COLLOQUIUM MATHEMATICUM

VOL. XV 1966 FASC. 2

*DEDICATED TO PROFESSOR*
*ALEXANDER DONIPHAN WALLACE*
*ON HIS SIXTIETH BIRTHDAY*

# *ON CONFLUENT MAPPINGS*

BY

A. LELEK (WROCŁAW)

Let $X$ and $Y$ be topological spaces. A mapping $f\colon X \to Y$ of $X$ onto $Y$ will be called *confluent* if for each connected closed subset $C$ of $Y$, and points $x \in f^{-1}(C)$ and $y \in C$, the set $f^{-1}(C)$ is connected between $\{x\}$ and $f^{-1}(y)$. In this definition we slightly modify the concept of confluent mappings, introduced recently by J. J. Charatonik, but nothing changes if we restrict ourselves to mappings of continua (see [1], p. 213). On the other hand, confluent mappings generalize the notion of quasi-monotone mappings of locally connected continua, due to A. D. Wallace (see [6], p. 138). Moreover, by a theorem of G. T. Whyburn, all open mappings of compacta are confluent (see [7], p. 148).

Recall that a space $X$ is said to be *contractible* relative to a space $Y$ if each mapping $f\colon X \to Y$ is null-homotopic. We shall denote by $R$ the set of all real numbers and by $S$ the set of complex numbers with module one. Since $0 \notin S$ but $1 \in S$, we shall follow on the lines of Eilenberg and Kuratowski in writing $f \sim 1$ to state that a mapping $f\colon X \to S$ is null-homotopic (see [4], p. 310). For an arbitrary topological space $X$ and a mapping $f\colon X \to S$, we have $f \sim 1$ if and only if there exists a mapping $\varphi\colon X \to R$ such that $f(x) = e^{2\pi i \varphi(x)}$ for $x \in X$ (see [5], p. 39). Any such continuous function $\varphi$ will be said to be *attached* to the mapping $f$.

It is known that the contractibility of compacta, i. e. compact metric spaces, relative to $S$ is an invariant under monotone or open mappings. Moreover, by a theorem of S. Eilenberg, if $g\colon X \to Y$ is a monotone (or open) mapping of a compactum $X$ onto a compactum $Y$ and $f\colon Y \to S$, then $fg \sim 1$ implies $f \sim 1$ (see [4], p. 331). In the present note we shall generalize this Eilenberg's theorem to confluent mappings. Thus, for not necessarily locally connected continua, confluent mappings seem to be a good analogue of both monotone and open mappings, like the contractibility relative to $S$ is an analogue of the unicoherence.

Suppose $C$ is a continuum, i. e. a connected compactum, and $f\colon C \to S$ is a null-homotopic mapping. Take a point $x_0 \in C$ and a number

$t_0 \in R$ such that $f(x_0) = e^{2\pi i t_0}$. Then a function $\varphi\colon C \to R$ attached to $f$ is uniquely determined by the condition $\varphi(x_0) = t_0$ (see [4], p. 309). It follows that for a pair of points $p$, $q \in C$ the number $\varphi(p) - \varphi(q)$ depends on neither $x_0$ nor $t_0$.

If $Y$ is a metric space, the distance between points $p$, $q \in Y$ will be denoted by $\varrho(p, q)$. The open ball with centre $p \in Y$ and radius $\varepsilon > 0$ will be denoted by $Q(p, \varepsilon)$. For a mapping $f\colon Y \to S$, the symbols $f$ *irr non*$\sim$1 will mean that $f$ *non*$\sim$1 but $f|C \sim 1$ for each closed proper subset $C$ of $Y$.

