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THE CATEGORY OF NON-INDEXED ALGEBRAS AND WEAK HOMOMORPHISMS

BY

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This paper deals with the category $\mathcal{N}$ of non-indexed algebras and weak homomorphisms. The monomorphisms and epimorphisms are described, and some questions concerning the existence of limits and colimits are solved. Some other remarks about $\mathcal{N}$ are also presented.

Section 1 includes the notation and definitions. In the next sections the individual basic categorical notions in $\mathcal{N}$ are successively studied. More information about non-indexed algebras and weak homomorphisms can be found in [1] and [2].

The impulse to write this paper was given by K. Głazek's report during the summer school on general algebra in Czechoslovakia in 1974. The following problem arose in the discussion: "Do the products in $\mathcal{N}$ exist and if they did, how would they look like?"

Some parts of this work were prepared during the second-named author's stay in Poland at the Copernicus scholarship. This enabled the author to have valuable conversations with K. Głazek and J. Michalski, the authors of [1] and [2].

1. Preliminaries. Let N be the set of all natural numbers (including zero) and $N^+ = N - \{0\}$.

Let A be a set. We will use the following notation:

$O^n(A)$ denotes the set of all n-ary operations on A $(n \in N)$, $O(A)$ — the set of all operations on A, i.e.

$$O(A) = \bigcup_{n \in N} O^n(A),$$

and $|A|$ — the cardinal number of A.

We define the operations ${}^A p_i^n \in O^n(A)$ for $n \in N^+$ and $i \in \{1, \ldots, n\}$ by the rule

$${}^A p_i^n(a_1, \ldots, a_n) = a_i \quad \text{for all } a_1, \ldots, a_n \in A.$$

For simplicity, we will often write only p_i^n.

If $f \in O^m(A)$ and $f_1, \ldots, f_m \in O^n(A)$, we define $f(f_1, \ldots, f_m) \in O^n(A)$ by the rule

$$\big(f(f_1, \ldots, f_m)\big)(a_1, \ldots, a_n) = f\big(f_1(a_1, \ldots, a_n), \ldots, f_m(a_1, \ldots, a_n)\big)$$

for all $a_1, \ldots, a_n \in A$.

An ordered pair (A, Φ) is called an *algebra* if A is a set and $\Phi \subseteq O(A)$.

For $F \subseteq O(A)$, we put $F^n = F \cap O^n(A)$.

$F \subseteq O(A)$ is called a *clone* on A if

(i) ${}^Ap_i^n \in F$ for all $n \in N^+$ and $i \in \{1, \ldots, n\}$,

(ii) $f \in F^m$ and $f_1, \ldots, f_m \in F^n$ implies $f(f_1, \ldots, f_m) \in F$.

We say that an algebra (A, F) is *non-indexed* if F is a clone on A.

Note that $(\emptyset, \{\emptyset\})$ is also a non-indexed algebra.

It is evident that the intersection of any family of clones on a set A is a clone as well. Therefore, for any $\Phi \subseteq O(A)$ the smallest clone F on A containing Φ exists; we denote it by ${}^A[\Phi]$ (briefly, $[\Phi]$).

For $\Phi = \{f\}$ we write only ${}^A[f]$.

Let $\mathfrak{A} = (A, \Phi)$ be an algebra and let $X \subseteq A$. The set X is called *closed* with respect to f if $x_1, \ldots, x_n \in X$ implies $f(x_1, \ldots, x_n) \in X$. In this case we denote the restriction of f on X by $f|X$. If X is closed with respect to f for every $f \in \Phi$, then X is called a *subuniverse* of $\mathfrak{A}$ and the algebra $(X, \Phi|X)$, where $\Phi|X = \{f|X \mid f \in \Phi\}$, is called a *subalgebra* of $\mathfrak{A}$.

If q is a congruence on $\mathfrak{A}$, then the class of q which includes the element $a \in A$ is denoted by a^q and the mapping $a \mapsto a^q$ by nat q.

Note that the algebras (A, Φ) and $(A, {}^A[\Phi])$ have the same subuniverses and the same congruence relations.

Let A and B be sets and let $\alpha : A \to B$ be any mapping. For $f \in O^n(A)$, $g \in O^n(B)$, $n \in N$, we write $(f, g) \in R_\alpha$ if

$$\alpha f(a_1, \ldots, a_n) = g(\alpha a_1, \ldots, \alpha a_n) \quad \text{for all } a_1, \ldots, a_n \in A.$$

Let $\mathfrak{A} = (A, F)$ and $\mathfrak{B} = (B, G)$ be non-indexed algebras. A mapping α: $A \to B$ is called a *weak homomorphism* if for all $n \in N$ we have:

(i) for every $f \in F^n$ there exists a $g \in G^n$ such that $(f, g) \in R_\alpha$,

(ii) for every $g \in G^n$ there exists an $f \in F^n$ such that $(f, g) \in R_\alpha$.

Evidently, in this case $\alpha A = \{\alpha a \mid a \in A\}$ is a subuniverse of $\mathfrak{B}$.

Any bijective weak homomorphism is called a *weak isomorphism*.

For any non-indexed algebra $\mathfrak{A} = (A, F)$, its subalgebras and its factor-algebras are non-indexed algebras, and natural embeddings of subalgebras into $\mathfrak{A}$ and natural mappings of $\mathfrak{A}$ onto factor-algebras are weak homomorphisms.

Let A and B be sets and let α: $A \to B$ be any mapping. We define the equivalence relation $\ker \alpha$ on A as follows:

$$(a, a') \in \ker \alpha \text{ iff } \alpha a = \alpha a' \quad \text{for all } a, a' \in A.$$

If $\Phi \subseteq O(A)$, α is onto, and $\ker \alpha$ is a congruence relation on (A, Φ), then for every $f \in \Phi$ there exists exactly one $g \in O(B)$ such that $(f, g) \in R_\alpha$; we denote it by $\bar{\alpha} f$.

