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DEPENDENCE IN GROUPS*

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1. Introduction. We shall survey a few problems, and fewer results, concerning relations of dependence in the theory of groups. For the most part there is little difficulty in extending these problems, if not their solutions, to more general algebraic systems; we do not discuss such extensions here. We are guided largely by analogy with two classical theories of dependence: linear dependence in the theory of vector spaces and algebraic dependence in the theory of fields. There arise also certain analogies with the relation of dependence, or consequence, in mathematical logic.

2. A first analog of linear dependence. Linear dependence in a vector space V has two dual aspects. We may define a vector v to be *dependent* on a set U of vectors if v is in every subspace containing U, or, alternatively, if v vanishes under every homomorphism that annihilates U. If we take V to be a group instead of a vector space, these two relations, that $v \in \operatorname{gp} U$ (the subgroup of V generated by U) and that $v \in \operatorname{ngp} U$ (the normal subgroup of V defined by U), are no longer equivalent. We look first at the relation $v \in \operatorname{gp} U$.

One can parallel the introduction of matrices in linear algebra. Let F be a free group of rank m, with basis X, and G a free group of rank n, with basis Y. The homomorphisms T from F into G are associated uniquely with the choice of values $x_iT = t_i(y_1, \ldots, y_n)$ in G as images of the elements x_i of X. It is natural to regard the m-tuple $M = \langle t_1(\xi_1, \ldots, \xi_n), \ldots$ $\ldots, t_m(\xi_1, \ldots, \xi_n)\rangle$ as an m-by-n matrix, where the ξ_j are indeterminates, that is, constitute a basis for a free group Φ. If H is a further free group, of rank p and with basis Z, and S is a homomorphism from G into H with n-by-p matrix $N = \langle s_1(\xi_1, \ldots, \xi_p), \ldots, s_n(\xi_1, \ldots, \xi_p)\rangle$, then the composite homomorphism TS has the m-by-p matrix $P = \langle p_1, \ldots, p_m\rangle$

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where each $p_i = t_i\big(s_1(\xi_1, \ldots, \xi_p), \ldots, s_n(\xi_1, \ldots, \xi_p)\big)$. It is natural to define the product to be $MN = P$.

It need hardly be remarked that we cannot view the matrix M as a rectangular array, and there does not appear to be any reasonable definition of a transpose of M, or of a group dual to the group Φ. In consequence there is no metatheorem of duality between left and right in the multiplicative theory of these matrices, but only an imperfect heuristic principle.

The invertible m-by-m matrices form a group A_m, analogous to the general linear group. This group, which is naturally viewed as the automorphism group of the free group Φ, was studied by Nielsen [28], who gave a description of it by generators and relations. There remains much to be done by way of determining the structure of A_m, either as abstract group or, more interestingly, its geometrical properties as a group of transformations of Φ. The richness of A_m is indicated by the observation that if Φ is replaced by the free abelian group $\Phi/[\Phi, \Phi]$ then A_m is replaced by the unimodular group of degree m, which is indeed a quotient group of A_m. More generally, if N is any characteristic subgroup of Φ, one may enquire into the kernel and image of the natural map from A_m into the group of automorphisms of Φ/N. Among recent work in this area is that of Andreadakis [1], Bachmuth [2, 3], and Mostowski [24].

Left equivalence of two m-by-n matrices M and N is defined by the condition that $N = PM$ for some invertible P. More intrinsically, left equivalence of $M = \langle t_1, \ldots, t_m\rangle$ and $N = \langle s_1, \ldots, s_m\rangle$ means that, in Φ, $\mathrm{gp}\{t_1, \ldots, t_m\} = \mathrm{gp}\{s_1, \ldots, s_m\}$. Nielsen's proof of the Subgroup Theorem [29] provides a complete and effective solution of the problem of left equivalence. It also provides a reduction of arbitrary M to a (semi) canonical form, $N = \langle s_1, \ldots, s_r, 1, \ldots, 1\rangle$ where $\{s_1, \ldots, s_r\}$ is a basis for a free group. Otherwise put, the r-by-n submatrix $N_1 = \langle s_1, \ldots, s_r\rangle$ is left independent in the sense that $QN_1 = 0$ iff $Q = 0$, where 0 denotes ambiguously any matrix $0 = \langle 1, \ldots, 1\rangle$. Indeed, Nielsen's algorithm enables one to determine all left annihilators $QM = 0$ of arbitrary M.