LEMMA 1. *Let $Y$ be a continuum and $f\colon Y \to S$ a mapping such that $f$ irr non$\sim$1. If $C$ is a proper subcontinuum of $Y$ and $\varepsilon > 0$, then there exists a proper subcontinuum $K$ of $Y$ and points $p$, $q \in K$ such that $C \subset K$ and*

$$\varrho(p, q) < \varepsilon, \quad |\varphi(p) - \varphi(q)| \geqslant \tfrac{2}{3},$$

*where $\varphi$ is a function attached to $f|K$.*

Proof. Suppose on the contrary that there exists a number $\varepsilon_0 > 0$ such that, for every proper subcontinuum $K$ of $Y$ containing $C$, and for each pair of points $p$, $q \in K$, the inequality

$$\varrho(p, q) < \varepsilon_0$$

implies the inequality

$$|\varphi(p) - \varphi(q)| < \tfrac{2}{3}.$$

Let $\eta < \frac{1}{3}\varepsilon_0$ be a positive number such that if $p$, $q \in Y$ and

$$\varrho(p, q) < \eta,$$

then

$$|f(p) - f(q)| < \sqrt{2 - \sqrt{2}}$$

(the number on the right side in the last inequality is the length of a chord in the unit circle $S$ that spans the angle $\frac{1}{4}\pi$). We can obviously find a proper subcontinuum $K_0$ of $Y$ such that $C \subset K_0$ and that there exists, for each $y \in Y$, a point $p \in K_0$ satisfying

$$\varrho(p, y) < \eta. \tag{1}$$

Since $f$ *irr non*$\sim$1, there is a function $\varphi_0$ attached to $f|K_0$. Take an arbitrary point $y \in Y$. If $p \in K_0$ and (1) holds, the radii from the centre of $S$ to $f(p)$ and $f(y)$ form an angle less than $\frac{1}{4}\pi$. Consequently, there is exactly one real number $\varphi^p(y)$ such that

$$|\varphi_0(p) - \varphi^p(y)| < \tfrac{1}{8}$$

and

$$f(y) = e^{2\pi i \varphi^p(y)}.$$

Now let $q \epsilon K_0$ be another point satisfying inequality (1) with $p = q$; then

$$\varrho(p, q) \leqslant \varrho(p, y) + \varrho(y, q) < 2\eta < \varepsilon_0,$$

whence

$$|\varphi^p(y) - \varphi^q(y)| \leqslant |\varphi^p(y) - \varphi_0(p)| + |\varphi_0(p) - \varphi_0(q)| + |\varphi_0(q) - \varphi^q(y)|$$
$$< \tfrac{1}{8} + \tfrac{2}{3} + \tfrac{1}{8} = \tfrac{11}{12}.$$

It follows from

$$e^{2\pi i \varphi^p(y)} = f(y) = e^{2\pi i \varphi^q(y)}$$

that $\varphi^p(y) = \varphi^q(y)$ and the number

$$\varphi(y) = \varphi^p(y)$$

is well-determined, independently on points $p \epsilon K_0$ satisfying (1). Thus we get a function $\varphi\colon Y \to R$, and observe that $\varphi$ is an extension of $\varphi_0$. It turns out however that $\varphi$ is attached to $f$, which contradicts the relation $f$ *non*$\sim 1$. In order to see this it remains to show that $\varphi$ is continuous.

For an arbitrary sequence of points $y_j \epsilon Y$ $(j = 0, 1, \ldots)$ with

$$y_0 = \lim_{j\to\infty} y_j$$

let us take points $p_j \epsilon K_0$ such that

$$\varrho(p_j, y_j) < \eta$$

for $j = 0, 1, \ldots$ and an index $j_0$ such that

$$\varrho(y_0, y_j) < \eta$$

for $j \geqslant j_0$. Then we have

$$|\varphi(p_j) - \varphi(y_j)| < \tfrac{1}{8}$$

for $j = 0, 1, \ldots$ and

$$\varrho(p_0, p_j) \leqslant \varrho(p_0, y_0) + \varrho(y_0, y_j) + \varrho(y_j, p_j) < 3\eta < \varepsilon_0$$

for $j \geqslant j_0$, whence

$$|\varphi(y_0) - \varphi(y_j)| \leqslant |\varphi(y_0) - \varphi(p_0)| + |\varphi(p_0) - \varphi(p_j)| + |\varphi(p_j) - \varphi(y_j)|$$
$$< \tfrac{1}{8} + \tfrac{2}{3} + \tfrac{1}{8} = \tfrac{11}{12}$$

