfrak{e}})' = \mathfrak{f}' = \mathfrak{e}$. Hence $\mathfrak{e} = \mathfrak{f}'$ is local (as pointed out when introducing local functions) and we have deduced (i) from (ii).

Assume the validity of the equivalent conditions (i) and (ii). Then we deduce from (ii.a, b) and Lemma 4.19 (b) that $\mathfrak{f}$ is second stage pre-local. If we let $\mathfrak{f}' = \mathfrak{e}$, then $\mathfrak{e}$ is a local property and $\mathfrak{f} = \mathfrak{h}_{\mathfrak{e}}$ by (i). Consider now an element g in G with the property:

(*) g belongs to $\mathfrak{f}\{g, \mathfrak{f}S\}$ for every subgroup S of G with finitely generated $\mathfrak{f}S$.

If E is a finitely generated $\mathfrak{e}$-subgroup of G, then $E = \mathfrak{f}E = \mathfrak{h}_{\mathfrak{e}}E$ [by $\mathfrak{f}' = \mathfrak{e}$ and $\mathfrak{f} = \mathfrak{h}_{\mathfrak{e}}$] and application of (*) shows that g belongs to $\mathfrak{f}\{g, \mathfrak{f}E\} = \mathfrak{h}_{\mathfrak{e}}\{g, E\}$. Since $\mathfrak{e}$ is local, we may apply Proposition 4.17 (iv.c) to show that g belongs to $\mathfrak{h}_{\mathfrak{e}}G = \mathfrak{f}G$. Hence (iii) is a consequence of the equivalent conditions (i) and (ii).

Assume finally the validity of (iii). If g belongs to $\mathfrak{f}G$, and if F is a finitely generated subgroup of G, then g belongs to

$$\{g, F\} \cap \mathfrak{f}G \subseteq \mathfrak{f}\{g, F\}$$

by (iii.a). Assume conversely that g belongs to $\mathfrak{f}\{g, F\}$ for every finitely generated subgroup F of G. Then g belongs to $\mathfrak{f}\{g, \mathfrak{f}S\}$ for every $S \subseteq G$ with finitely generated $\mathfrak{f}S$; and application of (iii.b) shows that g belongs to $\mathfrak{f}G$. Hence $\mathfrak{f}$ is local, proving (ii.a).

The validity of (ii.b) is an immediate consequence of (iii.a). Suppose now that the element g belongs to $\mathfrak{h}_{\mathfrak{f}'}G$. Then $\{g, E\}$ is by Lemma 4.2 (a) an $\mathfrak{f}'$-group for every $\mathfrak{f}'$-subgroup E of G; and $\mathfrak{f}S$ is always an $\mathfrak{f}'$-group (Lemma 4.5). Hence g belongs to $\{g, \mathfrak{f}S\} = \mathfrak{f}\{g, \mathfrak{f}S\}$. Application of (iii.b) shows that g belongs to $\mathfrak{f}G$. Hence $\mathfrak{h}_{\mathfrak{f}'}G \subseteq \mathfrak{f}G$; and we have derived (ii) from (iii).

REMARK 4.21. Assume that every group in $\mathfrak{D}$ is finitely generated — this implies, of course, that every group in $\mathfrak{D}$ is noetherian. If the function $\mathfrak{f}$ on $\mathfrak{D}$ is local, then we deduce from Lemma 4.19 (a) that

(1) $$U \cap \mathfrak{f}G \subseteq \mathfrak{f}U \quad \text{for} \quad U \subseteq G.$$

Assume conversely that $\mathfrak{f}$ meets requirement (1). If the element g in G belongs to $\mathfrak{f}G$, then we deduce from (1) that g belongs to

$$\{g, X\} \cap \mathfrak{f}G \subseteq \mathfrak{f}\{g, X\} \quad \text{for} \quad X \subseteq G.$$

If g belongs to $\mathfrak{f}\{g, F\}$ for every finitely generated subgroup F of G, then we recall that G itself is finitely generated so that g belongs to $\mathfrak{f}G$. Hence we have shown:

(1*) $\mathfrak{f}$ is local if, and only if, $\mathfrak{f}$ is first stage pre-local.

Combine this with Lemma 4.12 (2) to see that

(2) $\mathfrak{f}$ is second stage pre-local if, and only if, $\mathfrak{f}$ meets requirements (ii.a) and (ii.b) of Proposition 4.20.

Construction 4.14 shows now that (ii.c) is not a consequence of (ii.a) and (ii.b); and the indispensability of (ii.a) and (ii.b) is fairly obvious.

THE FIRST CLOSURE OPERATOR. If $\mathfrak{e}$ is a property (on $\mathfrak{D}$), then we define its (first) closure $\bar{\mathfrak{e}}$ by the rule:

The group G (in $\mathfrak{D}$) is an $\bar{\mathfrak{e}}$-group if, and only if, $G = \mathfrak{h}_{\mathfrak{e}} G$.

We note that $\mathfrak{e} = \bar{\mathfrak{e}}$ whenever $\mathfrak{e}$ is pre-local or local. Furthermore $\mathfrak{e}$-groups are always $\bar{\mathfrak{e}}$-groups, since $G = \mathfrak{h}_{\mathfrak{e}} G$ for $\mathfrak{e}$-groups G by the definition of $\mathfrak{h}_{\mathfrak{e}}$.

LEMMA 4.22. *If $\mathfrak{e}$ is subgroup-inherited, then*

(a) $\mathfrak{h}_{\bar{\mathfrak{e}}} \subseteq \mathfrak{h}_{\mathfrak{e}}$,

(b) *$\bar{\mathfrak{e}}$ is first stage pre-local.*

Proof. If S is a subgroup of the $\bar{\mathfrak{e}}$-group G, and if s is an element in S and E an $\mathfrak{e}$-subgroup of S, then $\{s, E\}$ is an $\mathfrak{e}$-group, since s belongs to $G = \mathfrak{h}_{\mathfrak{e}} G$ (Lemma 4.2, (a)). Hence s belongs to $\mathfrak{h}_{\mathfrak{e}} S$; and we have shown that $S = \mathfrak{h}_{\mathfrak{e}} S$ is an $\bar{\mathfrak{e}}$-group. Thus

(b′) *$\bar{\mathfrak{e}}$ is subgroup-inherited.*

Consider now an arbitrary group G and an element g in $\mathfrak{h}_{\bar{\mathfrak{e}}} G$. If E is an $\mathfrak{e}$-subgroup of G, then $E = \mathfrak{h}_{\mathfrak{e}} E$ is likewise an $\bar{\mathfrak{e}}$-subgroup of G so that $\{g, E\}$ is an $\bar{\mathfrak{e}}$-group. Hence $\{g, E\} = \mathfrak{h}_{\mathfrak{e}}\{g, E\}$. But then g is an $\mathfrak{e}$-hypercentral element of $\{g, E\}$; and as E is an $\mathfrak{e}$-group, we find that $\{g, E\}$ is an $\mathfrak{e}$-group (Lemma 4.2 (a)). Hence g belongs to $\mathfrak{h}_{\mathfrak{e}} G$; and we have verified $\mathfrak{h}_{\bar{\mathfrak{e}}} G \subseteq \mathfrak{h}_{\mathfrak{e}} G$ (i. e. (a)).

If G is an $\bar{\mathfrak{e}}$-group, then $G = \mathfrak{h}_{\bar{\mathfrak{e}}} G$ by the definition of $\mathfrak{h}_{\bar{\mathfrak{e}}}$. Assume conversely that $G = \mathfrak{h}_{\bar{\mathfrak{e}}} G$. Application of (a) shows that

$$G = \mathfrak{h}_{\bar{\mathfrak{e}}} G \subseteq \mathfrak{h}_{\mathfrak{e}} G = G.$$

Hence $G = \mathfrak{h}_{\mathfrak{e}} G$ is an $\bar{\mathfrak{e}}$-group; and we have shown.

(b″) *G is an $\bar{\mathfrak{e}}$-group if, and only if, $G = \mathfrak{h}_{\bar{\mathfrak{e}}} G$.*

Because of (b′), (b″) condition (i) of Proposition 4.4 is satisfied by $\bar{\mathfrak{e}}$. Hence $\bar{\mathfrak{e}}$ is first stage pre-local.

EXAMPLE 4.23. Let $\mathfrak{D}$ be the class of all groups. A group G has the property $\mathfrak{e}$ if, and only if,

(1) there exists a finite central chain connecting 1 and G (i. e. G is nilpotent of finite class), and

(2) there exists a positive integer n with $G^n = 1$ (i. e. G is of finite (positive) exponent).

It is clear that $\mathfrak{e}$ is subgroup-inherited.

