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ON THE STRUCTURE OF TRANCHES IN CONTINUOUSLY IRREDUCIBLE CONTINUA

BY

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1. Introduction. A continuum (compact, connected, metric space) X is said to be *irreducible* if there are points p and q in X such that no proper subcontinuum of X contains both p and q. The fundamental structure theorem for irreducible continua was proved by Kuratowski (see [10], p. 200): If X is an irreducible continuum such that every indecomposable subcontinuum of X has void interior, then X admits a finest monotone decomposition whose quotient space is an arc, i.e., there is a monotone mapping $m: X \twoheadrightarrow I$ such that every other monotone mapping of X onto the unit interval I can be factored through m. The continua $m^{-1}(t)$ for t in I are called *tranches* or *layers* of X. In 1935 Knaster [8] gave an example of an irreducible continuum K, each tranche of which is non-degenerate and such that the canonical map m is open or, equivalently, the continuum-valued map m^{-1} is continuous. We will call such continua *continuously irreducible*. Another well-known example is the Bing–Jones "arc of pseudo-arcs" (see [1]). Both of these continua contain tranches which are indecomposable and chainable (in the arc of pseudo-arcs every tranche is a pseudo-arc). The question whether every continuously irreducible continuum must contain a hereditarily indecomposable tranche was raised by Knaster [9]. Dyer [4] has shown that every continuously irreducible continuum must contain an indecomposable tranche and Oversteegen and Tymchatyn [13] have shown that any such continuum must contain an indecomposable tranche (in fact, a dense family of such tranches) containing indecomposable subcontinua of arbitrarily small diameter. However, the general answer to Knaster's question is "no", as we show below. In Section 2 we produce a fairly simple modification of Knaster's continuum, yielding a continuously irreducible continuum, every tranche of which contains an arc. We then proceed to a more complicated construction allowing us to modify any one-dimensional continuously irreducible continuum into one, every subcontinuum of which contains an arc.

The latter construction derives in a sense from Janiszewski's [7] famous construction in which he inserts a $\sin(1/x)$-curve "everywhere" in an arc to produce a rational curve containing no arcs. However, the method is modelled directly on a construction of Cook [3], used to produce continua admitting no non-degenerate self-maps. Cook's method has since been used to produce a variety of examples in papers by Ingram [6], Maćkowiak [11], and the authors [12].

An essential ingredient in the main construction is the notion of an atomic map, introduced by Cook in [3].

DEFINITION 1.1. Let X^* and X be continua. A mapping g of X^* onto X is said to be *atomic* if for every subcontinuum L of X^* such that $g(L)$ is non-degenerate we have $L = g^{-1}(g(L))$.

The canonical monotone map of a $\sin(1/x)$-curve onto an arc is an example of an atomic map. All atomic maps are known to be monotone (see [5]). There is also an affinity between atomic maps, open maps and irreducibility, as the following lemmas show. The first is easy to prove.

LEMMA 1.1. *Let X^* be a continuum admitting an atomic map onto the irreducible continuum X. Then X^* is irreducible.*

LEMMA 1.2. *Let X and Y be continua and let m be a monotone open mapping of X onto Y such that $m^{-1}(y)$ is non-degenerate for every $y \in Y$. Let X^* be a continuum and let g be an atomic map of X^* onto X. Then $m \circ g$ is an open map.*

Proof. Let $\{y_n\}$ be a sequence of points in Y converging to the point y. We must show that the sequence of continua $L_n = g^{-1}(m^{-1}(y_n))$ converges to the continuum $L = g^{-1}(m^{-1}(y))$ in X^*. Suppose not. Then some subsequence of the L_n's converges to a proper subcontinuum P of L. Without loss of generality we assume that the entire sequence L_n converges to P. Now since m is open, the continua $m^{-1}(y_n)$ converge to the continuum $m^{-1}(y)$. By the continuity of g, it follows that $g(P) = m^{-1}(y)$. But $m^{-1}(y)$ is non-degenerate and $L = g^{-1}(g(P))$, so this contradicts the fact that g is atomic.

COROLLARY 1.1. *If X is a continuously irreducible continuum and X^* is a continuum admitting an atomic mapping onto X, then X^* is continuously irreducible.*

2. Examples.

EXAMPLE 2.1. A continuously irreducible continuum, every tranche of which contains an arc. Let K be a copy of Knaster's continuously irreducible continuum [8], embedded in the upper half-plane (as in Knaster's construction) in such a way that each of its tranches meets the unit interval $[0, 1]$ on the x-axis. Let

$$K^* = \{(x, y, z) \in E^3 \colon (x, y) \in K,\ y > 0, \text{ and } z = \sin(1/y)\}$$
$$\cup \{(x, y, z) \in E^3 \colon x \in [0, 1] \cap K,\ y = 0, \text{ and } -1 \leqslant z \leqslant 1\}.$$

It is not difficult to verify that K^* has the desired properties(1).