If $\mathfrak{A} = (A, F)$ and $\mathfrak{B} = (B, G)$ are non-indexed algebras and $\alpha : \mathfrak{A} \to \mathfrak{B}$ is a weak homomorphism, then $\ker \alpha$ is a congruence relation on $\mathfrak{A}$. If α is onto, $\mathfrak{C} = (C, H)$ is another non-indexed algebra, and $\alpha' : \mathfrak{A} \to \mathfrak{C}$ a weak homomorphism which satisfies $\ker \alpha \subseteq \ker \alpha'$, then the unique $\beta : B \to C$, for which $\beta\alpha = \alpha'$, is a weak homomorphism of $\mathfrak{B}$ into $\mathfrak{C}$. If q is a congruence relation on $\mathfrak{A}$, which satisfies $\ker \alpha \subseteq q$, we denote the congruence relation induced by q on $\mathfrak{B}$ by $\tilde{\alpha} q$ ($(b_1, b_2) \in \tilde{\alpha} q$ iff $(a_1, a_2) \in q$ for all $a_1 \in \alpha^{-1} b_1$ and $a_2 \in \alpha^{-1} b_2$).

If an algebra $\mathfrak{B} = (B, \Psi)$ is a subalgebra of the algebra $\mathfrak{A} = (A, \Phi)$ and q is any congruence relation on $\mathfrak{A}$, we denote the restriction of q to B by $q \mid B$. Evidently, it is a congruence relation on $\mathfrak{B}$. The algebra $\mathfrak{B}$ is called a *proper subalgebra* of $\mathfrak{A}$ relative to q if the following condition does not hold:

For every $a \in A$ there exists a $b \in B$ such that $(a, b) \in q$.

For categorical notions see [3]. For our purposes, it is convenient to denote morphisms being the same mapping by the same symbol.

Let $\mathscr{N}$ be the category defined as follows:

a. objects are all non-indexed algebras,

b. morphisms are all weak homomorphisms,

c. the composition is the ordinary composition of mappings.

Let

$$\mathscr{N}_0 = \{(A, F) \in \mathscr{N} \mid F^0 \neq \emptyset\} \quad \text{and} \quad \mathscr{N}_1 = \{(A, F) \in \mathscr{N} \mid F^0 = \emptyset\}.$$

Let $\mathscr{N}_0$ and $\mathscr{N}_1$ denote also the full subcategories of the category $\mathscr{N}$ defined by the classes $\mathscr{N}_0$ and $\mathscr{N}_1$, respectively.

Evidently, if $\mathfrak{A} \in \mathscr{N}_0$ and $\mathfrak{B} \in \mathscr{N}_1$, then

$$\hom_{\mathscr{N}}(\mathfrak{A}, \mathfrak{B}) = \hom_{\mathscr{N}}(\mathfrak{B}, \mathfrak{A}) = \emptyset.$$

One-element non-indexed algebras are terminal objects in $\mathscr{N}_0$ and in $\mathscr{N}_1$. $(\emptyset, \{\emptyset\})$ is the initial object in $\mathscr{N}_1$.

Therefore, $\mathscr{N}$ is formed by two disjoint connected components $\mathscr{N}_0$ and $\mathscr{N}_1$.

Let I be any set and let $\big(\mathfrak{A}_i = (A_i, F_i)\big)_{i \in I}$ be a family of non-indexed algebras. The non-indexed product $\mathfrak{A} = (A, F)$ of the family $(\mathfrak{A}_i)_{i \in I}$ is defined as follows:

$$A = \prod_{i \in I} A_i, \quad F^n = \prod_{i \in I} F_i^n \text{ for } n \in N, \quad F = \bigcup_{n \in N} F^n,$$

$$(f_i)_{i \in I}\big((a_i^1)_{i \in I}, \ldots, (a_i^n)_{i \in I}\big) = \big(f_i(a_i^1, \ldots, a_i^n)\big)_{i \in I}$$

$$\text{for all } n \in N,\ f_i \in F_i^n \text{ and } a_i^1, \ldots, a_i^n \in A_i.$$

Evidently, $\mathfrak{A}$ is a non-indexed algebra.

We define the projections ε_i: $A \to A_i$ for $i \in I$ by the rule

$$\varepsilon_i\big((a_j)_{j\in I}\big) = a_i \quad \text{for all } (a_j)_{j\in I} \in A.$$

Obviously, $\ker \varepsilon_i$ is a congruence relation on $\mathfrak{A}$ for all $i \in I$, and if $\mathfrak{A}_i \in \mathcal{N}_1$ or $\mathfrak{A}, \mathfrak{A}_i \in \mathcal{N}_0$, then ε_i is a weak homomorphism of $\mathfrak{A}$ into $\mathfrak{A}_i$.

2. Monomorphisms and epimorphisms.

LEMMA 1. *Let $\mathfrak{A} = (A, \Phi)$ and $\mathfrak{B} = (B, \Psi)$ be algebras and let α: $A \to B$ be any mapping which satisfies:*

(i) *for every $f \in \Phi$ there exists a $g \in {}^B[\Psi]$ such that $(f, g) \in R_\alpha$,*

(ii) *for every $g \in \Psi$ there exists an $f \in {}^A[\Phi]$ such that $(f, g) \in R_\alpha$.*

Then α is a weak homomorphism of the non-indexed algebra $(A, {}^A[\Phi])$ into the non-indexed algebra $(B, {}^B[\Psi])$.

Proof. $({}^Ap_i^n, {}^Bp_i^n) \in R_\alpha$ for all $n \in N^+$ and $i \in \{1, \ldots, n\}$, and the relations

$$f \in [\Phi]^m, \quad f_1, \ldots, f_m \in [\Phi]^n, \quad g \in [\Psi]^m, \quad g_1, \ldots, g_m \in [\Psi]^n,$$
$$(f, g), (f_1, g_1), \ldots, (f_m, g_m) \in R_\alpha$$

mply

$$\big(f(f_1, \ldots, f_m), g(g_1, \ldots, g_m)\big) \in R_\alpha.$$

Now it is easy to see that (i) and (ii) of our lemma imply the conditions (i) and (ii) from the definition of weak homomorphism.