If G is an abelian torsion group, then one verifies $G = \mathfrak{h}_{\mathfrak{e}} G$ without any difficulty. On the other hand, there exists any number of abelian torsion groups not meeting requirement (2). Hence $\mathfrak{e}$ is not first stage pre-local.

Since $\mathfrak{e}$ is subgroup-inherited, its closure $\bar{\mathfrak{e}}$ is first stage pre-local (Lemma 4.22).

It is well known that there exists a p-group G with the following properties:

(a) G possesses a countably infinite normal subgroup A with

$$1 = A' = A^p, \quad A = \mathfrak{c}A.$$

Here as always $\mathfrak{c}U$ denotes the centralizer of U in G.

(b) There exists a countably infinite elementary abelian p-subgroup B of G with

$$G = AB, \quad 1 = A \cap B.$$

(c) If U is an infinite subgroup of B, then $A \cap \mathfrak{c}U = 1$ and in particular $\mathfrak{z}G = 1$.

(d) Finite subsets of G generate finite p-subgroups of G.

If $a \neq 1$ is an element in A, then $A \cap \{a, B\}$ is a normal subgroup, not 1, of $\{a, B\}$ which does not contain a center element, not 1, of $\{a, B\}$. Hence a is not $\mathfrak{e}$-hypercentral so that

$$A \cap \mathfrak{h}_{\mathfrak{e}}G = 1.$$

It follows in particular that the normal subgroups A and $\mathfrak{h}_{\mathfrak{e}}G$ of G centralize each other. But $A = \mathfrak{c}A$ so that

$$\mathfrak{h}_{\mathfrak{e}}G = 1.$$

Application of Lemma 4.22 (a) shows

$$\mathfrak{h}_{\bar{\mathfrak{e}}}G = 1.$$

In particular, G is not an $\bar{\mathfrak{e}}$-group. But G is locally an $\bar{\mathfrak{e}}$-group (by (d)) and hence locally an $\mathfrak{e}$-group. Consequently $\bar{\mathfrak{e}}$ *is not a local property*.

Naturally A as an abelian group is an $\mathfrak{e}$-subgroup and hence an $\bar{\mathfrak{e}}$-subgroup of G. If S is a subgroup of G with $A \subseteq S$, then we distinguish two cases:

1. S/A is finite.

Then S is of finite class and hence an $\mathfrak{e}$-group.

2. S/A is infinite.

Then we see as before that

$$\mathfrak{h}_{\bar{\mathfrak{e}}}S = \mathfrak{h}_{\mathfrak{e}}S = 1.$$

Hence S is not an $\bar{\mathfrak{e}}$-group.

Since G/A is an infinite elementary abelian p-group, there does not exist a maximal finite subgroup of G/A. If follows that A is not part of a maximal $\bar{\mathfrak{e}}$-subgroup of G. Consequently, $\bar{\mathfrak{e}}$ *is not third stage pre-local.*

We have not been able to decide whether $\bar{\mathfrak{e}}$ is or is not second stage pre-local (**P 534**).

THE SECOND CLOSURE OPERATOR. If $\mathfrak{e}$ is any property on $\mathfrak{D}$, then the property $\mathfrak{le}$ (= *locally* $\mathfrak{e}$) is defined by the rule:

The group G is an $\mathfrak{le}$-group if, and only if, every finitely generated subgroup of G is an $\mathfrak{e}$-group.

This important construction has been much used and investigated.

If S is a subgroup of the $\mathfrak{le}$-group G, then every finitely generated subgroup of S is an $\mathfrak{e}$-group as a finitely generated subgroup of G. Consequently, $\mathfrak{le}$ is subgroup-inherited.

If every finitely generated subgroup of G is an $\mathfrak{le}$-group, then every finitely generated subgroup of G is an $\mathfrak{e}$-group so that G itself is an $\mathfrak{le}$-group. In toto: we have shown that $\mathfrak{le}$ is a local property.

PROPOSITION 4.24. *If*

(a) *$\mathfrak{e}$-groups are finitely generated*

and

(b) *finitely generated subgroups of $\mathfrak{e}$-groups are $\mathfrak{e}$-groups,*

then

(1) *$\mathfrak{le}$ is a local property;*

(2) *a group is an $\mathfrak{e}$-group if, and only if, it is a finitely generated $\mathfrak{le}$-group;*

(3) $\mathfrak{h}_{\mathfrak{e}} = \mathfrak{h}_{\mathfrak{le}}$;

(4) *$\mathfrak{h}_{\mathfrak{e}}G$ is the intersection of all the maximal $\mathfrak{le}$-subgroups of G.*

Proof. We have shown before (without use of hypotheses (a) and (b)) that $\mathfrak{le}$ is always a local property.

If G is an $\mathfrak{e}$-group, then G is, by (a), finitely generated and, by (b), all its finitely generated subgroups are $\mathfrak{e}$-groups. Hence G is a finitely generated $\mathfrak{le}$-group. That conversely every finitely generated $\mathfrak{le}$-group is an $\mathfrak{e}$-group, is an immediate consequence of the definition of $\mathfrak{le}$. This shows the validity of (2).

If g is an element of $\mathfrak{h}_{\mathfrak{e}}G$ and if E is an $\mathfrak{le}$-subgroup of G, then consider a finitely generated subgroup S of $\{g, E\}$. Clearly there exists a finitely generated subgroup T of E with

$$S \subseteq \{g, T\}.$$

Since T is a finitely generated subgroup of the $\mathfrak{le}$-group E, it is an $\mathfrak{e}$-group. Since g belongs to $\mathfrak{h}_{\mathfrak{e}}G$ and T is an $\mathfrak{e}$-group, $\{g, T\}$ is an $\mathfrak{e}$-group. Since S is a finitely generated subgroup of the $\mathfrak{e}$-group $\{g, T\}$, it is an $\mathfrak{e}$-group. Thus every finitely generated subgroup of $\{g, E\}$ is an $\mathfrak{e}$-group. Hence $\{g, E\}$ is an $\mathfrak{le}$-group for every $\mathfrak{le}$-subgroup E of G. Consequently, g belongs to $\mathfrak{h}_{\mathfrak{le}}G$.

Assume conversely that g is an element in $\mathfrak{h}_{\mathfrak{le}}G$. If X is an $\mathfrak{e}$-subgroup of G, then X is, by (2), a finitely generated $\mathfrak{le}$-subgroup of G. Hence

$\{g, X\}$ is a finitely generated $\mathfrak{le}$-group. This implies, by (2), that $\{g, X\}$ is an $\mathfrak{e}$-group for every $\mathfrak{e}$-subgroup X of G. Hence g belongs to $\mathfrak{h}_{\mathfrak{e}}G$. Thus we have shown:

The element g in G belongs to $\mathfrak{h}_{\mathfrak{e}}G$ if, and only if, g belongs to $\mathfrak{h}_{\mathfrak{le}}G$.

This assertion is equivalent with $\mathfrak{h}_{\mathfrak{e}}G = \mathfrak{h}_{\mathfrak{le}}G$, proving (3).

(4) finally is an immediate consequence of (1), (3), Proposition 4.17 (iii.a) and Proposition 4.16 (a).

REMARK 4.25. Consider properties $\mathfrak{a}$ and $\mathfrak{b}$ meeting the following requirements:

(a) $\mathfrak{a}$-groups are finitely generated $\mathfrak{b}$-groups;

(b) finitely generated subgroups of $\mathfrak{a}$-groups are $\mathfrak{a}$-groups;

(c) $\mathfrak{b}$-groups are $\mathfrak{la}$-groups.

Then it follows from Proposition 4.24 (3) that

(d) $\mathfrak{h}_{\mathfrak{a}} = \mathfrak{h}_{\mathfrak{la}}$.

Suppose now that the element g in G belongs to $\mathfrak{h}_{\mathfrak{b}}G$. If A is an $\mathfrak{a}$-subgroup of G, then A is, by (a), a $\mathfrak{b}$-subgroup of G. Hence $\{g, A\}$ is a $\mathfrak{b}$-group. It follows from (c) that $\{g, A\}$ is a finitely generated $\mathfrak{la}$-group. Hence $\{g, A\}$ is an $\mathfrak{a}$-group. Consequently g belongs to $\mathfrak{h}_{\mathfrak{a}}G$. This proves

(e) $\mathfrak{h}_{\mathfrak{b}} \subseteq \mathfrak{h}_{\mathfrak{a}}$.

