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UNIFORM CONVERGENCE OF LACUNARY FOURIER SERIES

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1. A subset I of the integers Z is called a *set of uniform convergence*, or a *UC set*, if every Fourier series of the form

$$\sum_{n\in I} c_n e^{int}, \tag{1}$$

which represents a continuous function, converges uniformly. In [1] an example was given of a UC set which is not a Sidon set, i.e. such that there exist uniformly convergent series of form (1) which are not absolutely convergent. In this note we exhibit another example of a UC set which is not Sidon, by showing that the union of a set as in [1] and a finite number of Hadamard sets is still a UC set. To prove this fact we use convolutions with the de la Vallée Poussin kernels instead of the Riesz polynomials considered in [1]. The question of whether the union of two UC sets is again a UC set remains open in general. We do not know the answer even in the case where one of the sets is a Sidon set. For subsets of the dual of the Cantor group, some partial results are contained in [3] where general properties of UC sets are also discussed.

2. Let n_s be a sequence of positive integers such that

$$\frac{n_{s+1}}{n_s} > 1+\sqrt{3},$$

and let $E = \{n_i+n_j\colon i \neq j\}$. Let $E_l = \{m_j^{(l)}\}_{j=1}^{\infty}$ be sequences of positive integers such that $m_{j+1}^{(l)}/m_j^{(l)} \geqslant q > 1$ for some q and $l = 1, \ldots, M$. Finally, let

$$F = \bigcup_{l=1}^{M} E_l.$$

Then

THEOREM. *$E\cup F$ is a UC set which is not a Sidon set.*

Proof. Clearly, $E\cup F$ is not a Sidon set, since it contains infinite sets which are not Sidon (see [1]). Without loss of generality we may suppose that, for some $p \in Z$, $n_{s+1}/n_s \leqslant p$. Let h be a positive integer such that $q^h > p$. Then there are at most $hM = A$ elements of F between $n_s + n_{s-1}$ and $n_{s+1} + n_s$. Indeed, let $m_j^{(l)}$ be the smallest element of E_l such that $m_j^{(l)} > n_s + n_{s-1}$. Then

$$m_{j+h}^{(l)} \geqslant q^h m_j^{(l)} > p(n_s + n_{s-1}) \geqslant n_{s+1} + n_s.$$

Let now f be a continuous function on the circle group T such that $\hat{f}(n) = 0$ if $n \notin E \cup F$. For every positive integer N let

$$S_N(f) = \sum_{n=-N}^{N} \hat{f}(n) e^{int}.$$

The theorem will follow if we prove that for every N

$$\|S_N(f)\|_\infty \leqslant C \|f\|_\infty, \tag{2}$$

where C is a constant depending only on the set $E \cup F$. Clearly, we may suppose that $N \in E \cup F$. For every positive integer n, let

$$K_n(t) = \sum_{j=-n}^{n} \left(1 - \frac{|j|}{n+1}\right) e^{ijt}$$

be the Féjer kernel, and $V_n = 2K_{2n+1} - K_n$ the de la Vallée Poussin kernel. Then (see [2], p. 15) we get

(i) $\|V_n\|_{L^1} \leqslant 3$,

(ii) $\hat{V}_n(j) = 1$ if $|j| \leqslant n+1$, and $\hat{V}_n(j) = 0$ if $|j| > 2n+1$.

Firstly, let $N = n_s + n_{s-1}$. Then, according to (ii),

$$S_N(f) = V_{N-1} * f - \sum \hat{V}_{N-1}(j) \hat{f}(j) e^{ijt}, \tag{3}$$

where the summation is over all $j \in F$ such that

$$n_s + n_{s-1} < j \leqslant 2n_s + 2n_{s-1}.$$

Remark that $2n_s + 2n_{s-1} < n_{s+1} + n_1$, since $n_{s+1}/n_s > 1 + \sqrt{3}$.

Since, by definition, $|\hat{V}_n(j)| \leqslant 1$, and the summation on the right-hand side of (3) contains at most A terms, by (i) we get

$$\|S_N(f)\|_\infty \leqslant (3 + A) \|f\|_\infty. \tag{4}$$

Let $N = k \in F$ with $n_{s+1} + n_1 > k > n_s + n_{s-1}$. Then

$$S_N(f) = S_{n_s + n_{s-1}}(f) + \sum f(j) e^{ijt},$$

where the summation is extended to all $j \in F$ such that $n_s + n_{s-1} < j \leqslant k$. By (4) we have

$$\|S_N(f)\|_\infty \leqslant (3 + 2A) \|f\|_\infty. \tag{5}$$

Let now $N = n_{s+1}+n_r$ with $1 \leqslant r \leqslant s-1$. Since, by (ii),

$$\exp(in_{s+1}t)\,V_{n_r-1} * f = \sum_{|j-n_{s+1}|\leqslant 2n_r-1} \hat{V}_{n_r-1}(j-n_{s+1})\hat{f}(j)\exp(ijt)$$

and $\hat{V}_{n_r-1}(j-n_{s+1}) = 1$ for $|j-n_{s+1}| \leqslant n_r$, the following identity holds true:

$$(6)\qquad S_N(f) = S_{n_s+n_{s-1}}(f)+\exp(in_{s+1}t)\,V_{n_r-1} * f -$$
$$-\sum_{2n_r-1\geqslant|j-n_{s+1}|>n_r} \hat{V}_{n_r-1}(j-n_{s+1})\hat{f}(j)\exp(ijt)+\sum_{n_s+n_{s-1}<j<n_{s+1}-n_r} \hat{f}(j)\exp(ijt).$$

Remark that $\hat{f}(j) \neq 0$ in the summations on the right-hand side of (6) only if $j \in F$. Hence each sum contains at most A terms so that, by (4) and (i),

$$(7)\qquad \|S_N(f)\|_\infty \leqslant (3+A)\|f\|_\infty+3\|f\|_\infty+2A\|f\|_\infty = (6+3A)\|f\|_\infty.$$

Finally, suppose that $N = k \in F$ with $n_{s+1}+n_1 < k < n_{s+1}+n_s$. Then, if r is the largest integer such that $n_{s+1}+n_r < k$, we have

$$S_N(f) = S_{n_{s+1}+n_r}(f)+\sum f(j)e^{ijt},$$

where the summation is over all $j \in F$ such that $n_{s+1}+n_r < j \leqslant k$. Hence

$$(8)\qquad \|S_N(f)\|_\infty \leqslant (6+4A)\|f\|_\infty.$$

Therefore, by (4), (5), (7) and (8), inequality (2) holds with $C = 6+4A$.

REFERENCES

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