PROPOSITION 1. *If $\mathfrak{A} = (A, F)$, $\mathfrak{B} = (B, G) \in \mathcal{N}$, and α: $\mathfrak{A} \to \mathfrak{B}$, is a weak homomorphism, then the following conditions are equivalent:*

(i) *α is a monomorphism in $\mathcal{N}$,*

(ii) *α is one-to-one.*

Proof. The part (ii) $\Rightarrow$ (i) is satisfied in every concrete category.

Assuming that α is not one-to-one, we prove that α is not a monomorphism in $\mathcal{N}$. There exist $a_1, a_2 \in A$, $a_1 \neq a_2$, such that $\alpha a_1 = \alpha a_2$. Let $\overline{\mathfrak{A}}$ be the algebra of operations of $\mathfrak{A}$ (see [1]), i.e. $\overline{\mathfrak{A}} = (F, \bar{F})$, where $\bar{F} = \{\bar{f} \mid f \in F\}$ and, for $f \in F^n, f_1 \in F^{m_1}, \ldots, f_n \in F^{m_n}$, $\bar{f}(f_1, \ldots, f_n) \in F^{m_1+\ldots+m_n}$ is defined by

$$\big(\bar{f}(f_1, \ldots, f_n)\big)(a_{11}, \ldots, a_{1m_1}, \ldots, a_{n1}, \ldots, a_{nm_n})$$
$$= f\big(f_1(a_{11}, \ldots, a_{1m_1}), \ldots, f_n(a_{n1}, \ldots, a_{nm_n})\big) \quad \text{for all } a_{ij} \in A.$$

For $a \in A$ we define φ_a: $F \to A$ by the rule

$$\varphi_a f = f(\underbrace{a, \ldots, a}_{n \text{ times}}) \quad \text{for } f \in F^n \text{ and } n \in N.$$

Since $(\bar{f}, f) \in R_{\varphi_a}$, the conditions of Lemma 1 are satisfied. Therefore, φ_a is a weak homomorphism of the non-indexed algebra $\mathfrak{X} = (F, {}^F[\bar{F}])$

into $\mathfrak{A}$. In this case $\varphi_{a_1} \neq \varphi_{a_2}$, but $a\varphi_{a_1} = a\varphi_{a_2}$, which completes the proof of the proposition.

PROPOSITION 2. *Let $\mathfrak{A} = (A, F)$ be a non-indexed algebra. Then for every $X \subseteq A$ there exist a non-indexed algebra $\mathfrak{B}$ and weak homomorphisms a_1, a_2: $\mathfrak{A} \to \mathfrak{B}$ such that*

$$X = \{a \in A \mid a_1 a = a_2 a\}.$$

Proof. We write

$$B = (X \times \{0\}) \cup ((A - X) \times \{1\}) \cup ((A - X) \times \{2\}),$$

$$A_i = (X \times \{0\}) \cup ((A - X) \times \{i\}) \quad \text{for } i = 1 \text{ and } 2,$$

and define the mappings a_i: $A \to B$ for $i = 1$ and 2 by

$$a_i a = \begin{cases} (a, 0) & \text{for } a \in X, \\ (a, i) & \text{for } a \in (A - X). \end{cases}$$

Since $\ker a_1$ and $\ker a_2$ are congruence relations on $\mathfrak{A}$, we may define the clones $\bar{a}_i F$ on A_1 and A_2, respectively ($\bar{a}_i F = \{\bar{a}_i f \mid f \in F\}$ for $i = 1$ and 2). Now,

$$G = \{g \in O(B) \mid g|A_i \in \bar{a}_i F \text{ for } i = 1 \text{ and } 2\}$$

is a clone on B and a_1 and a_2 are weak homomorphisms of $\mathfrak{A}$ into $\mathfrak{B} = (B, G)$. Evidently, $X = \{a \in A \mid a_1 a = a_2 a\}$.

PROPOSITION 3. *If $\mathfrak{A}$ and $\mathfrak{B} \in \mathcal{N}$, and $a: \mathfrak{A} \to \mathfrak{B}$ is a weak homomorphism, then*

(i) *a is an epimorphism in $\mathcal{N}$ iff a is onto,*

(ii) *a is an isomorphism in $\mathcal{N}$ iff a is a bijection.*

Condition (i) follows from Proposition 2 and the proof of (ii) is trivial.

3. Limits.

PROPOSITION 4. *Let I be a set and let $\mathfrak{A}_i = (A_i, F_i) \in \mathcal{N}$ for all $i \in I$. If the family $(\mathfrak{A}_i)_{i \in I}$ has a (categorical) product in $\mathcal{N}$, then the non-indexed product $\mathfrak{A} = (A, F)$ of this family, together with the family of projections $(\varepsilon_i)_{i \in I}$, is also a product of the family $(\mathfrak{A}_i)_{i \in I}$.*

Proof. Let us suppose that $(\mathfrak{X} = (X, G), (\pi_i)_{i \in I})$ is a product of the family $(\mathfrak{A}_i)_{i \in I}$. Now ε_i is a weak homomorphism for every $i \in I$. Thus, there exists exactly one weak homomorphism φ: $\mathfrak{A} \to \mathfrak{X}$ such that $\pi_i \varphi = \varepsilon_i$ for all $i \in I$.

We define a: $X \to A$ as $ax = (\pi_i x)_{i \in I}$. Evidently, $\varepsilon_i a = \pi_i$ for all $i \in I$.

We need only to prove that φa is a weak homomorphism. (Then $\varphi a = \mathrm{id}_X$, $a\varphi = \mathrm{id}_A$, and hence φ is an isomorphism in $\mathcal{N}$.)