for $j \geqslant j_0$. It follows from

$$e^{2\pi i\varphi(y_0)} = f(y_0) = \lim_{j\to\infty} f(y_j) = \lim_{j\to\infty} e^{2\pi i\varphi(y_j)}$$

that

$$\varphi(y_0) = \lim_{j\to\infty}\varphi(y_j).$$

LEMMA 2. *If $Y$ is a continuum and $f\colon Y \to S$ is a mapping such that $f$ irr non$\sim$1, then there exist proper subcontinua $Y_1, Y_2, \ldots$ of $Y$ and points $y_0, y_1, y_2, \ldots$ such that for each $j = 1, 2, \ldots$ we have $y_0, y_j \in Y_j$ and*

$$y_0 = \lim_{j\to\infty} y_j, \qquad |\varphi_j(y_0)-\varphi_j(y_j)| \geqslant \tfrac{11}{12},$$

*where $\varphi_j$ is a function attached to $f|Y_j$.*

Proof. We first define proper subcontinua $C_1 \subset C_2 \subset \ldots$ of $Y$ and a sequence of points $a, p_1, p_2, \ldots$ converging to $a$ in $Y$, such that for $l = 1, 2, \ldots$ we have $p_l \in C_l$ and each function $\varphi$ attached to $f|C_{l+1}$ satisfies the inequality

$$(2) \qquad |\varphi(p_l)-\varphi(p_{l+1})| \geqslant \tfrac{1}{3}.$$

Indeed, applying successively Lemma 1 we get an increasing sequence of proper subcontinua $C_l$ of $Y$ and points $p_l', q_l' \in C_l$ such that

$$\varrho(p_l', q_l') < 1/l, \qquad |\varphi(p_l')-\varphi(q_l')| \geqslant \tfrac{2}{3},$$

where $l = 1, 2, \ldots$ and $\varphi$ is a function attached to $f|C_l$. Since $Y$ is a compactum, we can assume the points $p_1', p_2', \ldots$ converge in $Y$. Let us denote their limit by $a$. Thus $q_1', q_2', \ldots$ also converge to $a$. Put $p_1 = p_1'$ and suppose $p_l$ is already defined. If $\varphi$ is a function attached to $f|C_{l+1}$, then

$$|\varphi(p_l)-\varphi(p_{l+1}')| + |\varphi(p_l)-\varphi(q_{l+1}')| \geqslant |\varphi(p_{l+1}')-\varphi(q_{l+1}')| \geqslant \tfrac{2}{3}$$

and therefore (2) holds for $p_{l+1}$ equal to $p_{l+1}'$ or $q_{l+1}'$. We define $p_{l+1}$ accordingly.

Let $\eta > 0$ be a number such that if $p, q \in Y$ and $\varrho(p, q) < \eta$, then

$$|f(p)-f(q)| < \sqrt{2-\sqrt{3}}$$

(the number on the right side is the length of a chord in $S$ corresponding to the angle $\frac{1}{6}\pi$). Hence, for each closed proper subset $C$ of $Y$ containing $p$ and $q$ with $\varrho(p, q) < \eta$, a function $\varphi$ attached to $f|C$ must satisfy

$$|\varphi(p)-\varphi(q)| < \tfrac{1}{12} \quad \text{or} \quad |\varphi(p)-\varphi(q)| > \tfrac{11}{12}.$$

The continua $C_l$ are closed proper subsets of $Y$ and form an increasing sequence. Thus we see that, for a given function $\psi_1$ attached to $f|C_1$, there is a unique extension $\psi_l$ of $\psi_1$ such that $\psi_l$ is attached to $f|C_l$. In the sequel we shall use the symbol $\varphi$ to denote any of the functions $\psi_l$ $(l = 1, 2, \ldots)$.