It is impossible to prove equality in (e), as may be seen from the following example: Let $\mathfrak{D}$ be the universal class of all groups. Denote by $\mathfrak{a}$ the property of being a finitely generated group of finite class. Thus $\mathfrak{a}$-groups are exactly the noetherian groups G possessing a finite central chain connecting 1 and G. Finally define the property $\mathfrak{b}$ by the rule:

The group G is a $\mathfrak{b}$-group if (and only if) every epimorphic image $H \neq 1$ of G possesses a center $\mathfrak{z}H \neq 1$.

It is well known that these properties $\mathfrak{a}$ and $\mathfrak{b}$ meet the requirements (a), (b), (c); see Baer [5], p. 322, Satz 1. Note furthermore that $\mathfrak{la} = \mathfrak{lb}$ is the "local nilpotency" property. In Example 4.23 a group G has been discussed with the properties

$$G = \mathfrak{h}_{\mathfrak{a}}, \quad 1 = \mathfrak{h}_{\mathfrak{b}}.$$

AN APPLICATION. The following result may be useful in various situations.

PROPOSITION 4.26. *If $\mathfrak{e}$ is a local property and if the set Σ of subgroups of the group G (in $\mathfrak{D}$) is ordered by inclusion, then the compositum of all the subgroups $\mathfrak{h}_{\mathfrak{e}}X$ with X in Σ is an $\mathfrak{e}$-group.*

Proof. Consider first a finite subset Σ^* of Σ. Then we number the subgroups in Σ^* in such a way that

$$\Sigma^* = [X_1 \subseteq \ldots \subseteq X_i \subseteq X_{i+1} \subseteq \ldots \subseteq X_n].$$

Let

$$C_j = \mathfrak{h}_e X_1 \dots \mathfrak{h}_e X_j$$

be the ordered complex product of the $\mathfrak{h}_e X_i$ with $1 \leqslant i \leqslant j$. We prove by complete induction that each C_j with $1 \leqslant j \leqslant n$ is an $\mathfrak{e}$-subgroup of G. This is certainly true for $j = 1$. Hence we may assume that $1 \leqslant j < n$ and that C_j is an $\mathfrak{e}$-subgroup of G. Naturally C_j is an $\mathfrak{e}$-subgroup of $X_j \subseteq X_{j+1}$. But $\mathfrak{e}$ is a local property and as such $\mathfrak{e}$ is second stage pre-local. Application of Proposition 4.8 (iii.b) shows that

$$C_j \mathfrak{h}_e X_{j+1} = C_{j+1}$$

is an $\mathfrak{e}$-subgroup of X_{j+1}. This completes the inductive proof of the following fact:

(*) If Σ^* is a finite subset of Σ, then the composition of the subgroups $\mathfrak{h}_e X$ for X in Σ^* is an $\mathfrak{e}$-subgroup of G.

(Note that we have proved slightly more than claimed in (*)).

Denote now by C the compositum of all the $\mathfrak{h}_e X$ for X in Σ. If F is a finite subset of C, then there exists a finite subset Σ^* of Σ such that F is contained in the compositum C^* of the $\mathfrak{h}_e X$ for X in Σ^*. But C^* is by (*) an $\mathfrak{e}$-group. Since $\mathfrak{e}$ is subgroup-inherited (by Proposition 4.17 (ii.a)), the subgroup $\{F\}$ of C^* is likewise an $\mathfrak{e}$-group. Hence every finitely generated subgroup of C is an $\mathfrak{e}$-group so that C is, by Proposition 4.17 (i), an $\mathfrak{e}$-group.

5. *PRE-CO-LOCALITY AND CO-LOCALITY*

GENERAL REQUIREMENT. $\mathfrak{D}$ contains with any group all its epimorphic images.

CO-$\mathfrak{e}$-SUBGROUPS are normal subgroups E of G with $\mathfrak{e}$-quotient group G/E.

THE CO-$\mathfrak{e}$-HYPERCENTER. The normal subgroup N of G is termed *co-$\mathfrak{e}$-hypercentral, if $N \cap X$ is a co-$\mathfrak{e}$-subgroup of G for every co-$\mathfrak{e}$-subgroup X of G.*

Since $X = G$ is always a co-$\mathfrak{e}$-subgroup of G, every co-$\mathfrak{e}$-hypercentral subgroup is also a co-$\mathfrak{e}$-subgroup.

If A and B are co-$\mathfrak{e}$-hypercentral subgroups of G, and if X is a co-$\mathfrak{e}$-subgroup of G, then $B \cap X$ is a co-$\mathfrak{e}$-subgroup of G so that $A \cap (B \cap X) = (A \cap B) \cap X$ is likewise a co-$\mathfrak{e}$-subgroup of G. Hence $A \cap B$ is co-$\mathfrak{e}$-hypercentral. By an obvious induction we see that *the intersection of finitely many co-$\mathfrak{e}$-hypercentral subgroups is likewise co-$\mathfrak{e}$-hypercentral.*

This leads to the following definition:

The co-$\mathfrak{e}$-hypercenter ${}^{co}\mathfrak{h}_e G$ of the group G is the intersection of all the co-$\mathfrak{e}$-hypercentral normal subgroups of G.

Naturally this is a well determined characteristic subgroup.

PRE-CO-LOCALITY: THE FIRST STAGE. This will be characterized by the following

PROPOSITION 5.1. *The following properties of the property* $\mathfrak{e}$ *are equivalent*:

(i) $\begin{cases} \text{(a) } \mathfrak{e} \textit{ is epimorphism-inherited.} \\ \text{(b) } {}^{co}\mathfrak{h}_{\mathfrak{e}}G \textit{ is always co-}\mathfrak{e}\textit{-hypercentral.} \end{cases}$

(ii) $\begin{cases} \text{(a) } \mathfrak{e} \textit{ is epimorphism-inherited.} \\ \text{(b) } \textit{There exists always a minimal co-}\mathfrak{e}\textit{-hypercentral normal subgroup.} \end{cases}$

(iii) *The normal subgroup N of G is co-$\mathfrak{e}$-hypercentral, if (and only if) ${}^{co}\mathfrak{h}_{\mathfrak{e}}G \subseteq N$.*

Properties meeting these equivalent requirements will be termed *first stage pre-co-local.*

Proof. If (i) is valid, then ${}^{co}\mathfrak{h}_{\mathfrak{e}}G$ is, by definition and (i.b), the minimal co-$\mathfrak{e}$-hypercentral subgroup of G. Hence (i) implies (ii).

Assume next the validity of (ii) and denote by M some minimal co-e-hypercentral normal subgroup of G. Then we deduce

$$
{}^{co}\mathfrak{h}_{\mathfrak{e}}G \subseteq M
$$

from the definition of ${}^{co}\mathfrak{h}_{\mathfrak{e}}$. If N is any co-$\mathfrak{e}$-hypercentral subgroup of G, then $M \cap N$ is co-$\mathfrak{e}$-hypercentral as we have pointed out before. Hence $M = M \cap N \subseteq N$ by the minimality of M; and this implies that M is part of the intersection of all co-$\mathfrak{e}$-hypercentral subgroups of G. Consequently

$$
M \subseteq {}^{co}\mathfrak{h}_{\mathfrak{e}}G;
$$

and thus we have shown that ${}^{co}\mathfrak{h}_{\mathfrak{e}}G = M$ is co-$\mathfrak{e}$-hypercentral, proving the equivalence of (i) and (ii).

Assume next the validity of (i) and consider a normal subgroup N of G with ${}^{co}\mathfrak{h}_{\mathfrak{e}}G \subseteq N$. If X is a co-$\mathfrak{e}$-subgroup of G, then we have

$$
(*) \qquad X \cap {}^{co}\mathfrak{h}_{\mathfrak{e}}G \subseteq X \cap N;
$$

and we deduce from (i.b) that $X \cap {}^{co}\mathfrak{h}_{\mathfrak{e}}G$ is a co-$\mathfrak{e}$-subgroup of G. Hence $G/(X \cap {}^{co}\mathfrak{h}_{\mathfrak{e}}G)$ is an $\mathfrak{e}$-group; and we deduce from (i.a) and (*) that $G/(X \cap N)$ is an $\mathfrak{e}$-group. Hence $X \cap N$ is a co-$\mathfrak{e}$-subgroup of G so that N is co-$\mathfrak{e}$-hypercentral; and we have deduced (iii) from (i).

Assume the validity of (iii). Then ${}^{co}\mathfrak{h}_{\mathfrak{e}}G$ is always co-$\mathfrak{e}$-hypercentral so that (i.b) is a consequence of (iii). If G is an $\mathfrak{e}$-group, then 1 is a co-$\mathfrak{e}$-subgroup and 1 is even co-$\mathfrak{e}$-hypercentral. Hence ${}^{co}\mathfrak{h}_{\mathfrak{e}}G = 1$. Consequently ${}^{co}\mathfrak{h}_{\mathfrak{e}}G \subseteq N$ for every normal subgroup N of G, and it follows from (iii) that every normal subgroup of G is co-$\mathfrak{e}$-hypercentral. This implies in particular that every normal subgroup of G is a co-$\mathfrak{e}$-subgroup of G and that therefore every epimorphic image of G is an $\mathfrak{e}$-group. This proves (i.a) and completes the proof of the equivalence of (i)-(iii).