At first we prove that if $(g, f_i) \in R_{\pi_i}$ for $i \in I$ and $((f'_i)_{i\in I}, g) \in R_\varphi$, then $f_i = f'_i$ for all $i \in I$.

Let g, f_i and f'_i for $i \in I$ be n-ary. For all $x_1, \dots, x_n \in X$ and for all $i \in I$ the following equalities are satisfied:

$$\begin{aligned} f_i(\pi_i x_1, \dots, \pi_i x_n) &= f_i(\pi_i \varphi a x_1, \dots, \pi_i \varphi a x_n) \\ &= \pi_i g(\varphi a x_1, \dots, \varphi a x_n) = \pi_i \varphi (f'_i)_{i\in I}(a x_1, \dots, a x_n) \\ &= f'_i(\pi_i x_1, \dots, \pi_i x_n). \end{aligned}$$

Now, for every $i \in I$, π_i is onto since $\varepsilon_i = \pi_i \varphi$ is onto. Thus $f_i = f'_i$ for all $i \in I$. Further, for all $x_1, \dots, x_n \in X$ we have

$$\begin{aligned} \varphi a g(x_1, \dots, x_n) &= \varphi\big(\pi_i g(x_1, \dots, x_n)\big)_{i\in I} \\ &= \varphi\big(f_i(\pi_i x_1, \dots, \pi_i x_n)\big)_{i\in I} = \varphi\big(f'_i(\pi_i x_1, \dots, \pi_i x_n)\big)_{i\in I} \\ &= \varphi(f'_i)_{i\in I}\big((\pi_i x_1)_{i\in I}, \dots, (\pi_i x_n)_{i\in I}\big) \\ &= g\big(\varphi(\pi_i x_1)_{i\in I}, \dots, \varphi(\pi_i x_n)_{i\in I}\big) = g(\varphi a x_1, \dots, \varphi a x_n). \end{aligned}$$

Thus, for every $g \in G$ we obtain $(g, g) \in R_{\varphi a}$ and, therefore, φa is a weak homomorphism.

PROPOSITION 5. *Let I be a set and let $\mathfrak{A}_i = (A_i, F_i) \in \mathcal{N}$ for $i \in I$. Let $\mathfrak{A}_i \in \mathcal{N}_0$ for all $i \in I$ or $\mathfrak{A}_i \in \mathcal{N}_1$ for all $i \in I$. Let $\mathfrak{A} \in (A, F)$ denote the non-indexed product of this family and let $(\varepsilon_i)_{i\in I}$ be the family of projections. Then a product of the family $(\mathfrak{A}_i)_{i\in I}$ exists iff there exists no clone F' on A with* (i) *$F' \subset F$ and* (ii) *$\bar{\varepsilon}_i F' = F_i$ for all $i \in I$.*

Proof. 1. Let us suppose that there exists a clone F' on A with properties (i) and (ii). We put $\mathfrak{A}' = (A, F')$. Now the projection ε_i is a weak homomorphism of $\mathfrak{A}'$ into $\mathfrak{A}_i$ for all $i \in I$. Since id_A is not a weak homomorphism of $\mathfrak{A}'$ into $\mathfrak{A}$ and id_A is the unique mapping $a\colon A \to A$ for which $\varepsilon_i = \varepsilon_i a$ for all $i \in I$, $\big(\mathfrak{A}, (\varepsilon_i)_{i\in I}\big)$ is not a product of the family $(\mathfrak{A}_i)_{i\in I}$. By Proposition 4 the product of the family $(\mathfrak{A}_i)_{i\in I}$ does not exist.

2. Let us suppose that the product of the family $(\mathfrak{A}_i)_{i\in I}$ does not exist. Then there exist a non-indexed algebra $\mathfrak{X} = (X, G)$ and a family of weak homomorphisms $(\pi_i\colon \mathfrak{X} \to \mathfrak{A}_i)_{i\in I}$ such that there exists no weak homomorphism $a\colon \mathfrak{X} \to \mathfrak{A}$ for which $\varepsilon_i a = \pi_i$ for all $i \in I$ or there exists more than one weak homomorphism with these properties.

If we define $a\colon X \to A$ by $ax = (\pi_i x)_{i\in I}$, then a is the unique mapping for which $\varepsilon_i a = \pi_i$ for all $i \in I$.

Now we know that a is not a weak homomorphism. But if $(g, f_i) \in R_{\pi_i}$ for $i \in I$, then $\big(g, (f_i)_{i\in I}\big) \in R_a$. Thus there exists a family $(f_i^0)_{i\in I} \in F$ such that there exists no $g \in G$ with $\big(g, (f_i^0)_{i\in I}\big) \in R_a$. But

$$F' = \big\{(f_i)_{i\in I} \in F \mid \text{there exists a } g \in G \text{ such that } \big(g, (f_i)_{i\in I}\big) \in R_a\big\}$$

is a clone on A.

Finally, F' satisfies (ii) since π_i is a weak homomorphism for all $i \in I$.

COROLLARY 1. *Let I be a set, $\mathfrak{A}_i \in \mathcal{N}$ for all $i \in I$. Any of the following conditions implies that the product of the family $(\mathfrak{A}_i)_{i\in I}$ does not exist:*

(i) *There exists $J \subseteq I$ with $|J| \geqslant 2$ such that a product of the family $(\mathfrak{A}_i)_{i\in J}$ does not exist.*

(ii) *$I = \{1, 2\}$ and there exists a weak homomorphism of $\mathfrak{A}_1$ into $\mathfrak{A}_2$ which is not constant.*

(iii) *For every $i \in I$, $\mathfrak{A}_i \in \mathcal{N}_0$, and for at least two $i \in I$, $\mathfrak{A}_i$ has an operation which is neither a constant nor a projection.*

(iv) *For every $i \in I$, $\mathfrak{A}_i \in \mathcal{N}_1$, and for infinitely many $i \in I$, $\mathfrak{A}_i$ has an operation which is not a projection.*

To get the proofs, it suffices to find suitable clones F' of Proposition 5. It is easy and we omit it.