Right equivalence, $N = MP$ for some invertible P, means that the m-tuple $M = \langle t_1, \ldots, t_m\rangle$ can be carried into the m-tuple $N = \langle s_1, \ldots, s_m\rangle$ by some automorphism P of the free group Φ. The problem of right equivalence, which is much more difficult than that of left, was completely and effectively solved by Whitehead [43, 44] (see also [32, 11]).

[We take this occasion to sharpen the statement of Whitehead's theorem as given in [11]. For this we view the t_i and s_i as cyclic words (or as conjugate classes in Φ), and write $L(M)$ for the sum of the lengths of the t_i (minimum length of an element in the conjugate class), with $L(N)$ analogous. Suppose that $N = MP$ for some automorphism P, and that $L(N)$ has its minimum value for all N related thus to M. Then, for some $k \geqslant 0$, we have $P = P_1 \ldots P_k$ where

(1) each P_i is a Whitehead automorphism: for some fixed h, each $\xi_j P_i$ is one of ξ_j, $\xi_j \xi_h$, $\xi_h^{-1}\xi_j$, $\xi_h^{-1}\xi_j\xi_h$;

(2) the integers $L(MP_1 \ldots P_i)$, $0 \leqslant i \leqslant k$, decrease strictly until the value $L(N)$ is attained, and thereafter remain constant.]

The concept of a right independent matrix M, such that $MQ = 0$ iff $Q = 0$ is rather trivial, but the problem of determining all right annihilators of M is that of finding all solutions $Q = \langle q_1, \ldots, q_n \rangle$ in a free group of the system of m equations $t_i(\xi_1, \ldots, \xi_n) = 1$, $1 \leqslant i \leqslant m$, which will be mentioned later. The question of the existence of solutions Q of the corresponding non homogeneous system, $MQ = N$, that is, of extending Whitehead's results from automorphisms to endomorphisms, has been undertaken by P. Schupp([1]). We shall return also to the related question of solving $MQ = M$, that is, of determining the subgroup of A_n leaving fixed the m elements $t_1, \ldots, t_m$.

Two sided equivalence, $N = PMQ$ for P and Q both invertible, means that the two subgroups of Φ, generated by the t_i and by the s_i, are equivalent under an automorphism of Φ. I know of no work on this problem, which was mentioned to me by M. Hall, neither do I know of any work on the perhaps more tempting problem of similarity, $N = PMP^{-1}$, that is, on the classification of endomorphisms of a free group.

Thus far we have considered only the multiplication of matrices. It is clear that we should define the sum of two m-by-n matrices $M = \langle t_i \rangle$ and $N = \langle t_i \rangle$ to be $M+N = \langle t_i s_i \rangle$. Under this non commutative addition the set R_m of all m-by-m matrices becomes a near ring. This near ring R (for m infinite) and its ideal theory have been studied extensively by the Neumanns [26, 27] and independently by Šmelkin [39], in connection with the semigroup of varieties (or equational classes) of groups. There are two essential points in this connection. First, for an arbitrary group G, let $V_M(G)$ be the normal subgroup of G generated by all instances (homomorphic images) of the t_i in G, and let V_M be the variety of all G such that $V_M(G) = 1$. Then the two sided ideal $RMR = (M)$ consists of all $N = \langle s_i \rangle$ where the s_i range independently over $V_M(\Phi)$. Second, the product of two such ideals is the (principal) ideal $(M)(N) = (P)$ where the variety V_P is the product $V_M V_N = V_P$ of the varieties V_M and V_N, comprising all extensions of a group G in V_M by a group H in V_N. The central result of their work is a unique factorization theorem for varieties: the semigroup of all varieties of groups is a free unitary semigroup with zero.

