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ON TOPOLOGIES GENERATING THE EFFROS BOREL STRUCTURE AND ON THE EFFROS MEASURABILITY OF THE BOUNDARY OPERATION

BY

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In this paper it is shown that for an analytic 0-dimensional space (1) the boundary operation is measurable with respect to the Effros Borel structure(2) if and only if the space is σ-compact. However, this is not the case for an analytic space in general. Some results about topologies generating the Effros Borel structure are proved. The main result is an example showing that, in a non-locally compact space, Fell's topology and the convergence topology can coincide.

1. Notation and basic definitions. All spaces in this paper are separable metrizable; our terminology follows that in [4] and [6].

By $\mathfrak{F}(X)$ we denote the family of all closed subsets of a given space X. Analogously, $\mathfrak{K}(X)$ means the family of all compact subsets of X. Further

$$\langle F\rangle = \{A \in \mathfrak{F}(X) \colon A\cap F = \emptyset\} \quad \text{and} \quad)U(= \{A \in \mathfrak{F}(X) \colon A \cap U \neq \emptyset\}.$$

Let us define the z-convergence in $\mathfrak{F}(X)$ by

$$F_\sigma \xrightarrow{z} F$$

if for every $x \in X\setminus F$ there exists a neighbourhood which eventually does not intersect F and, moreover, for $x \in F$ every neighbourhood of x intersects F eventually. The convergence topology τ^z is defined by considering $\mathscr{A} \subset \mathfrak{F}(X)$ as closed if for every net $\{F_\sigma\}_{\sigma\in\Sigma}$ in $\mathscr{A}$ converging to $F \in \mathfrak{F}(X)$ we have $F \in \mathscr{A}$.

All sets $\langle K\rangle$ and $)U($, where U is open and K is compact in X, give a subbase of Fell's topology τ^F.

(1) X is an *analytic space* if there exists a continuous function mapping the irrationals onto X (see [8]).

(2) The Effros Borel structure was defined first by Effros (see [5]). This structure has extensively been studied by Christensen in [4] (see also [8]).

Let d be a precompact metric on X. By τ^{d^*} we denote the topology generated by the Hausdorff metric d^* defined by

$$d^*(A, B) = \sup_{a \in A, b \in B} \{d(a, B), d(A, b)\}.$$

Obviously, this topology depends on the metric d. We consider also the topology τ^p which is the intersection of all topologies τ^{d^*}, where d is running over all precompact metrics on X compatible with the topology; of course, τ^p depends only on the topology of X.

τ^V is the well-known Vietoris topology, where the sets of the form $\langle F \rangle$ and $)U($, F being closed and U open in X, form a subbase.

Definition. The *Effros Borel structure* (EBS) on $\mathfrak{F}(X)$ is the σ-algebra generated by all sets of the form $\{A \in \mathfrak{F}(X) : A \subset F\}$.

2. Auxiliary results. One can easily prove the following two lemmas:

LEMMA 1. *Let the topologies $\tau^V, \tau^{d^*}, \tau^p, \tau^s, \tau^F$ on $\mathfrak{F}(X)$ be defined as above. Then the following relations hold:*

$$\tau^V \supset \tau^{d^*} \supset \tau^p \supset \tau^s \supset \tau^F.$$

If X is a compact space, then all these topologies coincide.

LEMMA 2. *Let $(Z, \bar{d})$ be the compact d-completion of a precompact space (X, d). Then the mapping $\varphi: \mathfrak{F}(X) \to \mathfrak{F}(Z)$, defined by $\varphi(A) = \bar{A}^Z$, is a Borel isomorphism between $\mathfrak{F}(X)$ and $\varphi(\mathfrak{F}(X))$ with respect to the Effros Borel structure on $\mathfrak{F}(X)$ and $\mathfrak{F}(Z)$.*

PROPOSITION (cf. [4], p. 53). *The topologies τ^{d^*}, τ^p, τ^s, τ^F generate the EBS on $\mathfrak{F}(X)$.*

Proof. It is easy to see that τ^F generates the EBS. It follows from Lemma 1 that $\tau^{\bar{d}^*}$ generates the EBS on $\mathfrak{F}(Z)$. Since the mapping

$$\varphi: (\mathfrak{F}(X), \tau^{d^*}) \to (\mathfrak{F}(Z), \tau^{\bar{d}^*}),$$

defined in Lemma 2, is a homeomorphic embedding, we see that τ^{d^*} generates the EBS on $\mathfrak{F}(X)$. An application of Lemma 1 completes the proof.