(1) Note in particular that K^* admits an atomic mapping onto K. We also note in passing that K^* is chainable, and hence embeddable in the plane.

EXAMPLE 2.2. A continuously irreducible continuum, every subcontinuum of which contains an arc. The desired example is any continuum X^* as in the following theorem. X^* is continuously irreducible by Corollary 1.1.

THEOREM 2.1. *Let X be any continuously irreducible one-dimensional continuum. Then there is a continuously irreducible one-dimensional continuum X^*, every subcontinuum of which contains an arc and such that X^* admits an atomic mapping onto X.*

Proof. Let X be as above and let $\{X_i, f_i^j\}$ be an inverse sequence of finite graphs with piecewise linear, light bonding maps(2) whose inverse limit is X. Let $D = \{p_1, p_2, \ldots\}$ be a countable dense subset of X_1. (Note that then, for every n, $(f_1^n)^{-1}(D)$ is countable and dense in X_n.) To simplify the discussion, we further assume that the points p_k have been chosen so that, for every $n = 1, 2, \ldots$ and for every $x \in (f_1^n)^{-1}(p_k)$, x is neither a vertex nor an endpoint of X_n and there is a small open subinterval O_n of X_n containing x such that, for every $j = 2, 3, \ldots$, f_1^j is a homeomorphism on O_n.

The space X^* will be the inverse limit of a doubly infinite array of continua $X_{i,j}^*$. Roughly speaking, each $X_{i,j}^*$ will be a copy of the graph X_j with $\sin(1/x)$-curves inserted at the points $(f_1^k)^{-1}(p_k)$, $k = 1, 2, \ldots, i$. We will define the space $X_{i,j}^*$ one row at a time. Let $X_{1,1}^*$ be the graph X_1 with a small interval containing p_1, but no vertices or endpoints of X_1, replaced by two-sided $\sin(1/x)$-curve. Let $g_{1,1}\colon X_{1,1}^* \to X_1$ be the natural atomic mapping which sends the limit arc of the $\sin(1/x)$-curve to p_1 and is one-to-one elsewhere. $X_{1,2}^*$ is constructed from X_2 by inserting double $\sin(1/x)$-curves in small disjoint subintervals of X_2 containing the (finitely many) points $(f_1^2)^{-1}(p_1)$ and restricted to which, f_1^2 is a homeomorphism. $g_{1,2}\colon X_{1,2}^* \to X_2$ is the natural atomic map collapsing the limit arcs of the $\sin(1/x)$-curves to the points of $(f_2^1)^{-1}(p_1)$ and one-to-one elsewhere. The $\sin(1/x)$-curves should be copies of sub-curves of the $\sin(1/x)$-curve in $X_{1,1}^*$ obtained by mapping the subintervals of X_2 on which they are inserted into X_1 by f_2^1 and then inverting under $g_{1,1}$. Thus $f_{1,2}^*\colon X_{1,2}^* \to X_{1,1}^*$ may be defined to be a homeomorphism on these curves making the diagram

$$\begin{tikzcd} X_1 & X_2 \arrow[l, "f_2^1"'] \\ X_{1,1}^* \arrow[u, "g_{1,1}"] & X_{1,2}^* \arrow[l, "f_{1,2}^*"] \arrow[u, "g_{1,2}"'] \end{tikzcd}$$

commute. (The intervals in X_2 may also be chosen so that the $\sin(1/x)$-curves "cross the x-axis" at their endpoints.) Finally, extend $f_{1,2}^*$ to the rest of $X_{1,2}^*$ so that the above diagram continues to commute. The rest of the construction is very similar. The successive spaces $X_{i,n}^*$ are constructed by inserting

(2) The bonding maps may be taken piecewise linear and light by Brown's theorem [2].

two-sided $\sin(1/x)$-curves in X_n at each point of the set $(f_1^n)^{-1}(p_1)$. The insertions are made on little disjoint intervals containing no vertices or endpoints of X_n and restricted to which, f_{n-1}^n is a homeomorphism. Thus we may define an atomic map $g_{1,n}$: $X_{1,n}^* \to X_n$ which collapses the limit arcs of the inserted $\sin(1/x)$-curves to points and is one-to-one elsewhere and a map $f_{1,n}^*$: $X_{1,n}^* \to X_{1,n-1}^*$ which is a homeomorphism on each of the inserted $\sin(1/x)$-curves and such that the following diagram commutes:

$$\begin{tikzcd} X_{n-1} & X_n \arrow[l, "f_{n-1}^n"'] \\ X_{1,n-1}^* \arrow[u, "g_{1,n-1}"] & X_{1,n}^* \arrow[l, "f_{1,n}^*"] \arrow[u, "g_{1,n}"'] \end{tikzcd}$$