REMARK 5.2. The interest of (ii) stems from the fact that (ii.b) is certainly satisfied whenever the minimum condition is satisfied by the normal subgroups of the groups in $\mathfrak{D}$. In this case first stage pre-co-locality reduces to epimorphism-inheritance.

EXAMPLE 5.3. Assume that $\mathfrak{D}$ is the class of all abelian groups and $\mathfrak{e}$ the property of being a countable (abelian) group. Then $\mathfrak{e}$ is clearly epimorphism-inherited. If A and B are co-$\mathfrak{e}$-subgroups of the group G, then $G/(A \cap B)$ is isomorphic to a subgroup of the direct product $(G/A) \otimes (G/B)$ of the countable groups G/A and G/B so that $G/(A \cap B)$ is likewise countable. Hence $A \cap B$ is a co-$\mathfrak{e}$-subgroup of G, if A and B are co-$\mathfrak{e}$-subgroups; and now it follows that

(*) the subgroup S of G is a co-$\mathfrak{e}$-subgroup of G if, and only if, S is a co-$\mathfrak{e}$-hypercentral subgroup of G.

If $g \neq 1$ is an element in G, then there exist subgroups of G which do not contain g and among these a maximal one, M. One verifies that G/M is a torsiongroup of rank 1 and hence countable. Using (*) it follows that M is co-$\mathfrak{e}$-hypercentral and consequently

(**) ${}^{co}\mathfrak{h}_{\mathfrak{e}}G = 1$ for every G.

But if G is not countable, then ${}^{co}\mathfrak{h}_{\mathfrak{e}}G$ is not a co-$\mathfrak{e}$-subgroup and a fortiori not co-$\mathfrak{e}$-hypercentral.

Thus conditions (i.b) and (ii.b) of Proposition 5.1 are not consequences of epimorphism-inheritance.

COROLLARY 5.4. *Assume that the property $\mathfrak{e}$ is first stage pre-co-local.*

(a) *G is an $\mathfrak{e}$-group if, and only if, ${}^{co}\mathfrak{h}_{\mathfrak{e}}G = 1$.*

(b) *Epimorphisms map co-$\mathfrak{e}$-hypercentral subgroups upon co-$\mathfrak{e}$-hypercentral subgroups.*

(c) *${}^{co}\mathfrak{h}_{\mathfrak{e}}(G^{\sigma}) \subseteq [{}^{co}\mathfrak{h}_{\mathfrak{e}}G]^{\sigma}$ for every epimorphism σ of G.*

(d) *If the epimorphism σ of G induces an isomorphism in ${}^{co}\mathfrak{h}_{\mathfrak{e}}G$, then*

$$ {}^{co}\mathfrak{h}_{\mathfrak{e}}(G^{\sigma}) = [{}^{co}\mathfrak{h}_{\mathfrak{e}}G]^{\sigma}. $$

Proof. If G is an $\mathfrak{e}$-group, then 1 is a co-$\mathfrak{e}$-subgroup of G so that 1 is co-$\mathfrak{e}$-hypercentral. Hence $1 = {}^{co}\mathfrak{h}_{\mathfrak{e}}G$. If conversely $1 = {}^{co}\mathfrak{h}_{\mathfrak{e}}G$, then we note that ${}^{co}\mathfrak{h}_{\mathfrak{e}}G$ is co-$\mathfrak{e}$-hypercentral and hence a co-$\mathfrak{e}$-subgroup so that 1 is a co-$\mathfrak{e}$-subgroup and hence G an $\mathfrak{e}$-group, proving (a).

Consider an epimorphism σ of G upon H. If N is a co-$\mathfrak{e}$-hypercentral subgroup of G and X a co-$\mathfrak{e}$-subgroup of H, then $X^{\sigma^{-1}}$ is a co-$\mathfrak{e}$-subgroup of G, since σ induces an isomorphism of $G/X^{\sigma^{-1}}$ upon H/X. Hence $N \cap X^{\sigma^{-1}}$ is a co-$\mathfrak{e}$-subgroup of G. Since σ induces an epimorphism of $G/[N \cap X^{\sigma^{-1}}]$ upon $H/[N \cap X^{\sigma^{-1}}]^{\sigma}$ and since $\mathfrak{e}$ is epimorphism-inherited, $[N \cap X^{\sigma^{-1}}]^{\sigma}$ is a co-$\mathfrak{e}$-subgroup of H. Since

$$[N \cap X^{\sigma^{-1}}]^{\sigma} \subseteq N^{\sigma} \cap (X^{\sigma^{-1}})^{\sigma} = N^{\sigma} \cap X,$$

and since $\mathfrak{e}$ is epimorphism-inherited, $N^{\sigma} \cap X$ is likewise a co-$\mathfrak{e}$-subgroup. Hence N^{σ} is co-$\mathfrak{e}$-hypercentral, proving (b). In particular, ${}^{co}\mathfrak{h}_{\mathfrak{e}}G$ is co-$\mathfrak{e}$-hypercentral and thus $[{}^{co}\mathfrak{h}_{\mathfrak{e}}G]^{\sigma}$ is by (b) likewise co-$\mathfrak{e}$-hypercentral. Application of the definition of ${}^{co}\mathfrak{h}_{\mathfrak{e}}$ shows (c).

Assume that the epimorphism σ induces an isomorphism in ${}^{co}\mathfrak{h}_{\mathfrak{e}}G$. If we denote by K the kernel of σ, then this hypothesis is equivalent to

(1) $$K \cap {}^{co}\mathfrak{h}_{\mathfrak{e}}G = 1;$$

and we may assume without loss in generality that σ is the canonical epimorphism of G upon G/K. Denote by J the uniquely determined normal subgroup of G which contains K and satisfies

(2) $$J/K = {}^{co}\mathfrak{h}_{\mathfrak{e}}(G/K).$$

Then (c) implies

(3) $$K \subseteq J \subseteq K \cdot {}^{co}\mathfrak{h}_{\mathfrak{e}}G;$$

and application of Dedekind's modular law shows

(4) $$J = K[J \cap {}^{co}\mathfrak{h}_{\mathfrak{e}}G].$$

Assume now by way of contradiction that

(+) $$W = J \cap {}^{co}\mathfrak{h}_{\mathfrak{e}}G \subset {}^{co}\mathfrak{h}_{\mathfrak{e}}G.$$

Then W is a normal subgroup of G which is not co-$\mathfrak{e}$-hypercentral. Consequently there exists a co-$\mathfrak{e}$-subgroup X of G such that $W \cap X$ is not a co-$\mathfrak{e}$-subgroup of G. Since ${}^{co}\mathfrak{h}_{\mathfrak{e}}G$ is co-$\mathfrak{e}$-hypercentral, $Y = X \cap {}^{co}\mathfrak{h}_{\mathfrak{e}}G$ is a co-$\mathfrak{e}$-subgroup of G; and

$$W \cap Y = W \cap {}^{co}\mathfrak{h}_{\mathfrak{e}}G \cap X = W \cap X$$

is not a co-$\mathfrak{e}$-subgroup of G. Since $\mathfrak{e}$ is epimorphism-inherited, YK is likewise a co-$\mathfrak{e}$-subgroup of G. Hence KY/K is a co-$\mathfrak{e}$-subgroup of G/K. Since J/K is (by (2)) a co-$\mathfrak{e}$-hypercentral subgroup of G/K, it follows that

$$(KY/K) \cap (J/K) = (KY \cap J)/K$$

is a co-$\mathfrak{e}$-subgroup of G/K. Hence $KY \cap J$ is a co-$\mathfrak{e}$-subgroup of G. Application of Dedekind's modular law, (3), and (+) show

$$KY \cap J = K[Y \cap J] = K[Y \cap {}^{co}\mathfrak{h}_{\mathfrak{e}}G \cap J] = K[Y \cap W].$$

Since ${}^{co}\mathfrak{h}_{\mathfrak{e}}G$ is co-$\mathfrak{e}$-hypercentral and $K[Y \cap W]$ is a co-$\mathfrak{e}$-subgroup of G, it follows from Dedekind's modular law, $(+)$, and (1) that

$$ {}^{co}\mathfrak{h}_{\mathfrak{e}}G \cap K[Y \cap W] = [Y \cap W][K \cap {}^{co}\mathfrak{h}_{\mathfrak{e}}G] = Y \cap W $$

is a co-$\mathfrak{e}$-subgroup of G. This is the desired contradiction, derived from $(+)$. Hence

$$ {}^{co}\mathfrak{h}_{\mathfrak{e}}G = J \cap {}^{co}\mathfrak{h}_{\mathfrak{e}}G; $$

and it follows from (4) that

$$ J = K \cdot {}^{co}\mathfrak{h}_{\mathfrak{e}}G. $$

Apply (2) to see that

$$ {}^{co}\mathfrak{h}_{\mathfrak{e}}(G^{\sigma}) = {}^{co}\mathfrak{h}_{\mathfrak{e}}(G/K) = J/K = [K \cdot {}^{co}\mathfrak{h}_{\mathfrak{e}}G]/K = [{}^{co}\mathfrak{h}_{\mathfrak{e}}G]^{\sigma}, $$

proving (d).

