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ON COVERING OF BOUNDED SETS BY SETS WITH THE TWICE LESS DIAMETER

BY

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It is proved that every bounded plane set A is the union of 7 sets with diameters less than or equal to $\frac{1}{2}\delta(A)$ and that there exist bounded plane sets A which are not unions of 6 sets with diameters less than or equal to $\frac{1}{2}\delta(A)$.

1. For every $n = 1, 2, \ldots$ and for $0 < a \leqslant 1$ denote by $D_n(a)$ the smallest natural number q such that every bounded set A lying in the n-dimensional Euclidean space E^n is covered by q sets $A_1, \ldots, A_q$ with diameters $\delta(A_i) \leqslant a\delta(A)$ for $i = 1, \ldots, q$ (cf. [2]).

Since the geometric sphere $S^{n-1} \subset E^n$ with radius r cannot be covered by less than n sets with diameters less than $2r$ ([3], p. 178; see also [4]), we infer that

(1) $$D_n(a) > n \quad \text{for every } 0 < a < 1 \text{ and } n = 1, 2, \ldots$$

On the other hand, one shows easily ([2], p. 249) that

(2) $$D_n(a) \geqslant m_1 m_2 \ldots m_n,$$

where $m_1, \ldots, m_n$ are natural numbers such that

$$m_1^{-2} + m_2^{-2} + \ldots + m_n^{-2} \leqslant a^2.$$

Formulas (1) and (2) give an evaluation of $D_n(a)$. However, this evaluation is far to be satisfactory and the problem to compute the exact value of $D_n(a)$ remains open and seems to be hard.

The aim of the present note is to establish the following

THEOREM. *The number $D_2(\frac{1}{2})$ is equal to* 7.

2. First let us show that

(3) $$D_2\left(\frac{1}{2}\right) \leqslant 7.$$

It is evident that inequality (3) will be established if we show that every compact set $A \subset E^2$ with diameter 1 can be covered by 7 compact sets $A_1, \ldots, A_7$ such that $\delta(A_i) \leqslant \frac{1}{2}$ for $i = 1, \ldots, 7$.

It is known (see [1], p. 9) that in E^2 there exists a regular hexagon P_6 with diameter $\delta(P_6) = \frac{2}{3}\sqrt{3}$ containing A. Let $a_1, \ldots, a_6$ be vertices of P_6 (in a cyclic order) (see Fig. 1) and let c denote the center of P_6, and c_i — the center of the segment $\overline{a_i a_{i+1}}$ for $i = 1, \ldots, 6$ (where $a_7 = a_1$). Moreover, let c_i' denote the center of the segment $\overline{cc_i}$. Let T_i denote the regular triangle with vertices c, c_i, c_{i+1}, and T_i' — the triangle with vertices c_i, a_{i+1}, c_{i+1} (where $c_7 = c_1$). Denote by A_7 the circular disk with center c and radius $\varrho(c, c_1') = \frac{1}{4}$ and let

$$A_i = (T_i \cup T_i') \setminus A_7 \quad \text{for } i = 1, \ldots, 6.$$

Then $A \subset P_6 = A_1 \cup \ldots \cup A_7$ and one sees easily that $\delta(A_i) = \frac{1}{2}$ for $i = 1, \ldots, 7$. Thus inequality (3) is proved.

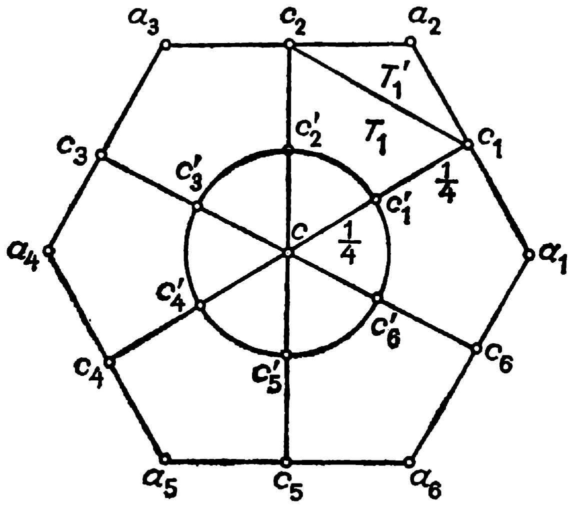

Fig. 1

3. In order to complete the proof of the Theorem, it remains to show that

$$D_2\left(\frac{1}{2}\right) \geqslant 7, \tag{4}$$

that is to show that there exists a set $A \subset E^2$ with diameter 1, for which every covering consisting of 6 sets contains at least one set with diameter greater than $\frac{1}{2}$.