Note that if there exist $i, j \in I$ such that $\mathfrak{A}_i \in \mathcal{N}_0$ and $\mathfrak{A}_j \in \mathcal{N}_1$, then the product of the family $(\mathfrak{A}_i)_{i\in I}$ does not exist.

Example 1. Let $(A, \{\cdot\})$ be a semigroup with neutral element e which satisfies $a\cdot a = e$ for all $a \in A$. Let $(B, \{q\})$ be a unary algebra with unique operation q which has the following property:

There exists an $m \in N^+$ such that $q^m(b) = b$ for all $b \in B$. We put $F = {}^A[\cdot]$ and $G = {}^B[q]$. Then the non-indexed algebras $\mathfrak{A} = (A, F)$ and $\mathfrak{B} = (B, G)$ have a product in the category $\mathcal{N}$.

Indeed, any $g \in G^2$ is of the form

$$g(b_1, b_2) = q^k(b_i) \quad \text{for a suitable } k \in N,\ i \in \{1, 2\}.$$

We write

$$H^n = F^n \times G^n \quad \text{and} \quad H = \bigcup_{n\in N} H^n.$$

Obviously,

$$^{A\times B}[({}^Ap_1^2, q^k(b_2))] = {}^{A\times B}[({}^Ap_2^2, q^k(b_1))] = H.$$

Thus

$$^{A\times B}[(\cdot, q^k(b_1)), ({}^Ap_1^2, q^l(b_1))] = H \quad \text{for any } k, l \in N,$$

since

$$\cdot({}^Ap_1^2, \cdot) = {}^Ap_2^2, \quad q^k(b_1)(q^l(b_1), q^k(b_1)) = q^{k+l}(b_1).$$

Analogously we have

$$^{A\times B}[(\cdot, q^k(b_2)), ({}^Ap_2^2, q^l(b_2))] = H \quad \text{for any } k, l \in N.$$

Thus there exists no clone F' on $A\times B$ satisfying (i) and (ii) from Proposition 5. Following this proposition the non-indexed algebras $\mathfrak{A}$ and $\mathfrak{B}$ have a product in $\mathcal{N}$.

Let I be a set. Let $\mathfrak{A}_i = (A_i, F_i), \mathfrak{B} = (B, G) \in \mathcal{N}$ for $i \in I$ and let a_i: $\mathfrak{A}_i \to \mathfrak{B}$ for $i \in I$ be a weak homomorphism. By a *pullback* of the family $(a_i\colon \mathfrak{A}_i \to \mathfrak{B})_{i\in I}$ we mean a limit of this diagram.

We set

$$A = \Big\{(a_i)_{i\in I} \in \prod_{i\in I} A_i \mid a_i\alpha_i = a_j\alpha_j \quad \text{for all } i, j \in I\Big\},$$

$$(F')^n = \Big\{(f_i)_{i\in I} \in \prod_{i\in I} F_i^n \mid \text{there exists a } g \in G^n$$

$$\text{such that } (f_i, g) \in R_{\alpha_i} \text{ for all } i \in I\Big\},$$

and

$$F' = \bigcup_{n\in N} (F')^n.$$

It is easy to see that F' is a clone on $\prod_{i\in I} A_i$ and A is a subuniverse of $\big(\prod_{i\in I} A_i, F'\big)$.

Finally, we put $F = F' | A$ and $\mathfrak{A} = (A, F)$, and we define, for $i \in I$, $\varepsilon_i\colon A \to A_i$ by $\varepsilon_i\big((a_j)_{j\in I}\big) = a_i$. Evidently, ε_i is a weak homomorphism of $\mathfrak{A}$ into $\mathfrak{A}_i$ for all $i \in I$ and $\alpha_i\varepsilon_i = \alpha_j\varepsilon_j$ for all $i, j \in I$.

PROPOSITION 6. *If the family* $(\alpha_i\colon \mathfrak{A}_i \to \mathfrak{B})_{i\in I}$ *has a pullback in* $\mathcal{N}$, *then* $\big(\mathfrak{A}, (\varepsilon_i)_{i\in I}\big)$ *described above is also a pullback of this family.*

The proof is analogous to that of Proposition 4.

PROPOSITION 7. *Using the notation as above, we have the following equivalence:*

A pullback of the family $(\alpha_i\colon \mathfrak{A}_i \to \mathfrak{B})_{i\in I}$ *exists iff there exists no clone* F'' *on* A *which satisfies*

(i) $F'' \subset F$ *and* (ii) $\bar{\varepsilon}_i F'' = \bar{\varepsilon}_i F$ *for all* $i \in I$.

The proof is analogous to that of Proposition 5.

Let $\mathfrak{A}, \mathfrak{B} \in \mathcal{N}$ and let α be a weak homomorphism of $\mathfrak{A}$ into $\mathfrak{B}$. By a *kernel pair* of α we mean a pullback of the diagram $(\alpha\colon \mathfrak{A} \to \mathfrak{B})_{i\in I}$ for $|I| = 2$.

PROPOSITION 8. *Let* $\mathfrak{A} = (A, F), \mathfrak{B} = (B, G) \in \mathcal{N}$ *and let* α *be a weak homomorphism of* $\mathfrak{A}$ *into* $\mathfrak{B}$. *A kernel pair of* α *exists iff* $\bar{\alpha}\colon F \to G|\alpha A$ *is one-to-one. Particularly, if* α *is one-to-one, then a kernel pair of* α *exists.*

The thesis follows from Proposition 7.