EXAMPLE 5.5. Denote by $\mathfrak{D}$ the class of all finite groups and by $\mathfrak{e}$ the property of being a (finite) cyclic group. Application of (Proposition 5.1 or) Remark 5.2 shows that $\mathfrak{e}$ is on $\mathfrak{D}$ first stage pre-co-local. If G is an elementary abelian group of order p^2, then every subgroup, not 1, of G is a co-$\mathfrak{e}$-subgroup of G; but 1 is not a co-$\mathfrak{e}$-subgroup of G. It follows that ${}^{co}\mathfrak{h}_{\mathfrak{e}}G = G$. If $K \neq 1$ is some cyclic subgroup of G, then G/K is cyclic of order p. Letting σ be the canonical epimorphism of G upon the $\mathfrak{e}$-group G/K, it follows that

$$ {}^{co}\mathfrak{h}_{\mathfrak{e}}(G^{\sigma}) = 1 \subset G/K = [{}^{co}\mathfrak{h}_{\mathfrak{e}}G]^{\sigma}; $$

and this shows that the extra-requirement, imposed by (d), is indispensable.

The function $\mathfrak{f}$ on $\mathfrak{D}$ is termed *first stage pre-co-local,* if

(a) $\mathfrak{f}(G^{\sigma}) \subseteq (\mathfrak{f}G)^{\sigma}$ *for every epimorphism* σ *of* G

and

(b) $\mathfrak{f}(G^{\sigma}) = (\mathfrak{f}G)^{\sigma}$ *for every epimorphism* σ *of* G *which induces an isomorphism in* $\mathfrak{f}G$.

We remind the reader that the group G (in $\mathfrak{D}$) is termed an $\mathfrak{f}^*$-group if, and only if, $\mathfrak{f}G = 1$.

PROPOSITION 5.6. *The following properties of the property* $\mathfrak{e}$ *are equivalent:*

(i) $\mathfrak{e}$ *is first stage pre-co-local.*

(ii) (a) ${}^{co}\mathfrak{h}_{\mathfrak{e}}$ *is first stage pre-co-local.*
(b) $\mathfrak{e} = [{}^{co}\mathfrak{h}_{\mathfrak{e}}]^*$.

Proof. If (i) is satisfied by $\mathfrak{e}$, then we deduce (ii.a) from Corollary 5.4 (c, d), and (ii.b) is a restatement of Corollary 5.4 (a).

Assume conversely the validity of (ii). We note that (ii.b) may be restated as

(b*) G is an $\mathfrak{e}$-group if, and only if, ${}^{\mathrm{co}}\mathfrak{h}_{\mathfrak{e}}G = 1$.

Consider an epimorphism σ of the $\mathfrak{e}$-group G. Then

$$ {}^{\mathrm{co}}\mathfrak{h}_{\mathfrak{e}}(G^{\sigma}) \subseteq ({}^{\mathrm{co}}\mathfrak{h}_{\mathfrak{e}}G)^{\sigma} = 1 $$

so that G^{σ} is, by (b*), an $\mathfrak{e}$-group. Hence $\mathfrak{e}$ *is epimorphism-inherited.*

Consider next some group G (in $\mathfrak{D}$) and a co-$\mathfrak{e}$-subgroup X of G. Let

$$ H = G/[X \cap {}^{\mathrm{co}}\mathfrak{h}_{\mathfrak{e}}G]; $$

and denote by σ the canonical epimorphism of G upon H. Let J be the uniquely determined normal subgroup of G with

$$ X \cap {}^{\mathrm{co}}\mathfrak{h}_{\mathfrak{e}}G \subseteq J \quad \text{and} \quad {}^{\mathrm{co}}\mathfrak{h}_{\mathfrak{e}}H = J/[X \cap {}^{\mathrm{co}}\mathfrak{h}_{\mathfrak{e}}G]. $$

Then

$$ J/[X \cap {}^{\mathrm{co}}\mathfrak{h}_{\mathfrak{e}}G] = {}^{\mathrm{co}}\mathfrak{h}_{\mathfrak{e}}H = {}^{\mathrm{co}}\mathfrak{h}_{\mathfrak{e}}(G^{\sigma}) \subseteq ({}^{\mathrm{co}}\mathfrak{h}_{\mathfrak{e}}G)^{\sigma} = {}^{\mathrm{co}}\mathfrak{h}_{\mathfrak{e}}G/[X \cap {}^{\mathrm{co}}\mathfrak{h}_{\mathfrak{e}}G] $$

so that

$$ (*) \qquad J \subseteq {}^{\mathrm{co}}\mathfrak{h}_{\mathfrak{e}}G. $$

Denote next by τ the canonical epimorphism of H upon G/X. If an element t in ${}^{\mathrm{co}}\mathfrak{h}_{\mathfrak{e}}H$ is mapped upon 1 by τ, then we note first that $t = j[X \cap {}^{\mathrm{co}}\mathfrak{h}_{\mathfrak{e}}G]$ with j in J. But $t^{\tau} = 1$ is equivalent with $jX = 1$ so that j belongs to

$$ J \cap X \subseteq {}^{\mathrm{co}}\mathfrak{h}_{\mathfrak{e}}G \cap X $$

by (*); and this implies $t = 1$. Thus an isomorphism is induced by τ on ${}^{\mathrm{co}}\mathfrak{h}_{\mathfrak{e}}H$; and this implies (by (b) of the definition of first stage pre-co-locality) that

$$ [{}^{\mathrm{co}}\mathfrak{h}_{\mathfrak{e}}H]^{\tau} = {}^{\mathrm{co}}\mathfrak{h}_{\mathfrak{e}}(H^{\tau}) = {}^{\mathrm{co}}\mathfrak{h}_{\mathfrak{e}}(G/X) = 1, $$

since X is a co-$\mathfrak{e}$-subgroup of G, so that G/X is an $\mathfrak{e}$-group and ${}^{\mathrm{co}}\mathfrak{h}_{\mathfrak{e}}(G/X) = 1$ by (b*). Since τ induces an isomorphism on ${}^{\mathrm{co}}\mathfrak{h}_{\mathfrak{e}}H$, our last result implies

$$ {}^{\mathrm{co}}\mathfrak{h}_{\mathfrak{e}}H = 1; $$

and it follows from (b*) that $H = G/[X \cap {}^{\mathrm{co}}\mathfrak{h}_{\mathfrak{e}}G]$ is an $\mathfrak{e}$-group. Hence $X \cap {}^{\mathrm{co}}\mathfrak{h}_{\mathfrak{e}}G$ is a co-$\mathfrak{e}$-subgroup of G for every co-$\mathfrak{e}$-subgroup X of G, so that

$$ {}^{\mathrm{co}}\mathfrak{h}_{\mathfrak{e}}G \text{ is co-}\mathfrak{e}\text{-hypercentral}. $$

Thus we have shown that $\mathfrak{e}$ meets requirement (i) of Proposition 5.1. Hence $\mathfrak{e}$ is first stage pre-co-local; and we have derived (i) from (ii).

Remark 5.7. When proving that a property $\mathfrak{e}$ is epimorphism-inherited if it meets requirement (ii) of Proposition 5.6, no use was made of condition (b) of first stage pre-co-locality of functions. We pointed out in Remark 5.2 that epimorphism-inheritance of a property implies its first stage pre-co-locality provided some weak sort of minimum condition is satisfied by the groups in $\mathfrak{D}$. In such a situation condition (b) of first stage pre-co-locality of functions is not needed for the validity of Proposition 5.6.