Consider a regular heptagon $P_7 \subset E^2$ with diameter 1 and let c denote its barycenter, and B — its boundary. Let $a_1, \ldots, a_7$ be vertices of P_7 in a cyclic order (see Fig. 2) and let us set, for every integer k,

$$a_{i+7k} = a_i \quad \text{for } i = 1, \ldots, 7.$$

Then

(5) $$1 = \delta(P_7) = \varrho(a_i, a_{i+3}) < 2\varrho(a_i, c) \quad \text{for every } i.$$

If c' denotes the center of the segment $\overline{a_1 a_4}$, then

$$\varrho(a_1, c) > \varrho(a_1, c') = \frac{1}{2}.$$

The segments $\overline{a_i a_{i+1}}$ are called *sides* of P_7. Let c_i denote the center of $\overline{a_i a_{i+1}}$. Two distinct sides of P_7 are said to be *adjacent* one to the other if they have a common vertex. They are said to be *opposite* one to the other if there exists no side adjacent to each of them.

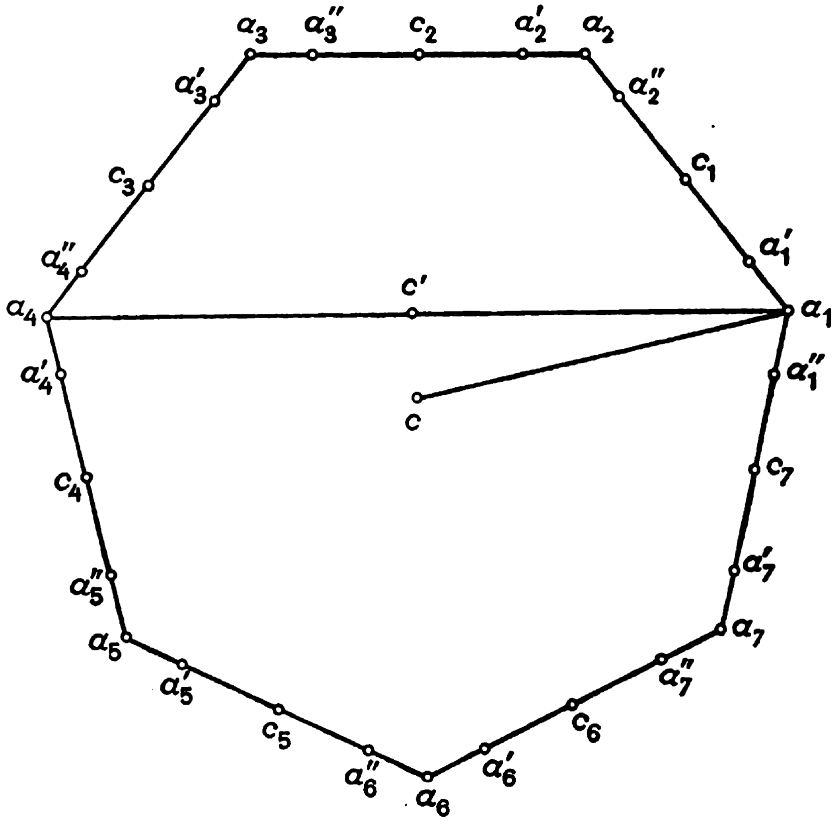

Fig. 2

Let a_i' denote the point of $\overline{a_i a_{i+1}}$ such that $\varrho(a_i', a_{i-1}) = \frac{1}{2}$, and a_i'' – the point of $\overline{a_i a_{i-1}}$ such that $\varrho(a_i'', a_{i+1}) = \frac{1}{2}$.

If x, y are two distinct points of B, then B is the union of two arcs with endpoints x, y. If the lengths of those arcs are not equal, then we denote by $\widehat{(x, y)}$ the shorter of those arcs.

Suppose now, contrary to (4), that there exists a covering of P_7 consisting of 6 sets $A_1, \ldots, A_6$ with diameters less than or equal to $\frac{1}{2}$. We may assume that the sets A_i $(i = 1, \ldots, 6)$ are closed and convex and that the center c of P_7 belongs to A_6. Then (5) implies that A_6 does not contain any vertex a_i of P_7. Now, let us distinguish two cases:

Case I. *Each of the sets* $A_1, \ldots, A_5$ *contains at least one vertex of* P_7.

Case II. *One of the sets* $A_1, \ldots, A_5$ *(say* A_5*) does not contain any vertex of* P_7.

It remains to show that in both cases our hypotheses lead to a contradiction.

Consider first the case I. Since $\varrho(a_i, a_{i+2}) > \frac{1}{2}$, none of the sets $A_1, \ldots, A_5$ contains three vertices. It follows that there exist two non-adjacent sides of P_7, each contained in one of the sets $A_1, \ldots, A_5$. We may assume that one of those sides lies in A_4 and the other in A_5. If those sides are adjacent to one side of P_7 (say to $\overline{a_2 a_3}$), then the center c_2 of $\overline{a_2 a_3}$ and the points a_5, a_6, a_7 belong to $A_1 \cup A_2 \cup A_3$. We may assume that $c_2 \in A_3$. Since $\varrho(c_2, a_i) > \frac{1}{2}$ for $i = 5, 6, 7$, we infer that three vertices a_5, a_6, a_7 belong to $A_1 \cup A_2$, whence one of the sets A_1, A_2 contains one of the sides $\overline{a_5 a_6}$, $\overline{a_6 a_7}$. But the sides $\overline{a_1 a_2}$ and $\overline{a_4 a_5}$ are opposite one to the other, as well the sides $\overline{a_3 a_4}$, $\overline{a_6 a_7}$ are opposite one to the other. Consequently, in the case I there exist two opposite sides of P_7 such that each of them is contained in one of the sets $A_1, \ldots, A_5$.