In the following we need 3 theorems due to Christensen (see [4], p. 54, 58 and 72).

THEOREM 1. *Let X be an analytic space and let d be a precompact metric on X. Then $(\mathfrak{F}(X), \tau^{d^*})$ is an analytic space.*

THEOREM 2. *The space X is completely metrizable if and only if $\mathfrak{K}(X)$ equipped with the Vietoris topology is an analytic space.*

THEOREM 3. *Let X be an analytic space. The intersection operation*

$$p: \mathfrak{F}(X) \times \mathfrak{F}(X) \to \mathfrak{F}(X),$$

given by $p(A, B) = A \cap B$, is measurable with respect to the EBS if and only if X is σ-compact.

Remark. It is easy to conclude Theorem 2 from Christensen's theorem which states that a compact-covering mapping preserves completeness. We are also able to give an easier proof of Theorem 3 and of Theorem 3.8 in [4], p. 69, than that in [4], using a natural Borel isomorphic embedding of $\mathfrak{F}(X)$ into the function space R^X with respect to the EBS on $\mathfrak{F}(X)$ and the compact-open topology on R^X and expanding Theorem 3.7 in [4], p. 66.

Later on we shall use the following two well-known theorems (see, e.g., [7]).

THEOREM 4. *$(\mathfrak{F}(X), \tau^F)$ is a compact T_1-space.*

THEOREM 5. *If X is locally compact, then $\tau^s = \tau^F$.*

3. Measurability of the boundary operation. Using the preceding results we now study the measurability of the boundary operation $\mathrm{Fr}\colon \mathfrak{F}(X) \to \mathfrak{F}(X)$, where $\mathrm{Fr}A = A \cap \overline{X \setminus A}$ is the boundary of $A \in \mathfrak{F}(X)$. Christensen noticed in [4], p. 74, the following fact, which generalizes a classical result on measurability of the boundary operation in compact spaces (see [8], § 43/8):

THEOREM 6. *If X is a σ-compact space, then the boundary operation $\mathrm{Fr}\colon \mathfrak{F}(X) \to \mathfrak{F}(X)$ is measurable with respect to the EBS.*

Christensen has conjectured that for an analytic space X the boundary operation is measurable if and only if X is σ-compact. The following theorem allows us to construct a counterexample to this hypothesis (cf. Example 1). However, we shall show in Theorem 8 that the hypothesis holds if the space is 0-dimensional.

THEOREM 7. *If the space X admits a metrizable compactification Z such that for every point $z \in Z \setminus X$ there exists a base $\{U_n(z)\}_{n=1}^{\infty}$ at this point with $U_n(z) \cap X$ connected for all natural n, then the boundary operation $\mathrm{Fr}\colon \mathfrak{F}(X) \to \mathfrak{F}(X)$ is measurable with respect to the EBS.*

Proof. Let $\psi\colon \mathfrak{F}(X) \to \mathfrak{F}(Z)$ be the mapping defined in Lemma 2 and let F be an arbitrary closed subset of X. For the generator $\{A \in \mathfrak{F}(X)\colon A \subset F\}$ of the EBS we have

$$\mathrm{Fr}_X^{-1}\{A \in \mathfrak{F}(X)\colon A \subset F\} = \{A \in \mathfrak{F}(X)\colon \mathrm{Fr}_X A \subset F\}.$$

(For the boundary operation with respect to the subset M we use the symbol Fr_M.) We prove the equation

$$(1)\qquad \psi\{A \in \mathfrak{F}(X)\colon \mathrm{Fr}_X A \subset F\} = \psi(\mathfrak{F}(X)) \cap \{A \in \mathfrak{F}(Z)\colon \mathrm{Fr}_Z A \subset \overline{F}^Z\}.$$

It follows from Theorem 6 that the set on the left-hand side in (1) is measurable with respect to the Borel structure on $\psi(\mathfrak{F}(X))$ generated by the EBS on $\mathfrak{F}(Z)$. Then, using Lemma 2, we can complete the proof.