Corollary 5.8. *If*

(a) ${}^{co}\mathfrak{h}_{\mathfrak{e}}(G^{\sigma}) \subseteq [{}^{co}\mathfrak{h}_{\mathfrak{e}}G]^{\sigma}$ *for every epimorphism* σ *of* G,
then the following properties of $\mathfrak{e}$ *are equivalent:*

(i) G *is an* $\mathfrak{e}$*-group if, and only if,* ${}^{co}\mathfrak{h}_{\mathfrak{e}}G = 1$.

(ii) $G/{}^{co}\mathfrak{h}_{\mathfrak{e}}G$ *is always an* $\mathfrak{e}$*-group.*

(iii) G *is an* $\mathfrak{e}$*-group, if* ${}^{co}\mathfrak{h}_{\mathfrak{e}}G = 1$.

Proof. Assume the validity of (i) and denote by σ the canonical epimorphism of G upon $G/{}^{co}\mathfrak{h}_{\mathfrak{e}}G$. Application of (a) shows

$${}^{co}\mathfrak{h}_{\mathfrak{e}}(G/{}^{co}\mathfrak{h}_{\mathfrak{e}}G) = {}^{co}\mathfrak{h}_{\mathfrak{e}}(G^{\sigma}) \subseteq [{}^{co}\mathfrak{h}_{\mathfrak{e}}G]^{\sigma} = {}^{co}\mathfrak{h}_{\mathfrak{e}}G/{}^{co}\mathfrak{h}_{\mathfrak{e}}G = 1;$$

and it follows from (i) that $G/{}^{co}\mathfrak{h}_{\mathfrak{e}}G$ is an $\mathfrak{e}$-group. Hence (ii) is a consequence of (i); and it is clear that (ii) implies (iii).

It is a consequence of the definition of ${}^{co}\mathfrak{h}_{\mathfrak{e}}$ that ${}^{co}\mathfrak{h}_{\mathfrak{e}}G = 1$ whenever G is an $\mathfrak{e}$-group. Consequently (iii) implies (i).

Remark 5.9. Condition (i) of Corollary 5.8 is a restatement of the condition

$$\mathfrak{e} = [{}^{co}\mathfrak{h}_{\mathfrak{e}}]^{*}$$

and (a) and (i) together imply that $\mathfrak{e}$ is epimorphism-inherited.

Remark 5.10. We have been unable to decide whether condition (b) of the definition of first stage pre-co-locality of functions can be omitted in Proposition 5.6 (ii.a), though it seems unlikely. (**P 535**)

We want to describe one of the difficulties arising when trying to solve this problem. The counter-examples of Example 5.3 had the feature that ${}^{co}\mathfrak{h}_{\mathfrak{e}}G$ was not always a co-$\mathfrak{e}$-subgroup. This suggests — dualising the first closure operator of § 4 — to introduce the following derived property:

If $\mathfrak{e}$ is a property, then the group G is an $\tilde{\mathfrak{e}}$-group if, and only if, ${}^{co}\mathfrak{h}_{\mathfrak{e}}G = 1$.

But this construction does not seem to be satisfactory, as may be seen from the following example:

Let $\mathfrak{D}$ be the class of all abelian groups and $\mathfrak{e}$ be the property of being an abelian group of positive exponent. Thus an abelian group A is an $\mathfrak{e}$-group if, and only if, there exists a positive integer n with $A^{n} = 1$.

Naturally $\mathfrak{e}$ is epimorphism-inherited (and subgroup-inherited and extension-inherited); and one verifies that

The subgroup X of the abelian group A is co-$\mathfrak{e}$-hypercentral if, and only if, X is a co-$\mathfrak{e}$-subgroup of A.

From this remark one deduces:

$$^{\mathrm{co}}\mathfrak{h}_{\mathfrak{e}} A = \bigcap_{n=1}^{\infty} A^n .$$

It follows that $\tilde{\mathfrak{e}}$ is the property of being free of elements $\neq 1$ of "infinite exponent"; and this property is not even epimorphism-inherited.

LEMMA 5.11. *If $\mathfrak{f}$ is first stage pre-co-local, then $\mathfrak{f}G$ is always a co-$\mathfrak{f}^*$-hypercentral subgroup of G and $^{\mathrm{co}}\mathfrak{h}_{\mathfrak{f}^*} \subseteq \mathfrak{f}$.*

Proof. Assume that X is a co-$\mathfrak{f}^*$-subgroup of G. This is equivalent to $\mathfrak{f}(G/X) = 1$. Let $Y = X \cap \mathfrak{f}G$ and denote by J the uniquely determined normal subgroup of G with

$$Y \subseteq J \quad \text{and} \quad \mathfrak{f}(G/Y) = J/Y .$$

Denote by α the canonical epimorphism of G upon G/Y and by β the canonical epimorphism of G/Y upon G/X. Then

$$J/Y = \mathfrak{f}(G/Y) = \mathfrak{f}(G^{\alpha}) \subseteq (\mathfrak{f}G)^{\alpha} = Y\cdot\mathfrak{f}G/Y = \mathfrak{f}G/Y$$

so that

$$Y \subseteq J \subseteq \mathfrak{f}G .$$

If b is an element in $\mathfrak{f}(G/Y)$ with $b^{\beta} = 1$, then $b = Yt$ and $1 = b^{\beta} = Xt$ so that t belongs to X. But b belongs to $\mathfrak{f}(G/Y) = J/Y$ so that t belongs to

$$X \cap J \subseteq X \cap \mathfrak{f}G = Y$$

implying $b = 1$. Hence β induces an isomorphism in $\mathfrak{f}(G/Y)$ and this implies

$$1 = \mathfrak{f}(G/X) = \mathfrak{f}[(G/Y)^{\beta}] = [\mathfrak{f}(G/Y)]^{\beta} = XJ/X$$

so that

$$Y \subseteq J \subseteq X \cap \mathfrak{f}G = Y .$$

Hence $J = X \cap \mathfrak{f}G$ proving

$$\mathfrak{f}(G/[X \cap \mathfrak{f}G]) = J/[X \cap \mathfrak{f}G] = 1 .$$

Hence $X \cap \mathfrak{f}G$ is a co-$\mathfrak{f}^*$-subgroup of G so that $\mathfrak{f}G$ is a co-$\mathfrak{f}^*$-hypercentral subgroup of G. This implies

$$^{\mathrm{co}}\mathfrak{h}_{\mathfrak{f}^*} G \subseteq \mathfrak{f}G$$

as was to be shown.

PROPOSITION 5.12. *The following properties of the function* $\mathfrak{f}$ *on* $\mathfrak{D}$ *are equivalent:*

(i) $\begin{cases} \text{(a) } \mathfrak{f} \textit{ is first stage pre-co-local.} \\ \text{(b) } \textit{If } N \textit{ is a normal subgroup of } G \textit{ with } N \subset \mathfrak{f}G, \textit{ then there exists} \\ \quad \textit{a normal subgroup } W \textit{ of } G \textit{ such that } W \subseteq \mathfrak{f}G,\ \mathfrak{f}(G/W) = 1, \\ \quad \mathfrak{f}(G/[N \cap W]) \neq 1. \end{cases}$

(ii) $\mathfrak{f}$ *is first stage pre-co-local and* $\mathfrak{f} \subseteq {}^{co}\mathfrak{h}_{\mathfrak{f}^*}$.

(iii) $\begin{cases} \text{(a) } \mathfrak{f}(G^\sigma) \subseteq (\mathfrak{f}G)^\sigma \textit{ for every epimorphism } \sigma \textit{ of } G. \\ \text{(b) } \mathfrak{f}G \textit{ is always a co-}\mathfrak{f}^*\textit{-hypercentral subgroup of } G. \\ \text{(c) } \mathfrak{f} \subseteq {}^{co}\mathfrak{h}_{\mathfrak{f}^*}. \end{cases}$

(iv) $\mathfrak{f}^*$ *is first stage pre-co-local and* $\mathfrak{f} = {}^{co}\mathfrak{h}_{\mathfrak{f}^*}$.

Proof. Assume first the validity of (i). Then it follows from Lemma 5.11 that $\mathfrak{f}G$ is co-$\mathfrak{f}^*$-hypercentral. Consider a co-$\mathfrak{f}^*$-hypercentral normal subgroup H of G. Then — as has been shown in the beginning of this § 5 — $H \cap \mathfrak{f}G = N$ is likewise co-$\mathfrak{f}^*$-hypercentral. Assume by way of contradiction that $N \neq \mathfrak{f}G$. Then $N \subset \mathfrak{f}G$; and we deduce from (i.b) the existence of a normal subgroup W of G with $W \subseteq \mathfrak{f}G$, $\mathfrak{f}(G/W) = 1$, $\mathfrak{f}(G/[N \cap W]) \neq 1$. But the last two statements are equivalent with the assertions:

W is a co-$\mathfrak{f}^*$-subgroup of G and $N \cap W$ is not a co-$\mathfrak{f}^*$-subgroup of G.