3. A second analog of linear dependence. We write $U \to v$ for this relation, that $v \in \operatorname{ngp} U$, to stress the similarity to the consequence relation in logic. Indeed, if we take a set of generators for the containing group G as logical constants, the study of this relation is just the restriction of the study of the elementary theory of G to the special class of those sentences that have the form of an equation. However, general methods from logic carry over very little, and even simple appearing propositions require considerable specific machinery from group theory for their proof.

The study of this relation is most naturally concerned with free groups, although there are reasons for trying to carry results over to relatively free groups, to free products, or to groups with perhaps a single special defining relation.

It was shown by Magnus [23] that the relation $u \to v$ is decidable. He showed also [22] that $u \leftrightarrow v$ is equivalent to conjugacy of u with v or v^{-1}. Karrass, Magnus, and Solitar [13] showed that $u \to v^n$ has no unexpected solutions. The well-known result of Novikov [30] and Boone [6] is that the relation $U \to v$ is not decidable, even for certain fixed and fairly small U.

We formulate two basic results in a slightly unfamiliar way, in order to emphasize their analogy with logic as well as the more obvious analogy with linear algebra. The first is the theorem of Schreier [37] on the existence of free products with amalgamation. Let the free group F have as basis the disjoint union of sets A, B, and C. Let $U \subseteq \mathrm{gp}(A \cup B)$, $V \subseteq \mathrm{gp}(B \cup C)$, and suppose that $\mathrm{ngp}\, U$ and $\mathrm{ngp}\, V$ have the same intersection with $\mathrm{gp} B$. If w is in $\mathrm{gp}(A \cup B)$ and $U \cup V \to w$, then $U \to w$. A corollary of this is the following analog [19] of a theorem of Craig [9]. For U and V as before, suppose that $U \to V$. Then there exists $W \subseteq \mathrm{gp} B$ such that $U \to W$ and $W \to V$.

The second basic result is the Freiheitssatz of Dehn and Magnus [21]. Its simplest form says that if every word equivalent to the element u of F contains a certain generator a, then every non trivial consequence of u contains a. By Schreier's theorem this can be greatly extended [21, 19]. For the next simplest case, let a and c be among a set of generators for F, and u and v elements of F such that u does not contain c while every conjugate of u contains a and that v does not contain a while every conjugate of v contains c. If $\{u, v\} \to w$ where w does not contain c, then $u \to w$. A refinement of this and related theorems, by Cohen and Lyndon [8], presents a close analogy with proof theory. The relation $U \to v$ holds iff v has a representation π as a product of factors $p_1, \ldots, p_n$, where each p_i is conjugate to u or u^{-1} for some u in U; it is natural therefore to call π a *proof* that $U \to v$. There are certain obvious transformations [31, 33] that carry any proof into a new proof of the same conclusion. Now, for u, v, and w as before, suppose given a proof π that $\{u, v\} \to w$. Then, by such transformations, π can be carried systematically into a proof π' that $u \to v$. A related result is that, if $u \to v$, then there exists, up to such rearrangements, only one proof of the fact. Theorems of this sort can be viewed as a sharpening of results concerning dependence in certain modules, and have cohomological implications.

4. Equations and forms in groups. Apart from the classical question of solutions of the equation $x^n = g$, for given G and g in G, the question of solving equations in groups stems, so far as I know, from the question raised by Tarski of the decidability of the elementary theory of free groups. With one exception, all results that I know of concern free groups. As a simple case of Tarski's problem, Vaught asked whether, for elements x, y, and z of a free group, $x^2y^2z^2 = 1$ implies that $xy = yx$. It has been shown [15, 4, 34, 38, 40, 20] that, for any $m, n, p \geqslant 2$, the equation $x^m y^n z^p = 1$ implies $xy = yx$. We shall come in a moment to a result of Baumslag which contains this.