Choose points $q_j \in C_j$ such that

$$\lim_{j\to\infty} q_j = \lim_{l\to\infty} p_l = a \neq q_j \neq p_l$$

for $j$, $l = 1, 2, \ldots$ We can assume that all the points $p_l$ and $q_j$ belong to the ball $Q(a, \frac{1}{4}\eta)$. Let $C_{jkl}$ be the connected component of the set

$$C_l \setminus Q\big(q_j, \varrho(p_l, q_j)/k\big),$$

to which the point $p_l$ belongs, for $k = 1, 2, \ldots$ Then there is a common point $r_{jkl}$ of $C_{jkl}$ and the boundary of the ball $Q\big(q_j, \varrho(p_l, q_j)/k\big)$. It follows that

$$\varrho(q_j, r_{jkl}) = \varrho(p_l, q_j)/k \tag{3}$$

for $j, k, l = 1, 2, \ldots$ and

$$\varrho(p_l, r_{jkl}) \leqslant \varrho(p_l, q_j) + \varrho(q_j, r_{jkl}) \leqslant 2\varrho(p_l, q_j) < \eta,$$

whence

$$|\varphi(p_l) - \varphi(r_{jkl})| < \tfrac{1}{12} \quad \text{or} \quad |\varphi(p_l) - \varphi(r_{jkl})| > \tfrac{11}{12}.$$

Case 1. There exists $j_0$ such that, for each $k$, there exists $l_k \geqslant j^0$ such that, for each $l \geqslant l_k$, we have

$$|\varphi(p_l) - \varphi(r_{j_0kl})| < \tfrac{1}{12}.$$

If the inequality

$$|\varphi(q_{j_0}) - \varphi(r_{j_0kl})| < \tfrac{1}{12}$$

held for $l \geqslant l_k$, we should get

$$|\varphi(p_l) - \varphi(q_{j_0})| < \tfrac{1}{6}$$

for $l \geqslant l_k$, and consequently

$$|\varphi(p_l) - \varphi(p_{l+1})| < \tfrac{1}{3}$$

which contradicts (2). Thus there exists $m_k \geqslant l_k$ such that

$$|\varphi(q_{j_0}) - \varphi(r_{j_0km_k})| > \tfrac{11}{12},$$

according to the condition

$$\varrho(q_{j_0}, r_{j_0km_k}) \leqslant \varrho(p_{m_k}, q_{j_0}) < \eta$$

which follows from (3). We define

$$y_0 = q_{j_0}, \quad y_k = r_{j_0 k m_k}, \quad Y_k = C_{m_k}$$

for $k = 1, 2, \ldots$ Since $m_k \geqslant l_k \geqslant j_0$, we have $y_0 \in Y_k$ for $k = 1, 2, \ldots$ Moreover $y_k \in Y_k$ and (3) yields

$$\varrho(y_0, y_k) = \varrho(p_{m_k}, q_{j_0})/k$$

for $k = 1, 2, \ldots$ Hence the points $y_1, y_2, \ldots$ converge to $y_0$, the numbers $\varrho(p_{m_k}, q_{j_0})$ being less than or equal to the diameter of $Y$. If we take $\varphi_k = \varphi|C_{m_k}$ for $k = 1, 2, \ldots$, we obtain

$$|\varphi_k(y_0) - \varphi_k(y_k)| > \tfrac{11}{12}.$$

Case 2. For each $j$, there exists $k_j$ such that, for each $l \geqslant j$, there exists $n_{jl} \geqslant l$ such that

$$|\varphi(p_{n_{jl}}) - \varphi(r_{jk_j n_{jl}})| > \tfrac{11}{12}.$$

The space $Y$ is a compactum, and so we can assume the sequence of points $r_{jk_j n_{jl}}$ $(l = j, j+1, \ldots)$ converges in $Y$. We define

$$y_0 = a, \quad y_j = \lim_{l\to\infty} r_{jk_j n_{jl}}, \quad Y_j = \operatorname*{Ls}_{l\to\infty} C_{jk_j n_{jl}}$$