PROPOSITION 9. *For* $\mathfrak{A} = (A, F), \mathfrak{B} = (B, G) \in \mathcal{N}$ *and weak homomorphisms* $\alpha_1, \alpha_2\colon \mathfrak{A} \to \mathfrak{B}$ *there exists an equalizer of morphisms* α_1 *and* α_2 *iff among the subuniverses* X *of* $\mathfrak{A}$ *with the property*

$$X \subseteq A' = \{a \in A \mid \alpha_1 a = \alpha_2 a\}$$

the greatest one exists.

The proof is obvious.

Note that if $\mathfrak{A}$ is idempotent (i.e. $f(a, \ldots, a) = a$ for all $f \in F$ and all $a \in A$), then the equalizer of the pair $\alpha_1, \alpha_2\colon \mathfrak{A} \to \mathfrak{B}$ exists iff A' is a subuniverse of $\mathfrak{A}$.

COROLLARY 2. *Every monomorphism of $\mathcal{N}$ is an equalizer of a suitable pair of morphisms of $\mathcal{N}$.*

The corollary follows from Propositions 2 and 9.

4. Colimits.

PROPOSITION 10. *For any $\mathfrak{A} = (A, F), \mathfrak{B} = (B, G) \in \mathcal{N}$ and for any weak homomorphisms $a_1, a_2\colon \mathfrak{A} \to \mathfrak{B}$ there exists a coequalizer of the pair a_1 and a_2.*

Proof. If q is the smallest congruence relation on $\mathfrak{B}$ for which the inclusion $\{(a_1 a, a_2 a) \mid a \in A\} \subseteq q$ holds, then $\operatorname{nat} q\colon \mathfrak{B} \to \mathfrak{B}/q$ is the coequalizer of the pair a_1 and a_2 in $\mathcal{N}$.

In the next considerations it will be convenient to understand the notion of colimit as follows:

Let $\mathcal{K}$ be a small category (i.e. the class of objects of $\mathcal{K}$ is a set). Let $D\colon \mathcal{K} \to \mathcal{N}$ be a functor. For every $\mathfrak{A} \in \mathcal{N}$ there is a constant functor $D_{\mathfrak{A}}\colon \mathcal{K} \to \mathcal{N}$. For $\mathfrak{X}, \mathfrak{A} \in \mathcal{N}$ every weak homomorphism $\varphi\colon \mathfrak{X} \to \mathfrak{A}$ determines the natural transformation $\bar{\varphi}\colon D_{\mathfrak{X}} \dot{\to} D_{\mathfrak{A}}$. A natural transformation $\pi\colon D \dot{\to} D_{\mathfrak{X}}$ will be called a *colimit* of the functor D if for every $\mathfrak{A} \in \mathcal{N}$ and for every natural transformation $\mu\colon D \dot{\to} D_{\mathfrak{A}}$ there exists exactly one weak homomorphism $\varphi\colon \mathfrak{X} \to \mathfrak{A}$ such that $\bar{\varphi}\pi = \mu$.

LEMMA 2. *Let $\mathcal{K}$ be any small category and $D\colon \mathcal{K} \to \mathcal{N}$ a functor. Let J denote the set of all objects of $\mathcal{K}$ and let $D(j) = \mathfrak{A}_j = (A_j, F_j)$ for $j \in J$. Assume that there exist a set A and a family $(\mu_j\colon A_j \to A)_{j \in J}$ of mappings such that*

(i) $\bigcup_{j \in J} \mu_j A_j = A$,

(ii) $\left|\{j \in J \mid (\mu_j A_j - \bigcup_{i \in J - \{j\}} \mu_i A_i) \neq \emptyset\}\right| \geqslant 2$,

(iii) $\ker \mu_j$ *is a congruence relation on $\mathfrak{A}_j$ for all $j \in J$,*

(iv) *for any morphism $a\colon i \to j$ of $\mathcal{K}$ we have $\mu_j D(a) = \mu_i$.*

Then the colimit of the functor D does not exist.

Proof. The proof is illustrated by Fig. 1.

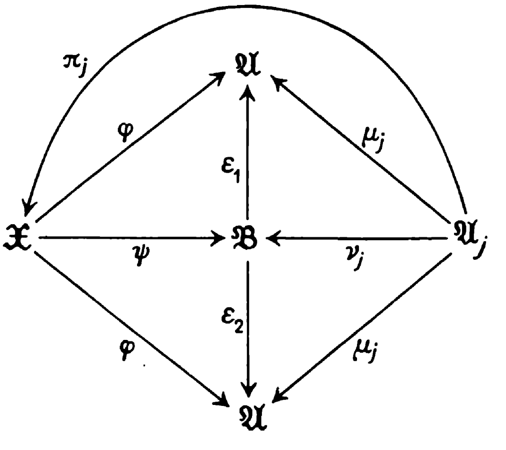

Fig. 1

We put

$$F = \{f \in O(A) \mid f|\mu_j A_j \in \bar{\mu}_j F_j \quad \text{for all } j \in J\}.$$

$\mathfrak{A} = (A, F)$ is a non-indexed algebra and μ_j: $\mathfrak{A}_j \to \mathfrak{A}$ is a weak homomorphism for all $j \in J$. Further, we set $B = A \times A$,

$$G^n = \{(f, f') \in F^n \times F^n \mid f|\mu_j A_j = f'|\mu_j A_j \text{ for all } j \in J\}$$

and

$$G = \bigcup_{n \in N} G^n.$$

Evidently, $\mathfrak{B} = (B, G)$ is a non-indexed algebra.

Finally, we define $\varepsilon_1, \varepsilon_2$: $B \to A$ by

$$\varepsilon_1(a, a') = a, \qquad \varepsilon_2(a, a') = a',$$

and ν_j: $A_j \to B$ by

$$\nu_j a = (\mu_j a, \mu_j a) \quad \text{for all } j \in J.$$

It is easy to see that $\varepsilon_1, \varepsilon_2$ and ν_j for all $j \in J$ are weak homomorphisms and $\varepsilon_1 \nu_j = \mu_j$, $\varepsilon_2 \nu_j = \mu_j$ for all $j \in J$.