Of course, $\bar{F}^Z \supset \mathrm{Fr}_Z \bar{A}^Z \supset \mathrm{Fr}_X(\bar{A}^Z \cap X) = \mathrm{Fr}_X A$, whence $\mathrm{Fr}_X A \subset F$. Thus the set on the left-hand side of (1) contains the set on the right-hand side. To show the other inclusion, let A be a closed subset of X such that $\mathrm{Fr}_X A \subset F$. Suppose that there exists a point $x \in \mathrm{Fr}_Z \bar{A}^Z \setminus \bar{F}^Z$. If $x \in A$, then x is an interior point of $\bar{A}^Z$ in Z, which is impossible. Suppose now that $x \notin A$. Then $x \in Z \setminus X$ and there exists an element U of the base $\{U_n(x)\}_{n=1}^{\infty}$ such that

$$U \cap A \neq \emptyset \quad \text{and} \quad U \cap \mathrm{Fr}_X A = \emptyset. \tag{2}$$

By the definition of the point x we have

$$U \cap (X \setminus \mathrm{Int}_X A) \neq \emptyset. \tag{3}$$

Now conditions (2) and (3) contradict the connectedness of $U \cap X$.

From Theorem 7 we obtain the following

Example 1. *There exists an $F_{\sigma\delta}$-subset X of the Euclidean plane R^2 such that*

(a) *X is not σ-compact;*

(b) *X is not complete;*

(c) *the boundary operation Fr_X is measurable with respect to the EBS.*

Proof. Let

$$X = [0,1] \times (0,1] \cup ([0,1/2] \cap P \times \{0\}) \cup ([1/2,1] \cap Q \times \{0\}),$$

where P are the irrationals and Q are the rationals.

THEOREM 8. *Let X be a 0-dimensional analytic space. Then the following conditions are equivalent:*

(i) *X is σ-compact.*

(ii) *The boundary operation $\mathrm{Fr}_X: \mathfrak{F}(X) \to \mathfrak{F}(X)$ is measurable with respect to the EBS.*

(iii) *The set $\mathscr{P}$ of closed-open subsets of X belongs to the EBS.*

Proof. (i) $\Rightarrow$ (ii) is a consequence of Theorem 6. (ii) $\Rightarrow$ (iii) is immediate. To prove (iii) $\Rightarrow$ (i), let $(Z, \bar{d})$ be the closure of X embedded in the Cantor set. It follows from Theorem 6 and Lemma 1 that the superposition $\mathrm{Fr}_Z \circ \psi: \mathfrak{F}(X) \to \mathfrak{F}(Z)$ is measurable with respect to the EBS. By Theorem 1, $\mathscr{P}$ is an analytic set. Now it is sufficient to show that

$$\mathrm{Fr}_Z \circ \psi(\mathscr{P}) = \mathfrak{K}(Z \setminus X).$$

Then $\mathfrak{K}(Z \setminus X)$, as the image of an analytic set, is also analytic and it follows from Theorem 2 that $Z \setminus X$ is complete metrizable. Hence X is σ-compact.