for $j = 1, 2, \ldots$ Since $n_{jl} \geqslant l$ for $l = j, j+1, \ldots$, we have

$$\lim_{l\to\infty} p_{n_{jl}} = a,$$

whence $y_0 \in Y_j$ for $j = 1, 2, \ldots$ Moreover $y_j \in Y_j$ and (3) yields

$$\begin{aligned}
\varrho(y_0, y_j) &= \lim_{l\to\infty} \varrho(a, r_{jk_j n_{jl}}) \\
&\leqslant \varrho(a, q_j) + \lim_{l\to\infty} \varrho(q_j, r_{jk_j n_{jl}}) \\
&= \varrho(a, q_j) + \lim_{l\to\infty} \varrho(p_{n_{jl}}, q_j)/k_j \\
&= \varrho(a, q_j) + \varrho(a, q_j)/k_j \leqslant 2\varrho(a, q_j)
\end{aligned}$$

for $j = 1, 2, \ldots$ Therefore the points $y_1, y_2, \ldots$ converge to $y_0$. But we also have

$$\begin{aligned}
Y_j &\subset \operatorname*{Ls}_{l\to\infty} [C_{n_{jl}} \setminus Q(q_j, \varrho(p_{n_{jl}}, q_j)/k_j)] \\
&\subset Y \setminus Q(q_j, \varrho(a, q_j)/k_j),
\end{aligned}$$

which gives $q_j \notin Y_j$ for $j = 1, 2, \ldots$ Consequently $Y_1, Y_2, \ldots$ are proper subcontinua of $Y$, whence $f|Y_j \sim 1$ for $j = 1, 2, \ldots$ It follows that there exist open subsets $G_j$ of $Y$ such that $Y_j \subset G_j$ and $f|G_j \sim 1$ (see [4], p. 311).

Let $\varphi_j$ be a function attached to $f|G_j$. For almost all indices $l = j, j+1, \ldots$ the continuum $C_{jk_jn_{jl}}$ lies in $G_j$ and contains the points $p_{n_{jl}}$ and $r_{jk_jn_{jl}}$. Since these points converge to $y_0$ and $y_j$, respectively, the inequality

$$|\varphi_j(p_{n_{jl}}) - \varphi_j(r_{jk_jn_{jl}})| > \tfrac{11}{12}$$

implies, for $j = 1, 2, \ldots$, the required inequality

$$|\varphi_j(y_0) - \varphi_j(y_j)| \geqslant \tfrac{11}{12}.$$

THEOREM. *If $g\colon X \to Y$ is a confluent mapping of a countably compact space $X$ onto a metrizable space $Y$ and $f\colon Y \to S$ is a mapping of $Y$ into the circle $S$, then $fg \sim 1$ implies $f \sim 1$.*

Proof. Let us suppose the contrary, i. e. that $fg \sim 1$ but $f$ *non*$\sim 1$, and let $\varphi\colon X \to R$ be a function attached to $fg$. Since $Y$ is countably compact like $X$, i. e. $Y$ is a compactum, we can find a continuum $Y' \subset Y$ such that

$$f|Y' \ \mathit{irr}\ \mathit{non}\!\sim\!1$$

(see [4], p. 325), and then we can apply Lemma 2 to $Y'$ and $f|Y'$. Take proper subcontinua $Y_1, Y_2, \ldots$ of $Y'$ and points $y_0, y_1, y_2, \ldots$ according to Lemma 2. Thus $y_0, y_j \in Y_j$, the points $y_j$ converge to $y_0$ and we can assume that

$$\varphi_j(y_0) - \varphi_j(y_j) \geqslant \tfrac{11}{12}, \tag{4}$$

where $j = 1, 2, \ldots$ and $\varphi_j$ is a function attached to $f|Y_j$. In order to get the last inequality it suffices first to choose an infinite subsequence of the sequence $y_1, y_2, \ldots$ such that the difference on the left side of the inequality has a fixed sign. We may agree the subsequence is the all sequence and the sign is "plus", the further proof for "minus" being quite similar. The set