This contradicts the fact that N is co-$\mathfrak{f}^*$-hypercentral; and this contradiction shows that $\mathfrak{f}G = N \subseteq H$. It follows that $\mathfrak{f}G \subseteq {}^{co}\mathfrak{h}_{\mathfrak{f}^*}G$, since ${}^{co}\mathfrak{h}_{\mathfrak{f}^*}G$ is the intersection of all the co-$\mathfrak{f}^*$-hypercentral subgroups H of G. Hence (ii) is a consequence of (i).

Assume next the validity of (ii). Then the validity of (iii.a) and (iii.c) is immediately clear; and the validity of (iii.b) is a consequence of Lemma 5.11.

Assume next the validity of (iii). Then we deduce

$${}^{co}\mathfrak{h}_{\mathfrak{f}^*}G \subseteq \mathfrak{f}G \quad \text{for every } G$$

from (iii.b) and the definition of ${}^{co}\mathfrak{h}_{\mathfrak{f}^*}$. Combining this with (iii.c) we obtain

$$(*) \qquad \mathfrak{f} = {}^{co}\mathfrak{h}_{\mathfrak{f}^*}.$$

If F is an $\mathfrak{f}^*$-group, then $\mathfrak{f}F = 1$. Hence it follows from (iii.a) that $\mathfrak{f}H = 1$ for every epimorphic image H of F. The property $\mathfrak{f}^*$ is consequently epimorphism-inherited. Since ${}^{co}\mathfrak{h}_{\mathfrak{f}^*}G$ is by (*) and (iii.b) always co-$\mathfrak{f}^*$-hypercentral, condition (i) of Proposition 5.1 is satisfied by the property $\mathfrak{f}^*$. Consequently

(**) $\mathfrak{f}^*$ is first stage pre-co-local.

Combination of (*) and (**) shows the validity of (iv).

If (iv) is satisfied by $\mathfrak{f}$, then $\mathfrak{e} = \mathfrak{f}^*$ is first stage pre-co-local. It follows from Proposition 5.6 that

$$(+) \qquad {}^{co}\mathfrak{h}_{\mathfrak{e}} = {}^{co}\mathfrak{h}_{\mathfrak{f}^*} = \mathfrak{f}$$

is first stage pre-co-local, showing the validity of (i.a). Assume now that N is a normal subgroup of G with $N \subset \mathfrak{f}G$. Because of $(+)$ this N cannot be co-$\mathfrak{e}$-hypercentral. Hence there exists a co-$\mathfrak{e}$-subgroup V of G such that $V \cap N$ is not a co-$\mathfrak{e}$-subgroup of G. Let $W = V \cap \mathfrak{f}G$. Then $W \subseteq \mathfrak{f}G = {}^{co}\mathfrak{h}_{\mathfrak{e}}G$ by $(+)$. Since $\mathfrak{e}$ is first stage pre-co-local, ${}^{co}\mathfrak{h}_{\mathfrak{e}}G$ is co-$\mathfrak{e}$-hypercentral (Proposition 5.1 (i.b)). Consequently W is with V a co-$\mathfrak{e}$-subgroup of G. Since

$$W \cap N = V \cap \mathfrak{f}G \cap N = V \cap N,$$

the normal subgroup W of G meets all the requirements of (i.b). Thus we have shown he equivalence of (i)-(iv).

CONSTRUCTION 5.13. Denote by $\mathfrak{e}$ some epimorphism-inherited property on $\mathfrak{D}$ and define the function $\mathfrak{f} = \mathfrak{e}_0$ on $\mathfrak{D}$ by the rule

$$\mathfrak{f}G = \begin{cases} 1, & \text{if } G \text{ is an } \mathfrak{e}\text{-group}, \\ G, & \text{if } G \text{ is not an } \mathfrak{e}\text{-group}. \end{cases}$$

It is easily checked that $\mathfrak{f}^* = \mathfrak{e}$ and that $\mathfrak{f}$ is first stage pre-co-local. If we assume furthermore, as we may, that $\mathfrak{e}$ is actually first stage pre-co-local, then all the requirements of Proposition 5.12 are satisfied with the possible exception of (i.b) and

$$(*) \qquad \mathfrak{f} \subseteq {}^{co}\mathfrak{h}_{\mathfrak{f}^*}.$$

But $(*)$ will, in general, be false, as may be seen from many easily constructed examples. The simplest one is obtained as follows:

Let $\mathfrak{D}$ be the class of all finite groups and $\mathfrak{e}$ be the property of being finite and nilpotent. Then ${}^{co}\mathfrak{h}_{\mathfrak{f}^*} = {}^{co}\mathfrak{h}_{\mathfrak{e}}$ is just the terminal member of the descending central chain; and thus we shall have

$$1 \subset {}^{co}\mathfrak{h}_{\mathfrak{f}^*}G \subset G = \mathfrak{f}G$$

whenever the group G is soluble, but not nilpotent.

Thus we have shown the indispensability of (i.b), (iii.c), and of the second part of condition (ii) of Proposition 5.12.

PRE-CO-LOCALITY: THE SECOND STAGE. The property $\mathfrak{e}$ will be termed *second stage pre-co-local, if*

(a) $\mathfrak{e}$ *is epimorphism-inherited*

and

(b) *every co-$\mathfrak{e}$-subgroup of G contains a minimal co-$\mathfrak{e}$-subgroup of G.*

The terminology is justified as may be seen from the following

PROPOSITION 5.14. *If the property* $\mathfrak{e}$ *is second stage pre-co-local, then*

(A) $\mathfrak{e}$ *is first stage pre-co-local*

and

(B) ${}^{\mathrm{co}}\mathfrak{h}_{\mathfrak{e}}G$ *is the product of all minimal co-*$\mathfrak{e}$*-subgroups of* G.

Proof. Denote by P the product of all minimal co-$\mathfrak{e}$-subgroups of G. If M is a minimal co-$\mathfrak{e}$-subgroup of G, and if X is a co-$\mathfrak{e}$-hypercentral subgroup of G, then $M \cap X$ is a co-$\mathfrak{e}$-subgroup of G. Apply the minimality of M to show

$$M = M \cap X \subseteq X .$$

Apply the definition of ${}^{\mathrm{co}}\mathfrak{h}_{\mathfrak{e}}$ to show

$$M \subseteq {}^{\mathrm{co}}\mathfrak{h}_{\mathfrak{e}}G;$$

and now it follows from the definition of P that

(1) $$P \subseteq {}^{\mathrm{co}}\mathfrak{h}_{\mathfrak{e}}G .$$

If X is some co-$\mathfrak{e}$-subgroup of G, then X contains (by (b)) a minimal co-$\mathfrak{e}$-subgroup Y of G. It follows that

$$Y \subseteq X \cap P;$$

and since $\mathfrak{e}$ is, by (a), epimorphism-inherited, $X \cap P$ is likewise a co-$\mathfrak{e}$-subgroup of G. Hence

(2) P is co-$\mathfrak{e}$-hypercentral.

Combining (2) and the definition of ${}^{\mathrm{co}}\mathfrak{h}_{\mathfrak{e}}$ we find that

(3) $${}^{\mathrm{co}}\mathfrak{h}_{\mathfrak{e}}G \subseteq P .$$

Now (B) is a consequence of (1), (3). Because of (2) and (B) (and (a)), condition (i) of Proposition 5.1 is satisfied by $\mathfrak{e}$, proving (A).

We are now in a position to improve slightly Corollary 5.4 (d).

COROLLARY 5.15. *If the property* $\mathfrak{e}$ *is second stage pre-co-local, and if the epimorphism* σ *of* G *induces an isomorphism in* XY *whenever* X *and* Y *are minimal co-*$\mathfrak{e}$*-subgroups of* G, *then*

(+) σ *maps minimal co-*$\mathfrak{e}$*-subgroups of* G *upon minimal co-*$\mathfrak{e}$*-subgroups of* G^{σ} *and*

(++) $${}^{\mathrm{co}}\mathfrak{h}_{\mathfrak{e}}(G^{\sigma}) = [{}^{\mathrm{co}}\mathfrak{h}_{\mathfrak{e}}G]^{\sigma}.$$

Proof. Suppose that A is a minimal co-$\mathfrak{e}$-subgroup of G. Then σ induces an epimorphism of the $\mathfrak{e}$-group G/A upon G^{σ}/A^{σ}. Since $\mathfrak{e}$ is epimorphism-inherited, A^{σ} is a co-$\mathfrak{e}$-subgroup of G^{σ}. Suppose that X is a co-$\mathfrak{e}$-subgroup of G^{σ} with $X \subseteq A^{\sigma}$. If $Y = X^{\sigma^{-1}}$ is the inverse image of X, then G/Y is isomorphic to the $\mathfrak{e}$-group G^{σ}/X. Hence Y is a co-$\mathfrak{e}$-subgroup of G. Since $\mathfrak{e}$ is second stage pre-co-local, there exists a minimal co-$\mathfrak{e}$-subgroup B of G with $B \subseteq Y$. Clearly

$$B^\sigma \subseteq Y^\sigma = X^\sigma \subseteq A^\sigma.$$

By hypothesis, an isomorphism is induced by σ in AB. Thus $B \subseteq A$ may be deduced from $B^\sigma \subseteq A^\sigma$. Since B is a co-$\mathfrak{e}$-subgroup of G and A is a minimal co-$\mathfrak{e}$-subgroup of G, we conclude that $A = B$ and that therefore $X = A^\sigma$. Hence A^σ is a minimal co-$\mathfrak{e}$-subgroup of G^σ, proving (+).