We have mentioned already that the question of existence of solutions of a homogeneous system $MQ = 0$ is essentially trivial. Apart from the trivial solution, $Q = 0 = \langle 1, \ldots, 1 \rangle$, there will exist nearly trivial commutative solutions, with all the q_j in a cyclic group, provided that the matrix of the exponent sums e_{ij} of the ξ_j in the t_i has rank less than n. One wants some account of the complexity of the set of all solutions, but a full account seems no more possible in general than a full geometric description of an arbitrary algebraic variety. At least one can define an analog of dimension: we define the *rank* $\boldsymbol{r}(M)$ of the system M to be the maximum of the ranks of free subgroups $\mathrm{gp}\{q_1, \ldots, q_n\}$ of Φ, generated by the components of a solution Q of $MQ = 0$. In the case of a single equation $t(\xi_1, \ldots, \xi_n) = 1$, we write $\boldsymbol{r}(t)$ for the rank of the system. Since $\boldsymbol{r}(t)$ is altered if we view t as depending vacuously on another generator ξ_{n+1}, it is sometimes preferable to pass to the nullity $\boldsymbol{n}(t) = n - \boldsymbol{r}(t)$, which is an invariant of the word t alone, and can be thought of as measuring the number of degrees of constraint imposed by t.

Unless $t(\xi_1, \ldots, \xi_n)$ reduces to the trivial element of Φ, elements $q_1, \ldots, q_n$ satisfying the relation $t(q_1, \ldots, q_n) = 1$ cannot, by Nielsen's argument [29], generate a subgroup of rank as great as n, whence $\boldsymbol{n}(t) \geqslant 1$. On the other hand, if t is primitive, that is, a member of some basis for Φ, then setting $t = 1$ and leaving the remaining elements of this basis unchanged yields a solution of rank $n-1$, whence $\boldsymbol{n}(t) = 1$. Steinberg [42] has shown that $\boldsymbol{n}(t) = 1$ iff $u \to t$ for some primitive u. The solution of Vaught's problem was first obtained in the equivalent form of showing that, for $t = \xi_1^2 \xi_2^2 \xi_2^3$, one has $\boldsymbol{n}(t) > 1$.

Baumslag's result [5] deals with a word of the form $t = s(\xi_1, \ldots, \xi_{n-1})\xi_n^k$, where s is neither primitive nor a proper power, $s = u^h$, $h > 1$, of another word, and where $k > 1$, and asserts then that $\boldsymbol{n}(t) > 1$. An extension in turn of Baumslag's result has been obtained by Steinberg

[42], and rests on an interesting lemma, giving broad conditions under which a relation $u \to t\big(s_1(\xi_{11}, \ldots, \xi_{1p}), \ldots, s_n(\xi_{n1}, \ldots, \xi_{np})\big)$ implies that u has the form $u = u(\xi_{11}, \ldots, \xi_{np}) = v(s_1, \ldots, s_n)$.

Quadratic words t, in which each ξ_i occurs exactly twice, with exponent $+1$ or -1, provide a manageable generalization of the case $t = \xi_1^2 \xi_2^2 \xi_3^2$. It is easy to see that each such word is equivalent either to a product of squares or to a product of commutators, and, from this, that $\boldsymbol{r}(t) = [n/2]$; see [45]. Steinberg [41] has extended these considerations to free nilpotent groups. Analogy with the classical groups suggests examining the subgroup $\boldsymbol{B}$ of $\boldsymbol{A}_n$ leaving fixed $t = \Pi_1^n \xi_i^2$ or $t = \Pi_1^k [\xi_i, \eta_i]$ for $n = 2k$. Whitehead's algorithm provides in principle a method for determining the group $\boldsymbol{B}$ leaving fixed any m-tuple $M = \langle t_1, \ldots, t_m \rangle$. Zieschang [46] has determined $\boldsymbol{B}$ for a single $t = \xi_1^{e_1} \ldots \xi_n^{e_n}$, provided all $e_i > 2$; in the most interesting case, that all e_i are equal, B is essentially the Artin braid group. He has also determined the group B_2 leaving fixed $t = \xi_1^2 \xi_2^2$, where a new generator appears in addition to those to be expected from the case $e_i > 2$. Frisch is investigating the conjecture, based on examination of analogous $\boldsymbol{B}_3$ and $\boldsymbol{B}_4$, that no further unexpected generators appear for large n (2).