$$F = \varphi g^{-1}(y_0)$$

is compact in $R$. Let $t_0$ be the minimum number which belongs to $F$, and $x_0 \in g^{-1}(y_0)$ a point such that $\varphi(x_0) = t_0$. Setting

$$X_j = g^{-1}(Y_j)$$

we have $x_0 \in X_j$ for $j = 1, 2, \ldots$ Consider the continuous function $\psi$ : $X_j \to R$ defined by the formula

$$\psi_j(x) = \varphi(x) - \varphi_j g(x) + \varphi_j(y_0) - t_0$$

for $x \in X_j$. The functions $\varphi$ and $\varphi_j$ are attached to $fg$ and $f|Y_j$, respectively, and so

$$e^{2\pi i\varphi(x)} = fg(x) = e^{2\pi i\varphi_j g(x)}$$

for $x \in X_j$. But, since $y_0 = g(x_0)$ and $t_0 = \varphi(x_0)$, we get

$$e^{2\pi i \psi_j(x)} = 1$$

for $x \in X_j$, i. e. the function $\psi_j$ is integer-valued. However, the mapping $g$ is confluent, thus the set $X_j$ is connected between $\{x_0\}$ and $g^{-1}(y_j)$.

Consequently, it follows from

$$\psi_j(x_0) = 0$$

that there exists a point $x_j \in g^{-1}(y_j)$ satisfying

$$\psi_j(x_j) = 0,$$

whence, by (4), we obtain

$$t_0 - \varphi(x_j) = \varphi_j(y_0) - \varphi_j g(x_j) \geqslant \tfrac{11}{12} \tag{5}$$

for $j = 1, 2, \ldots$ Since the space $X$ is countably compact, there exists a cluster point $\bar{x}$ of the sequence $x_1, x_2, \ldots$ The points $g(x_j) = y_j$ converging to $y_0$, it must be $g(\bar{x}) = y_0$, and therefore $\varphi(\bar{x}) \in F$. But (5) implies

$$t_0 - \varphi(\bar{x}) \geqslant \tfrac{11}{12},$$

whence $\varphi(\bar{x}) < t_0$. This shows that $t_0$ is, in fact, not the minimum in $F$.

COROLLARY 1. *The contractibility of compacta relative to the circle is invariant under confluent mappings.*

A question of J. J. Charatonik is answered by Corollary 1 (see [1], p. 219).

COROLLARY 2. *Each confluent mapping $f: X \to Y$ of a compactum $X$ onto a compactum $Y$ induces a monomorphism*

$$f^*: H^1(Y) \to H^1(X)$$

*of cohomology groups* (we take here the Čech cohomology based on arbitrary open coverings and with integer coefficients).

To deduce Corollary 2 from our theorem it suffices to apply a classical theorem of N. Bruschlinsky, modified by C. H. Dowker, which establishes a relation between the elements of 1-dimensional Čech cohomology group and the mappings into $S$, for paracompact normal spaces (see [3], p. 226).

Remarks. First observe that, for compacta, all monotone mappings are confluent. This, of course, need not be true for a space which is not a compactum.

(I) Evidently, each mapping $f: X \to Y$ of $X$ onto $Y$ induces a monomorphism

$$f^*: H^0(Y) \to H^0(X).$$

(II) Although there are confluent mappings $f\colon I^2 \to S^2$ of the square $I^2$ onto a 2-sphere $S^2$, for instance a monotone mapping that sends the boundary of $I^2$ into a point of $S^2$, none of them can induce a monomorphism

$$f^*\colon H^2(S^2) \to H^2(I^2).$$

(III) The condition that the space $X$ is countably compact cannot be omitted in our theorem. Really, take $X = R$, $Y = S$ and $f\colon S \to S$ the identity mapping. Define $g\colon R \to S$ setting $g(x) = e^{2\pi i x}$ for $x \in R$; then $fg \sim 1$ but $f$ *non* $\sim 1$.