The inclusion $\mathrm{Fr}_Z \circ \psi(\mathscr{P}) \subset \mathfrak{K}(Z \setminus X)$ is obvious. To show the converse inclusion, let $K \in \mathfrak{K}(Z \setminus X)$. There exists a sequence $\{S_n\}_{n=1}^{\infty}$ of finite subsets of X such that

$$S_n \xrightarrow{\bar{d}^*} K.$$

Moreover, we may assume that $S_n \cap S_m = \emptyset$ for $n \neq m$. Let $S_n = \{x_1^n, \ldots, x_{k_n}^n\}$. For all $n \geqslant 1$ and for all $i \leqslant k_n$ we find a closed-open subset of $Z \setminus U_i^n$ such that the following conditions are satisfied:

$x_i^n \in U_i^n$;

$\mathrm{diam}(U_i^n) < 1/n$;

$U_i^n \cap U_j^m = \emptyset$ if $n \neq m$ or $i \neq j$.

Obviously, $\overline{U_i^n \cap X}^Z = U_i^n$, whence

$$\mathrm{Fr}_X\Big(\bigcup_{n=1}^{\infty} \bigcup_{i=1}^{k_{2n}} (U_i^{2n} \cap X)\Big) = \emptyset.$$

If we put

$$A = \bigcup_{n=1}^{\infty} \bigcup_{i=1}^{k_{2n}} (U_i^{2n} \cap X),$$

then $A \in \mathscr{P}$ and $\mathrm{Fr}_Z(A) = K$.

4. Some results on the topologies generating the EBS. Several facts about topologies generating the EBS are contained in Lemma 1, Proposition 1 and Theorems 4 and 5. Here we obtain further results, answering some questions from [4]. It is easy to show that τ^V equals τ^d if and only if X is compact and, similarly, that τ^p equals τ^s if and only if X is compact. The relation between Fell's topology τ^F and the convergence topology τ^s is much more complicated. F. Topsøe has conjectured (see [4], p. 53) that τ^F equals τ^s if and only if the space is locally compact. This hypothesis is not true as we show in the following example.

Example 2. *Let*

$$X = \{0\} \cup \bigcup_{n=1}^{\infty} X_n,$$

where $X_n = \{1/n + 1/n^2 m\}_{m=1}^{\infty}$ is a subset of the reals. Then X is not locally compact but $\tau^F = \tau^s$.

Proof. The inclusion $\tau^s \supset \tau^F$ follows from Lemma 1. Suppose now that $\mathscr{A}$ is a subset of $\mathfrak{F}(X)$ which is convergence closed but not closed in Fell's topology τ^F, i.e.,

$$\mathscr{A} = \overline{\mathscr{A}}^{\tau^s} \tag{4}$$

and

$$A \in \overline{\mathscr{A}}^{\tau^F} \setminus \mathscr{A} \quad \text{for some } A \in \mathfrak{F}(X). \tag{5}$$

Let A be fixed. Then either $0 \in A$ or $0 \neq A$. Suppose that $0 \in A$; let $A \setminus \{0\} = \{x_1, x_2, \ldots\}$, $X \setminus A = \{y_1, y_2, \ldots\}$ and let

$$U_i = \{0\} \cup \bigcup_{n=i}^{\infty} X_n.$$

For every $i \geqslant 1$ let us fix a set P_i such that

$$P_i \in \langle y_1, \ldots, y_i \rangle \cap)x_1(\cap \ldots \cap)x_i(\cap)U_i(.$$

The existence of P_i follows from (5). Obviously, $P_i \xrightarrow{s} A$, which contradicts (4).

If, conversely, $0 \notin A$, then

$$A \subset \bigcup_{n=1}^{t} X_n$$

for some natural t. The subspace topology on $\bigcup_{n=1}^{t} X_n$ is discrete and, of course, the topologies τ^F and τ^s do not depend on the metric of the given space. Hence we may assume simply that $A = X_1$ or $A = \{1+1/m\}_{m=1}^{k}$ for A infinite or k-element, respectively. If A has k elements, then we change the notation and write

$$X_1 = A = \{1+1/m\}_{m=1}^{k} \quad \text{and} \quad X_2 = \{1/2+1/4m\}_{m=1}^{\infty} \cup \{1+1/m\}_{m>k}.$$

This observation guarantees that the following construction works also if A is finite.