An extension of the problem of solving $MQ = 0$ involves equations with constant coefficients, for example, to solve $t(a_1, \ldots, a_k; \xi_1, \ldots, \xi_n) = 1$ in a group G, where $a_1, \ldots, a_k$ are given elements of G. This has been solved [16, 17] in the case that G is free and $n = 1$. For a simple case, let $t = a_1 \xi_1 a_1^{-1} \xi_1^{-1}$ where a_1 is a member of a basis for G; then the general solution q_1 is $q_1 = a_1^{\nu}$, for ν an arbitrary integer. In the general case mentioned, the full solution is given by a finite number of expression, each containing a finite number of parameters ranging over the integers. For G a Lie group, certain conclusions of this general nature follow from the work of Gerstenhaber and Rothaus [10].

5. The problem of adjunction. The last matter we shall discuss is the solution of a system of equations $t_i(a_1, \ldots, a_k;\ \xi_1, \ldots, \xi_n)$, with coefficients a_i in a group G, in some group G' containing G. It follows from Schreier's theorem that one can always adjoin a root, that is, solve an equation $\xi^k = a$, provided $k \neq 0$. A theorem of Higman, Neumann and Neumann [12] shows that any isomorphism between two subgroups of G can be effected under conjugation by an element of a larger group; in particular, $a_1 \xi_1 a_2 \xi_1^{-1} = 1$ has a solution iff a_1 and a_2 have the same order. The broadest result in this area was obtained by Gerstenhaber and Rothaus [10] by topological means. If G can be embedded in a compact connected Lie group, then n equations in n unknowns have a solution, provided the determinant of the exponent sums e_{ij} of the ξ_j in the t_i does not vanish. Levin [14] has given a group theoretic proof for the case that $n = 1$ and t contains ξ_1 only with positive exponents.

A natural approach to the problem of adjunction, say for $m = n = 1$, is to form the free product $P = G*Q$ of G with an infinite cyclic group on generator q, to let N be the normal closure in P of the element $t(a_1, \ldots, \ldots, a_k; q)$, and to ask whether N has trivial intersection with G. The theorem of Gerstenhaber and Rothaus suggests that $G \cap N = 1$ provided the exponent sum e_{11} does not vanish. The example $t = a_1 q a_2 q^{-1}$, where $e_{11} = 0$, and $G \cap N$ need not be trivial, shows that we cannot expect for free products a literal analog of the Freiheitssatz; however, this is essentially the only counterexample that I know of. Schiek [35, 36] has studied the case of exponent sum zero, where a further reduction is possible. In this case t lies in the free product of a certain finite set of the groups $G_n = q^{-h}Gq^h$, and one wants to know whether t has any non trivial consequence lying in the free product of any proper subset of these groups. Schiek has observed that a solution exists in case the indices h on the factors of t, in order of occurrence in t, fall into two separable blocks. In various other cases, suitable hypotheses on the indices h enable one to deduce the existence of a solution by a theorem of Britton [7]. In all, however, it does not seem likely that the methods available at present can yield a full solution of the adjunction problem.

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Footnotes (*added in proof*)

(1) Even the general question of whether given w is an instance (endomorphic image) $w = t(q_1, \dots, q_n)$ of given $t(\xi_1, \dots, \xi_n)$ seems very difficult. A simple, but not obvious, answer for the case that $t = [\xi_1, \xi_2]$, a simple commutator, is given in the following paper: M. T. Wicks, *Commutators in free products*, Journal of the London Mathematical Society 37 (1962), p. 433-444. Schupp has extended this to obtain a method for deciding if given w is an instance of a given compound commutator t.

(3) As pointed out by Zieschang, further information on these groups may be derived from the following papers:

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