Now we suppose that the natural transformation π: $D \dot{\to} D_{\mathfrak{X}}$, where $\pi = (\pi_j)_{j \in J}$ and $\mathfrak{X} = (X, H)$, is a colimit of the functor D. Then there exists exactly one weak homomorphism φ: $\mathfrak{X} \to \mathfrak{A}$ such that $\varphi \pi_j = \mu_j$ for all $j \in J$, and there exists exactly one weak homomorphism ψ: $\mathfrak{X} \to \mathfrak{B}$ such that $\psi \pi_j = \nu_j$ for $j \in J$.

We have $\varepsilon_1 \psi \pi_j = \varepsilon_1 \nu_j = \mu_j$, and thus $\varepsilon_1 \psi = \varphi$. Similarly, $\varepsilon_2 \psi = \varphi$. Thus $\psi X \subseteq A' = \{(a, a) \mid a \in A\}$. Following (i), for any $a \in A$ there exist $j \in J$ and $a_j \in A_j$ such that $\mu_j a_j = a$. Then

$$\psi(\pi_j a_j) = \nu_j a_j = (\mu_j a_j, \mu_j a_j) = (a, a).$$

Thus $\psi X = A'$.

For $a \in A$ we define $f_a \in O^2(A)$ as follows:

$$f_a(a_1, a_2) = \begin{cases} a_1 & \text{if there exists a } j \in J \text{ such that } a_1, a_2 \in \mu_j A_j, \\ a & \text{in other case.} \end{cases}$$

Evidently, for all $a \in A$ we have $f_a \in F^2$.

Let $a, a' \in A$ be any elements. We have $(f_a, f_{a'}) \in G$. Following (ii), there exist $j_1, j_2 \in J$ and $a_1 \in \mu_{j_1} A_{j_1}$, $a_2 \in \mu_{j_2} A_{j_2}$ such that

$$a_i \in \mu_{j_i} A_{j_i} - \bigcup_{j \in J - \{j_i\}} \mu_j A_j \quad \text{for } i = 1 \text{ and } 2.$$

Then

$$(f_a, f_{a'})\big((a_1, a_1), (a_2, a_2)\big) = \big(f_a(a_1, a_2), f_{a'}(a_1, a_2)\big) = (a, a').$$

Thus A' generates $\mathfrak{B}$ and this is a contradiction with the facts that A' is a subuniverse of $\mathfrak{B}$ and $A' \subset B$.

PROPOSITION 11. *Let I be a set and let $\mathfrak{A}_i = (A_i, F_i) \in \mathcal{N}_1$ for all $i \in I$. If $|\{i \in I \mid A_i \neq \emptyset\}| \geqslant 2$, then the sum of the family $(\mathfrak{A}_i)_{i\in I}$ does not exist.*

For the proof, in Lemma 2 we put $J = A = I$ and $\mu_i A_i = \{i\}$ for all $i \in I$.

PROPOSITION 12. *Let I be a set and let $\mathfrak{A}_i = (A_i, F_i) \in \mathcal{N}_0$ for all $i \in I$. Let q_i for $i \in I$ denote the smallest congruence relation on $\mathfrak{A}_i$, for which F_i^0 is included in a single class. If*

$$|\{i \in I \mid |A_i/q_i| \geqslant 2\}| \geqslant 2,$$

then the sum of the family $(\mathfrak{A}_i)_{i\in I}$ does not exist.

Proof. We define the equivalence relation r on the set

$$A' = \bigcup_{i\in I} (A_i/q_i \times \{i\})$$

as follows:

$$\big((a, i), (a', j)\big) \in r \quad \text{iff} \quad a = (F_i^0)^{q_i},\ a' = (F_j^0)^{q_j} \text{ or } i = j \text{ and } a = a'.$$

We put $A = A'/r$ and define $\mu_i\colon A_i \to A$ for $i \in I$ by $\mu_i x = (x^{q_i}, i)^r$. Evidently, $\ker \mu_i = q_i$ for all $i \in I$ and, therefore, we may apply Lemma 2 for $J = I$.

Example 2. Let $\mathfrak{A} = (A, F)$ be a non-indexed algebra for which $|A| = 1$ and $F^0 = A$. Then $\big(\mathfrak{A}, (\mathrm{id}_A)_{i\in I}\big)$ is the sum of the family $(\mathfrak{A}_i)_{i\in I}$.

Let I be a set and let $\mathfrak{B} = (B, G), \mathfrak{A}_i = (A_i, F_i) \in \mathcal{N}$ for $i \in I$. Let $(a_i\colon \mathfrak{B} \to \mathfrak{A}_i)_{i\in I}$ be a family of weak homomorphisms.

By a *pushout* of the family $(a_i)_{i\in I}$ we mean a colimit of the functor $D\colon \mathcal{K} \to \mathcal{N}$, where

the set of objects of $\mathcal{K}$ is $\{0\} \cup I$, $0 \notin I$,

$\mathcal{K}$ has no other morphisms than $k_i\colon 0 \to i$ for $i \in I$ and identities,

$D(0) = \mathfrak{B}$, $D(\mathrm{id}_0) = \mathrm{id}_B$,

$D(k_i) = a_i$, $D(\mathrm{id}_i) = \mathrm{id}_A$ for all $i \in I$.

For all $i \in I$ we put $B_i = a_i B$ and $\mathfrak{B}_i = (B_i, F_i \mid B_i)$.

Between the congruence relations q on $\mathfrak{B}$ with the properties

(i) $q \supseteq \ker a_i$ for all $i \in I$,

(ii) for every $i \in I$ there exists a congruence relation q_i on $\mathfrak{A}_i$ such that $q_i \mid B_i = \tilde{a}_i q$

there exists a relation, denoted by q^0, which is the smallest one.