Fix $l \geqslant 1$ and let

$$\mathscr{P}_j = \Big\{P \in \mathscr{A} \colon P \supset \{1+1/m\}_{m=1}^{l} \text{ and } P \subset \bigcup_{n=1}^{i} X_n\Big\}.$$

It follows from (5) that the set $\mathscr{P}$ defined by

(6) $$\mathscr{P} = \bigcup_{j=2}^{\infty} \mathscr{P}_j$$

satisfies

(7) For any compact $K \subset X \setminus A$ we can find a $P \in \mathscr{P}$ such that $P \cap K = \emptyset$.

Now we construct a sequence $\{P_i\}_{i=1}^{\infty}$ such that $P_i \xrightarrow{s} A(l)$ as $i \to \infty$ for some $A(l) \in \mathfrak{F}(X)$. Then condition (4) implies that $A(l) \in \mathscr{A}$ and, moreover, $A(l) \xrightarrow{s} A$ as $l \to \infty$ ($A(k) = A$ if A has k elements). Hence $A \in \mathscr{A}$, which contradicts (5) and completes the proof.

Suppose, conversely, that such a sequence does not exist. Then by induction we construct a sequence of finite sets $K_n \subset X_n$, $n \geqslant 2$, such that

$$P \cap \bigcup_{n=2}^{j} K_n \neq \emptyset \quad \text{for all } P \in \mathcal{P}_j. \tag{8}$$

Let $n = 2$. Suppose that K_2 does not exist. Let $X_2 = \{x_1, x_2, \ldots\}$ and fix $r \geqslant 1$. We can find $P_r \in \mathcal{P}_2$ such that $P_r \cap \{x_1, \ldots, x_r\} = \emptyset$. By the definition of $\mathcal{P}_2, P_r \subset X_1 \cup X_2$ for all r. Notice that $X_1 \cup X_2$ is a locally compact space and a closed-open subset of X. That is why we can use Theorems 4 and 5. By Theorem 4 there exists a convergent subsequence $\{P_{r_s}\}_{s=1}^{\infty}$ of the sequence $\{P_r\}_{r=1}^{\infty}$:

$$P_{r_s} \xrightarrow{\tau^F} F \text{ as } s \to \infty \quad \text{for some } F \in \mathfrak{F}(X).$$

Of course, $A \supset F \supset \{1+1/m\}_{m=1}^{l}$. Now Theorem 5 implies that

$$P_{r_s} \xrightarrow{s} F \quad \text{as } s \to \infty.$$

This is a contradiction if we set $\{P_i\}_{i=1}^{\infty} = \{P_{r_s}\}_{s=1}^{\infty}$ and $A(l) = F$. Suppose now that the sets $K_2, \ldots, K_n$ have been defined. Let

$$\mathcal{P}'_{n+1} = \{P \in \mathcal{P}_{n+1} : P \cap \bigcup_{j=1}^{n} K_j = \emptyset\} \quad \text{and} \quad X_{n+1} = \{x_1^{n+1}, x_2^{n+1}, \ldots\}.$$

If K_{n+1} does not exist, then we construct a sequence $\{P'_r\}_{r=1}^{\infty} \subset \mathcal{P}'_{n+1}$ such that for all r we have $P'_r \cap \{x_1^{n+1}, \ldots, x_r^{n+1}\} = \emptyset$. We use the local compactness of $X_1 \cup \ldots \cup X_{n+1}$ and, as previously, we choose a subsequence $\{P'_{r_s}\}_{s=1}^{\infty}$ such that

$$P'_{r_s} \xrightarrow{\tau^F} F \text{ and } P'_{r_s} \xrightarrow{s} F \text{ as } s \to \infty \quad \text{for some } F \in \mathfrak{F}(X). \tag{9}$$

Obviously, from (9) and (5) it follows that $F \in \mathcal{A}$ and, in particular, $F \in \mathcal{P}_{n+1}$. By the definition of $\mathcal{P}'_{n+1}$ we have

$$P'_{r_s} \cap \bigcup_{j=2}^{n} K_j = \emptyset \quad \text{for all } s,$$

whence

$$F \cap \bigcup_{j=2}^{n} K_j = \emptyset.$$

Using these two facts and (9), we can conclude that $A \supset F \supset \{1+1/m\}_{m=1}^{l}$. This gives a contradiction, since we can take $\{P_i\}_{i=1}^{\infty} = \{P'_{r_s}\}_{s=1}^{\infty}$ and $A(l) = F$. Hence the set K_{n+1} exists, and so the induction is completed.