Thus we may assume that

$$\overline{a_1 a_2} \subset A_5, \quad \overline{a_4 a_5} \subset A_4 \quad \text{and} \quad a_3 \in A_3. \tag{6}$$

Now let us consider the points $a_2' \in \overline{a_2 a_3}$ and $a_4'' \in \overline{a_3 a_4}$ and let us distinguish the following subcases:

Subcase I_1. $A_6 \cap \widehat{(a_2', a_4'')} = \emptyset$.

Subcase I_2. $A_6 \cap \widehat{(a_2', a_4'')} \neq \emptyset$.

In the subcase I_1, we infer by the inequality $\varrho(a_2', a_4'') > \frac{1}{2}$ and by (6) that at least one of the sets A_1, A_2 (say A_2) intersects $\widehat{(a_2', a_4'')}$. Then the sets A_2, A_3, A_4 and A_5 do not intersect the interior of the arc $\widehat{(a_5', a_1'')}$ and we infer that this last arc is a subset of $A_1 \cup A_6$. But $a_6, a_7 \in \widehat{(a_5', a_1'')} \setminus A_6$, whence $a_6, a_7 \in A_1$. It follows that $c_5, c_7 \in A_6$, which contradicts the inequality $\varrho(a_5, c_7) > \frac{1}{2}$. Thus the subcase I_1 is impossible.

In the subcase I_2, the sets A_6 and $\widehat{(a_5', a_1'')}$ are disjoint. Then (6) implies that the arc $\widehat{(a_5', a_1'')}$ must be covered by A_1, A_2, A_3 and, since $a_3 \in A_3$, the arc $\widehat{(a_5', a_1'')}$ lies in $A_1 \cup A_2$. We may assume that $a_5' \in A_1$ and $a_1'' \in A_2$. Since $\varrho(c_6, a_1'') > \frac{1}{2}$ and $\varrho(c_6, a_6') > \frac{1}{2}$, we infer that the point $c_6 \in \widehat{(a_5', a_1'')}$ does not belong to $A_1 \cup A_2$, which contradicts the inclusion $\widehat{(a_5', a_1'')} \subset A_1 \cup A_2$. Thus the subcase I_2 is also impossible.

In the case II, all vertices $a_1, \ldots, a_7$ belong to the set $A_1 \cup A_2 \cup A_3 \cup A_4$. Since none of the sets A_i contains three vertices, we infer that there exist three sides of P_7 such that each of them lies in one of the sets $A_1, \ldots$ $\ldots, A_4$ and no two of those sides are adjacent one to the other. We may assume that

$$\overline{a_1 a_2} \subset A_4, \quad \overline{a_4 a_5} \subset A_3 \quad \text{and} \quad \overline{a_6 a_7} \subset A_2.$$

Consider the points a_2' and a_4'' (see Fig. 2) and the arc $(\widehat{a_2', a_4''})$. Then the vertex a_3 does not belong to $A_2 \cup A_3 \cup A_4$, whence $a_3 \in A_1$. Observe that both points c_2, c_3 do not belong to $A_2 \cup A_3 \cup A_4$ and that at least one of them does not belong to A_1. Denote this last point by d. Consequently, d belongs to one of the sets A_5, A_6, say to A_6. Then A_6 and the set $\overline{a_5 a_6} \cup \overline{a_1 a_7}$ are disjoint and we infer that

$$\overline{a_5 a_6} \cup \overline{a_1 a_7} \subset A_2 \cup A_3 \cup A_4 \cup A_5.$$

But it is clear that the centers c_5 of $\overline{a_5 a_6}$ and c_7 of $\overline{a_1 a_7}$ do not belong to $A_2 \cup A_3 \cup A_4$. Consequently, $c_5, c_7 \in A_5$, which contradicts the inequality $\varrho(a_5, a_7) > \frac{1}{2}$. Thus the case II is also impossible and the proof of the Theorem is complete.

REFERENCES

[1] В. Г. Болтянский и И. Ц. Гохберг, *Разбиение фигур на меньшие части*, Москва 1971.

[2] K. Borsuk, *Some remarks on covering of bounded subsets of the Euclidean n-space with sets of smaller diameter*, Demonstratio Mathematica 11 (1978), p. 247-251.

[3] — *Drei Sätze über die n-dimensionale euklidische Sphäre*, Fundamenta Mathematicae 20 (1933), p. 177-190.

[4] L. Lusternik et L. Schnirelmann, *Méthodes topologiques dans les problèmes variationnels*, Moscou 1930.

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