COLLOQUIUM MATHEMATICUM

VOL. XXXII 1975 FASC. 2

ON NON-SMOOTHLY EQUIVALENT DIFFEOMORPHISMS OF THE CIRCLE

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In one of his papers Arnold [1] states the following conjecture:

If two structurally stable C^2-diffeomorphisms φ and ψ of the unit circle S^1 have the same

1. rotation numbers $\varrho_\varphi = \varrho_\psi$,
2. numbers of periodic points $m_\varphi = m_\psi = m$,
3. prime periods of periodic points $p_\varphi = p_\psi = p$,
4. values of the first derivatives $d\varphi^p(x_i) = d\psi^p(y_i)$ at the corresponding periodic points x_i and y_i for $i = 1, \ldots, m$,

then φ and ψ are C^1-diffeomorphically conjugate (loc. cit., p. 81).

It is easy to show that such φ and ψ are topologically conjugate.

In this note we shall show that the conjecture is not true. We shall construct two structurally stable C^∞-diffeomorphisms φ and ψ of the unit circle such that they have the same numbers defined in 1-3, and $d\varphi^r(x_i) = d\psi^r(y_i)$ for all $r = 1, 2, \ldots$, where x_i and y_i are the corresponding periodic points, $i = 1, 2, \ldots, m$, and φ and ψ are not C^1-diffeomorphically conjugate.

Let $J = \langle 0, 1\rangle$ be the unit interval on the real line. It suffices to find two C^∞-diffeomorphisms φ and ψ of J such that they have no fixed points inside J, $|d\varphi(0)|, |d\psi(0)|, |d\varphi(1)|, |d\psi(1)| \neq 1$ (the necessary condition for the structural stability), φ and ψ are equal each to other in some neighborhoods of the end points 0 and 1, and they are not C^1-diffeomorphically conjugate.

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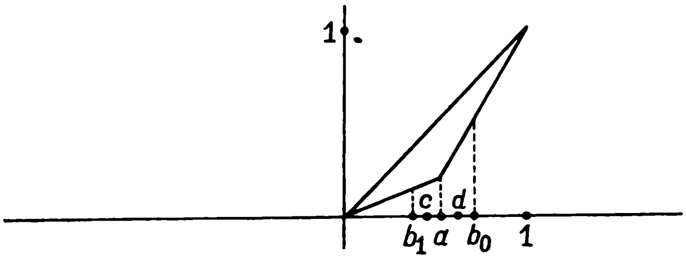
(1) В. И. Арнольд, *Малые знаменатели, I. Об отображениях окружности на себя,* Известия Академии наук СССР 25 (1961), p. 21-86.

Denote by $f\colon \langle 0,1\rangle \to \langle 0,1\rangle$ a continuous, piecewise linear function with the following graph:



The function f is linear on the intervals $\langle 0, a\rangle$ and $\langle a, 1\rangle$. Let b_0 be a point from the interval $(a, 1\rangle$ such that $f(b_0) = b_1 \in \langle 0, a)$, and let c, d be two numbers such that $b_1 < c < a < d < b_0$. Write $I = (b_1, b_0)$. We set $\psi(x) = f(x)$ for $x \notin I$ and we extend it to a C^∞-diffeomorphism of the whole $\langle 0, 1\rangle$, not linear on any subinterval of I. Now we set $\varphi(x) = f(x)$ for $x \notin (c, d)$ and we extend it to a C^∞-diffeomorphism of the whole $\langle 0, 1\rangle$.

Suppose that φ and ψ are smoothly conjugate by a diffeomorphism $h\colon \langle 0,1\rangle \to \langle 0,1\rangle$, i.e.

$$h\circ\varphi^n(x) = \psi^n\circ h(x), \qquad n = 0, \pm 1, \pm 2, \ldots$$

The function h is linear on the interval $\langle 0, \min(b_1, h^{-1}(b_1))\rangle$. Indeed, let $x < \min(b_1, h^{-1}(b_1))$. Then

$$h'(\varphi^n(x))\cdot\varphi^{n\prime}(x) = \psi^{n\prime}(h(x))\cdot h'(x) \qquad \text{for } n = 0, 1, 2, \ldots$$

and $\varphi^{n\prime} = \psi^{n\prime} = f^{n\prime}$, so $h'(\varphi^n(x)) = h'(x)$. Letting $n \to +\infty$, we get $h'(x) = h'(0)$. It means there exists a linear function l such that $h(x) = l(x)$ for all $x \in \langle 0, \min(b_1, h^{-1}(b_1))\rangle$. In the same way we show that there exists a linear function k such that $h(x) = k(x)$ for $x \in \langle \max(b_0, h^{-1}(b_0)), 1\rangle$.

Let p be an integer such that

$$\varphi^p(I) \subset \langle 0, \min(b_1, h^{-1}(b_1))\rangle \qquad \text{and} \qquad \varphi^{-p}(I) \subset \langle \max(b_0, h^{-1}(b_0)), 1\rangle.$$

Then, for $x \in I$,

$$l\circ\varphi^p(x) = h\circ\varphi^p(x) = \psi^p\circ h(x)$$

which yields

$$h(x) = \psi^{-p}\circ l\circ\varphi^p(x). \tag{1}$$

On the other hand, for $y \in \varphi^{-p}(I)$, $y = \varphi^{-p}(x)$ and $x \in I$, we have

$$h\circ\varphi^p(y) = \psi^p\circ h(y) = \psi^p\circ k(y);$$

hence

$$h(x) = \psi^p\circ k\circ\varphi^{-p}(x). \tag{2}$$

Therefore, by (1) and (2),

$$\psi^{-p}\circ l\circ\varphi^p(x) = \varphi^p\circ k\circ\varphi^{-p}(x) \qquad \text{for } x \in I.$$

The latter equality we rewrite in the following way:

$$l \circ \varphi^p(x) = \psi^{2p} \circ k \circ \varphi^{-p}(x).$$

For $b_1 < x < c$ the functions $l \circ \varphi^p$ and $k \circ \varphi^{-p}$ are linear by construction; therefore, ψ^{2p} is linear on the interval $k \circ \varphi^{-p}\big((b_1, c)\big)$. Let $P \subset k \circ \varphi^{-p}\big((b_1, c)\big)$ be an interval such that there exists an integer i for which $\psi^i(P) \subset I$ (such an interval exists in view of $\psi(b_0) = b_1$). Since $P \subset (b_0, 1\rangle$ and $\psi^{2p}(P) \subset \langle 0, b_1)$, the index i is less than $2p$. Therefore, the composition $\psi^{2p} = \psi \circ \psi \circ \ldots \circ \psi$ contains $2p-1$ linear functions and one non-linear function (the $(i+1)$-st term), so ψ^{2p} cannot be linear on P. This completes the proof.

QUESTION. Is the conjecture true for analytic functions? (**P 932**)

Reçu par la Rédaction le 13. 5. 1973