The set

$$K = \bigcup_{n=1}^{\infty} K_n \cup \{0\}$$

is a compact subset of X and, therefore, conditions (6) and (8) contradict (7). This completes the proof of Example 2.

Flachsmeyer [7] has given an example of a σ-compact space with $\tau^F \neq \tau^s$. This result can slightly be improved by Example 3. We shall use this construction in Theorem 9.

Example 3. *Let $Y = X \oplus X$, where X is the space defined in Example 2. Then Fell's topology τ^F and the convergence topology τ^s do not coincide.*

Proof. Let

$$Y = \{0\} \cup \{1/n + 1/n^2 m\}_{n,m \geq 1} \cup \{2\} \cup \{1/n + 1/n^2 m + 2\}_{n,m \geq 1}.$$

The set

$$\mathscr{A} = \big\{\{0, 2\}, \{1/n + 1/n^2 m, 1/n + 1/n^2 m + 2\}_{n,m \geq 1}\big\} \subset \mathfrak{F}(Y)$$

is closed in τ^s. We show that $\{0\} \in \overline{\mathscr{A}}^{\tau^F}$. Let

$$\{0\} \in)U_1(\cap \ldots \cap)U_n(\cap \langle K \rangle,$$

where U_i is open in Y and K is compact in Y. There exists a k such that

$$U_1 \cap \ldots \cap U_n \supset \{0\} \cup \{1/n + 1/n^2 m\}_{n>k}^{m=1,2,\ldots}.$$

Since K is compact, the set

$$(\{2\} \cup \{1/n^2 m + 1/n + 2\}_{n>k}^{m=1,2,\ldots}) \setminus K$$

is infinite, however, the set

$$(\{1/n + 1/n^2 m\}_{n>k}^{m=1,2,\ldots}) \cap K$$

is finite. Now, it is easily seen that every neighbourhood in τ^F of the set $\{0\}$ contains an element of $\mathscr{A}$.

THEOREM 9. *If a space X contains two points without any compact neighbourhood, then the topologies τ^F and τ^s do not coincide.*

For the proof it is enough to note that X contains the space Y defined in Example 3 as a closed subset.

Remark. It seems to be undecided if there exists a space X with only one point without a compact neighbourhood such that the topologies τ^F and τ^s on $\mathfrak{F}(X)$ are unequal. If one can give a negative answer to this question, then τ^F equals τ^s if and only if the space contains at most one point without a compact neighbourhood. **(P 1182)**

The topologies τ^F and τ^s are non-Hausdorff when X is not locally compact. (Moreover, they are Hausdorff if and only if X is locally compact.) Christensen supposed (see [4], p. 75) that it might be that the intersection topology τ^p gives a Hausdorff topology on $\mathfrak{F}(X)$ with reasonable properties also for a non-locally compact space. We shall see that this is not the case.

THEOREM 10. *The following conditions are equivalent for the space X:*

(i) *X is locally compact;*

(ii) *$(\mathfrak{F}(X), \tau^p)$ is Hausdorff;*

(iii) *$(\mathfrak{F}(X), \tau^p)$ is metrizable.*

Proof. (i) $\Rightarrow$ (iii). Let $Z_0 = X \cup \{z_0\}$ be the one-point Alexandroff compactification of X and let $\bar{d}_0$ be a compatible metric on Z_0. We show that $\tau^p = \tau^{d_0}$, where d_0 is the metric on X induced by $\bar{d}_0$. Let d be an arbitrary precompact metric on X. Now it is sufficient to show that if a sequence $\{F_i\}_{i=1}^{\infty} \subset \mathfrak{F}(X)$ converges to $F \in \mathfrak{F}(X)$ relatively to d, then $\{F_i\}_{i=1}^{\infty}$ converges to F relatively to d_0.