It is an immediate consequence of (+) and Proposition 5.14 (B) that $[{}^{co}\mathfrak{h}_\mathfrak{e} G]^\sigma$ is a product of minimal co-$\mathfrak{e}$-subgroups of G^σ and that therefore

$$[{}^{co}\mathfrak{h}_\mathfrak{e} G]^\sigma \subseteq {}^{co}\mathfrak{h}_\mathfrak{e}(G^\sigma).$$

Combination of Proposition 5.14, (A) and Corollary 5.4 (c) gives

$${}^{co}\mathfrak{h}_\mathfrak{e}(G^\sigma) \subseteq [{}^{co}\mathfrak{h}_\mathfrak{e} G]^\sigma,$$

proving (++).

THE THIRD STAGE: CO-LOCALITY. The principal property of co-locality is described by the following

LEMMA 5.16. *The following properties of the property $\mathfrak{e}$ on $\mathfrak{D}$ are equivalent*:

(i) *If θ is a set of normal subgroups of G such that every intersection of finitely many subgroups in θ is a co-$\mathfrak{e}$-subgroup of G, then $\bigcap_{X \in \theta} X$ is a co-$\mathfrak{e}$-subgroup of G.*

(ii) *If θ is a set of co-$\mathfrak{e}$-subgroups of G such that $X \subseteq Y$ or $Y \subseteq X$ for X, Y in θ, then $\bigcap_{X \in \theta} X$ is a co-$\mathfrak{e}$-subgroup of G.*

The proof of this well-known equivalence may be indicated for the convenience of the reader. It is firstly clear that (ii) is just a weak form and hence a consequence of (i). If conversely (i) is false, then there exist sets θ of normal subgroups of G such that

(a) every intersection of finitely many subgroups in θ is a co-$\mathfrak{e}$-subgroup of G

and

(b) $\bigcap_{X \in \theta} X$ is not a co-$\mathfrak{e}$-subgroup of G.

Among these sets θ there is one ψ of minimal cardinality. ψ is infinite because of (a) and consequently there exists a set $\mathfrak{S}$ of subsets of ψ with the following three properties:

(c) $\Xi' \subseteq \Xi''$ or $\Xi'' \subseteq \Xi'$ for Ξ', Ξ'' in $\mathfrak{S}$.

(d) $\psi = \bigcup_{\Xi \in \mathfrak{S}} \Xi$ is the join of the sets Ξ in $\mathfrak{S}$.

(e) Every set in $\mathfrak{S}$ meets requirement (a) and has cardinality less than the cardinality of ψ.

Because of (e) and the minimality of ψ we have

(f) $\bigcap_{X \in \Xi} X$ is a co-$\mathfrak{e}$-subgroup of G for every Ξ in $\mathfrak{S}$.

If we denote now by Φ the set of all the $\bigcap_{X \in \Xi} X$ for Ξ in $\mathfrak{S}$, then it follows from (c) and (f) that every subgroup Y in Φ is a co-$\mathfrak{e}$-subgroup of G and that $X \subseteq Y$ or $Y \subseteq X$ for X, Y in Φ. Because of (d) we have

$$\bigcap_{X \in \Phi} X = \bigcap_{Y \in \psi} Y.$$

But this is, by (b), not a co-$\mathfrak{e}$-subgroup of G; and so (ii) is not satisfied either. Hence (i) is a consequence of (ii).

The property $\mathfrak{e}$ is termed *co-local,* if

(1) $\mathfrak{e}$ *is epimorphism-inherited*

and

(2) $\mathfrak{e}$ *meets the equivalent requirements* (i) *and* (ii) *of Lemma* 5.16.

It is clear that (2) may be omitted whenever the minimum condition is satisfied by the co-$\mathfrak{e}$-subgroups of the groups in $\mathfrak{D}$.

PROPOSITION 5.17. *The following properties of the property* $\mathfrak{e}$ *on* $\mathfrak{D}$ *are equivalent*:

(i) $\mathfrak{e}$ *is co-local.*

(ii) (a) $\mathfrak{e}$ *is second stage pre-co-local.*
(b) *If the group* G *possesses a set* θ *of co-*$\mathfrak{e}$*-subgroups such that* θ *contains* $X \cap Y$ *with* X *and* Y *and such that* $1 = \bigcap_{X \in \theta} X$, *then the number of minimal co-*$\mathfrak{e}$*-subgroups of* G *is finite.*

Proof. Assume first the validity of (i) and consider a co-$\mathfrak{e}$-subgroup E of the group G. Denote by $\mathfrak{E}$ the set of all co-$\mathfrak{e}$-subgroups X of G with $X \subseteq E$. Because of Lemma 5.16 (ii), the Maximum Principle of Set Theory may be applied — as a Minimum Principle — on $\mathfrak{E}$. Hence there exists a minimal subgroup in $\mathfrak{E}$ and this shows that $\mathfrak{e}$ is second stage pre-co-local. Suppose next that θ is a set of co-$\mathfrak{e}$-subgroups of G such that $X \cap Y$ belongs to θ whenever X and Y belong to θ and $\bigcap_{X \in \theta} X = 1$.

If $X_1, \ldots, X_n$ are finitely many subgroups in θ, then it follows by complete induction that $\bigcap_{i=1}^{n} X_i$ belongs to θ and is consequently a co-$\mathfrak{e}$-subgroup of G. Application of Lemma 5.16 (i) shows that $1 = \bigcap_{X \in \theta} X$ is a co-$\mathfrak{e}$-subgroup of G. Hence G is an $\mathfrak{e}$-group so that 1 is the one and only one minimal co-$\mathfrak{e}$-subgroup of G. Thus we have derived (ii.b) and hence (ii) from (i).

Assume conversely the validity of (ii). Then $\mathfrak{e}$ is, by (ii.a), epimorphism-inherited. Consider next a set θ of co-$\mathfrak{e}$-subgroups of G such that $X \subseteq Y$ or $Y \subseteq X$ for every pair of subgroups X, Y in θ. Let

$$J = \bigcap_{X \in \theta} X \quad \text{and} \quad H = G/J.$$

Denote by θ^* the set of all subgroups X/J with X in θ. Because of

$$G/X \simeq (G/J)/(X/J) \quad \text{for} \quad X \text{ in } \theta,$$

every subgroup in θ^* is a co-$\mathfrak{e}$-subgroup of H and we have $U \subseteq V$ or $V \subseteq U$ for U, V in θ^*. Furthermore it is clear that

$$1 = \bigcap_{W \in \theta^*} W.$$

Application of (ii.b) shows that the number of minimal co-$\mathfrak{e}$-subgroups of H is finite. Denote by $M_1, \ldots, M_m$ the totality of minimal co-$\mathfrak{e}$-subgroups of H. Assume by way of contradiction that none of the M_i belongs to all the subgroups in θ^*. Then there exists to every i a subgroup S_i in θ^* which does not contain M_i; and among the S_i there is one T with $T \subseteq S_i$ for every i. Clearly T does not contain any of the M_i. But T is a co-$\mathfrak{e}$-subgroup of H and as such it contains by (ii.a) a minimal co-$\mathfrak{e}$-subgroup of H. This is a contradiction proving that one of the minimal co-$\mathfrak{e}$-subgroups of H, say M, belongs to all the subgroups S in θ^*. Then

$$M \subseteq \bigcap_{S \in \theta^*} S = 1,$$

proving that 1 is a co-$\mathfrak{e}$-subgroup of $H = G/J$ and that therefore J is a co-$\mathfrak{e}$-subgroup of G. This shows the validity of Lemma 5.16 (ii) and we have shown the co-locality of $\mathfrak{e}$.

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