The successive rows are defined from the previous rows in a similar way. Each space $X_{i,j}^*$ is obtained from $X_{i-1,j}^*$ by inserting little $\sin(1/x)$-curves at each point of the set

$$(g_{i-1,j})^{-1} \circ (g_{i-2,j})^{-1} \circ \ldots \circ (g_{1,j})^{-1} \circ (f_1^j)^{-1}(p_i).$$

$g_{i,j}$: $X_{i,j}^* \to X_{i-1,j}^*$ is the natural atomic map which collapses the limit arcs of the newly inserted $\sin(1/x)$-curves to points and is one-to-one elsewhere. The $\sin(1/x)$-curves are inserted on intervals so that the map $f_{i,j}^*$: $X_{i,j}^* \to X_{i,j-1}^*$ may be defined to be a homeomorphism on them and so that in general the following diagram commutes:

$$\begin{tikzcd} X_{i-1,j-1}^* & X_{i-1,j}^* \arrow[l, "f_{i-1,j}^*"'] \\ X_{i,j-1}^* \arrow[u, "g_{i,j-1}"] & X_{i,j}^* \arrow[l, "f_{i,j}^*"] \arrow[u, "g_{i,j}"'] \end{tikzcd}$$

Note that no new insertions are ever made in the limit bars of the previously inserted $\sin(1/x)$-curves (unlike the constructions of Janiszewski and Cook) so that all bonding maps on components of pre-images of these arcs in factor spaces farther out in the inverse limit system are homeomorphisms.

It is straightforward to show that the inverse limit X^* of this array admits an atomic map g: $X^* \to X$ such that the inverse image of each point is either an arc or a point. Moreover, let L be any non-degenerate subcontinuum of X^*. Then L must have a non-degenerate projection in some factor space $X_{i,j}^*$. If L projects into the limit arc of some inserted $\sin(1/x)$-curve, then, by the last sentence of the previous paragraph, L is an arc. Otherwise, some projection of L in a factor space $X_{i+k,j}^*$ will contain a $\sin(1/x)$-curve and, by the same reasoning, L will contain a homeomorphic copy of the limit bar of this $\sin(1/x)$-curve.

3. Generalizations and questions. The construction in Theorem 2.1 can be used to insert not only arcs, but any continuum K "everywhere" in any one-dimensional continuum X. And if K has some hereditary property (e.g., hereditary unicoherence, atriodicity or is a pseudo-arc; the hereditary property in 2.1 was "is an arc"), then every subcontinuum of the new continuum X^* will contain a continuum with that property. One may also ask whether various properties of X will be preserved by X^*, as was the property of continuous irreducibility in our example. We offer a theorem, without proof, summarizing some fairly straightforward results.

THEOREM 3.1. *Let X be a one-dimensional continuum and let X^* be a continuum constructed from X by inserting continua, as in Theorem* 2.1. *If X and each of the inserted continua has property $\mathscr{P}$, where $\mathscr{P}$ is any of the following, then X^* will also have property $\mathscr{P}$: arclikeness, treelikeness, atriodicity, span* 0, *hereditary unicoherence.*

The reason that X must have dimension 1 in our construction is that in order for our construction to yield inserted continua in every subcontinuum of X^*, the factor spaces X_i must have the property that each of their non-degenerate subcontinua (in particular, projections of non-degenerate continua in X^*) has non-void interior. However, two of the above properties will be preserved by any continuum X^* admitting an atomic map onto any other continuum X, where X and every non-degenerate point inverse have property $\mathscr{P}$; namely, treelikeness and span 0. In light of the above remark, we may ask:

QUESTION 3.1. **(P 1330)** Is there an inverse limit construction for atomically inserting a continuum in every subcontinuum of a continuum X of arbitrary dimension?

A fairly radical type of atomic map is one in which every point inverse is non-degenerate. For example, the Bing–Jones arc of pseudo-arcs admits an atomic mapping onto the unit interval such that every point inverse is a pseudo-arc. The following question seems quite hard. We note in particular that the space S^* would have to have dimension 1.

QUESTION 3.2. **(P 1331)** Does there exist a continuum S^* admitting an atomic mapping onto the 2-sphere, such that every point inverse is a pseudo-arc?

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