If $z_0 \in \overline{F}^{Z_0}$, then we can use Lemma 1, since we may assume that for some $\varepsilon > 0$ and for every $i \geqslant 1$ the set F_i is contained in the compact set $Z_0 \setminus K_{\bar{d}_0}(z_0, \varepsilon)$. If, conversely, $z_0 \in \overline{F}^{Z_0}$, then

(10) for every $\varepsilon > 0$ there exists an index i_0 such that

$$F_i \cap K_{\bar{d}_0}(z_0, \varepsilon) = \emptyset \quad \text{for } i \geqslant i_0.$$

By Lemma 1 we have

$$F_i \setminus K_{\bar{d}_0}(z_0, \varepsilon) \xrightarrow{d_0^*} F \setminus K_{\bar{d}_0}(z_0, \varepsilon). \tag{11}$$

From (10) and (11) we obtain $\overline{F}_i^{Z_0} \xrightarrow{\bar{d}_0^*} \overline{F}^{Z_0}$, whence $F_i \xrightarrow{d_0^*} F$.

Implication (iii) $\Rightarrow$ (ii) is clear. To show (ii) $\Rightarrow$ (i), let d be a precompact metric on X and let $(Z, \bar{d})$ be the d-completion of X. Suppose that there is a point x_0 without any compact neighbourhood and let $z \in Z \setminus X$. Let us choose neighbourhoods U and V of x_0 and z, respectively, open in Z and such that $\overline{U}^Z \cap \overline{V}^Z = \emptyset$. We choose a sequence $\{x_n\}_{n=1}^{\infty} \subset X \cap U$ converging to z. Let $\{U_i\}_{i=1}^{\infty}$ be a base at the point x_0. We may assume that $\overline{U}_i^Z \subset U$ for all $i \geqslant 1$. For every $i \geqslant 1$ let us fix a sequence $\{y_k^i\}_{k=1}^{\infty} \subset U_i$ such that

$$y_k^i \xrightarrow{\bar{d}} y^i \text{ as } k \to \infty \quad \text{for some } y^i \in Z \setminus X. \tag{12}$$

If d_i is a metric on the quotient space $Z/\{z, y^i\}$, then

$$\{x_n\}_{n=1}^{\infty} \cup \{y_k^i\} \xrightarrow{d_i^*} \{x_n\}_{n=1}^{\infty} \in \mathfrak{F}(X) \quad \text{as } k \to \infty. \tag{13}$$

Now, let $\mathscr{P}$ be a neighbourhood of the point $\{x_n\}_{n=1}^{\infty}$ in the intersection topology τ^p. Notice that

(14) for every $i \geqslant 1$ there exists an index k_i such that $\{x_n\}_{n=1}^{\infty} \cup \{y_{k_i}^i\} \in \mathscr{P}$.

From (12)-(14) and from the definition of y_k^i it follows that

$$\{x_n\}_{n=1}^{\infty} \cup \{y_{k_i}^i\} \xrightarrow{d^*} \{x_n\}_{n=1}^{\infty} \cup \{x_0\} \quad \text{as } i \to \infty,$$

whence

$$\{x_n\}_{n=1}^{\infty} \cup \{y_{k_i}^i\} \xrightarrow{\tau^p} \{x_n\}_{n=1}^{\infty} \cup \{x_0\} \quad \text{as } i \to \infty.$$

Thus arbitrary neighbourhoods $\mathscr{P}$ and $\mathscr{P}'$ of the points $\{x_n\}_{n=1}^{\infty}$ and $\{x_n\}_{n=1}^{\infty} \cup \{x_0\}$ both contain a point of the form $\{x_n\}_{n=1}^{\infty} \cup \{y_k^i\}$. Hence the intersection topology τ^p is non-Hausdorff.

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