The smallest relation, denoted by q_i^0, exists also between the congruence relations q_i on $\mathfrak{A}_i$ $(i \in I)$ with the property

(iii) $q_i \mid B_i = \tilde{a}_i q^0$.

PROPOSITION 13. *Using the notation as above, we have the following equivalence:*

There exists a pushout of the family $(a_i\colon \mathfrak{B} \to \mathfrak{A}_i)_{i\in I}$ *iff*

$$I_0 = \left|\{i \in I \mid \mathfrak{B}_i \text{ is a proper subalgebra of } \mathfrak{A}_i \text{ relative to } q_i^0\}\right| \leqslant 1.$$

Particularly, if all a_i's, with a possible exception of one, are onto, then the pushout of the family $(a_i)_{i\in I}$ *exists.*

Proof. We define the equivalence relation r on the set

$$A' = \bigcup_{i\in I} (A_i/q_i^0 \times \{i\})$$

as follows: $\big((a, i), (a', j)\big) \in r$ iff there exists a $b \in B$ such that $a \in (a_i b)^{q_i^0}$, $a' = (a_j b)^{q_j^0}$ or $i = j$ and $a = a'$.

We put $A = A'/r$ and for $i \in I$ define $\mu_i\colon A_i \to A$ by

$$\mu_i x = (x^{q_i^0}, i)^r \quad \text{for all } x \in A_i.$$

Now $\ker \mu_i = q_i^0$ and

$$F = \{f \in O(A) \mid f|\mu_i A_i \in \bar{\mu}_i F_i \text{ for all } i \in I\}$$

is a clone on A. Evidently, μ_i is a weak homomorphism of $\mathfrak{A}_i$ into $\mathfrak{A} = (A, F)$ for all $i \in I$ and $\mu_i a_i = \mu_j a_j$ for all $i, j \in I$. If $I_0 \geqslant 2$, we may apply Lemma 2 for $J = \{0\} \cup I$ and $\mu_0 = \mu_i a_i$.

Now let $I_0 \leqslant 1$. It follows easily from the construction of $q^0, q_i^0, \mathfrak{A}$ and μ_i for $i \in I$ that the natural transformation $\mu = (\mu_j)_{j\in J}$ is a pushout of the family $(a_i)_{i\in I}$ ($J = \{0\} \cup I$ and $\mu_0 = \mu_i a_i$).

5. Miscellanea. We define the functor $\mathscr{F}\colon \mathscr{N}_0 \to \mathscr{N}_1$ as follows: $\mathscr{F}(A, F) = (A, F - F^0)$ for all $(A, F) \in \mathscr{N}_0$ and $\mathscr{F}(a) = a$ for all morphisms of $\mathscr{N}_0$.

PROPOSITION 14. *The functor* $\mathscr{F}\colon \mathscr{N}_0 \to \mathscr{N}_1$ *is a full embedding.*

Proof. Let $\mathfrak{A} = (A, F) \in \mathscr{N}_0$. For $a \in A$ and $n \in N$ we denote by f_a^n the n-ary operation on A defined by

$$f_a^n(a_1, \ldots, a_n) = a \quad \text{for all } a_1, \ldots, a_n \in A.$$

Now, if $a \in F^0$, then $f_a^n \in F^n$ for all $n \in N$, and, conversely, if $f_a^n \in F^n$ for some $n \in N$, then $a \in F^0$. Hence the functor $\mathscr{F}$ is one-to-one on objects.

Now we prove that $\mathscr{F}$ is full. Let $\mathfrak{A} = (A, F), \mathfrak{B} = (B, G) \in \mathscr{N}_0$ and let $a\colon \mathscr{F}(\mathfrak{A}) \to \mathscr{F}(\mathfrak{B})$ be a weak homomorphism. We need only to prove that a satisfies conditions (i) and (ii) from the definition of weak homomorphism for $n = 0$.

Let $a \in F^0$ and $b \in G^0$. There exist $f \in F^1$ and $g \in G^1$ such that $(f_a^1, g), (f, g_b^1) \in R_a$. Thus for all $x \in A$ we have

$$aa = af_a^1(x) = g(ax) \quad \text{and} \quad af(x) = g_b^1(ax) = b.$$

Since F^0 and G^0 are subuniverses of $\mathfrak{A}$ and $\mathfrak{B}$, respectively, we have $aa \in G^0$ (because $aa = g(b)$) and $f(a) \in F^0$. Finally, $(a, aa), \big(f(a), b\big) \in R_a$.

In the natural manner $((A, F) \mapsto A)$ there are defined forgetful functors $\mathscr{G}_0$: $\mathscr{N}_0 \to \mathrm{Set}$ and $\mathscr{G}_1$: $\mathscr{N}_1 \to \mathrm{Set}$.

PROPOSITION 15. *The forgetful functors $\mathscr{G}_0$ and $\mathscr{G}_1$ do not have left adjoints.*

The proposition follows from the fact that there exist non-indexed algebras with a single generator and with supports of arbitrary cardinality.

PROPOSITION 16. *The non-indexed product forms monoidal categories from the categories $\mathscr{N}_0$ and $\mathscr{N}_1$.*

Proof (for $\mathscr{N}_1$). We need only to prove the mapping $(\mathfrak{A}, \mathfrak{B}) \to \mathfrak{A} \times \mathfrak{B}$ can be extended to a functor of $\mathscr{N}_1 \times \mathscr{N}_1$ into $\mathscr{N}_1$. But, for weak homomorphisms α: $\mathfrak{A} \to \mathfrak{A}'$ and β: $\mathfrak{B} \to \mathfrak{B}'$, the mapping

$$\alpha \times \beta \colon \mathfrak{A} \times \mathfrak{B} \to \mathfrak{A}' \times \mathfrak{B}'$$

defined by $(a, b) \mapsto (\alpha a, \beta b)$ is a weak homomorphism as well.

The unit is a one-element non-indexed algebra. The associativity and coherence follow from the monoidality of the category Set relative to the Cartesian product.

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