(IV) The condition that the space $Y$ is metrizable also cannot be omitted in the theorem. One sees it from an example which follows.

Example. Denote by $\Omega$ the minimum uncountable ordinal and consider the set $A$ consisting of all ordinals $a \leqslant \Omega$ with the order topology. Thus $A$ is a compact space. Let $I$ be the unit segment on the real line $R$.

Take the quotient mappings

$$h\colon A \times I \to A \times I / A \times \{1\} = B,$$
$$g'\colon B \to B/\{h(\Omega, 0), h(\Omega, 1)\}$$

and a mapping

$$f'\colon B/\{h(\Omega, 0), h(\Omega, 1)\} \to S$$

defined by the formula

$$f'(y) = e^{2\pi i \chi h^{-1} g'^{-1}(y)}, \tag{6}$$

where $\chi\colon A \times I \to I$ is the projection $\chi(a, t) = t$. In order to see that the function $f'$ is genuinely defined, i. e. that $f'(y)$ is a point for each $y \in g'h(A \times I)$, it suffices to observe that $\chi h^{-1} g'^{-1}(y)$ is a one-point set for $y \neq p$, where

$$p = g'h(\Omega, 0) = g'h(\Omega, 1),$$

and that $\chi h^{-1} g'^{-1}(p) = \{0, 1\}$. The continuity of the function $f'$ now follows, since $h$ and $g'$ are closed mappings. Put

$$A' = \{a\colon a < \Omega\} = A \setminus \{\Omega\}$$

and define spaces $X$, $Y$ and mappings

$$g\colon X \to Y, \quad f\colon Y \to S$$

by means of the formulas

$$X = h(A' \times I), \qquad g = g'|X,$$
$$Y = g'(X), \qquad f = f'|Y.$$

The space $A'$ being countably compact, so are $A'\times I$ and $X$ (see [2], p. 185). Let

$$I_a = gh(\{a\}\times I)$$

for $a < \Omega$. Then we have

$$Y = \bigcup_{a<\Omega} I_a$$

and each $I_a$ is an arc, since $gh$ topologically maps $\{a\}\times I$ onto $I_a$, for $a < \Omega$. The point $p$ is a common end point of all arcs $I_a$ which are mutually disjoint outside $p$. Suppose $C$ is a connected closed subset of $Y$. It readily follows that each set $C \cap I_a$ is connected. If $p \notin C$, the set $C$ is a subset of some $I_a$. If $p \in C$, the set $C$ is the union of connected sets having a point in common. But the mapping $g$ being clearly one-to-one, its inverse $g^{-1}$ is a homeomorphism on each arc $I_a$, for $a < \Omega$. Consequently, in both cases the set $g^{-1}(C)$ is connected, and we have shown that $g$ is a confluent mapping.

The space $X$ is topologically a cone over $A'$. Thus $X$ is contractible relative to itself, and we get $fg \sim 1$. However, we are going to prove that

$$f\, non \sim 1.$$

Suppose, on the contrary, $f \sim 1$ holds. Hence there exists a mapping $\varphi\colon Y \to R$ such that

$$f'(y) = f(y) = e^{2\pi i\varphi(y)} \tag{7}$$

for $y \in Y$. Let $a < \Omega$ be an ordinal. Put $\varphi_a = \varphi|I_a$ and take a mapping $\psi_a\colon I_a \to R$ defined by the formula

$$\psi_a(y) = \chi h^{-1} g'^{-1}(y)$$

for $y \in I_a$. This is a genuine definition because $g'$, like $g$, is one-to-one on $X$, and $\chi$ sends the set $A\times\{1\}$ into 1. Writing

$$q_a = g'h(a, 0),$$

we have $\psi_a(q_a) = 0$ and $\psi_a(p) = 1$. Since the functions $\varphi_a$ and $\psi_a$ are both attached to the function $f'|I_a$, by (6) and (7), and since $I_a$ is a continuum, we obtain

$$\varphi(p) - \varphi(q_a) = \varphi_a(p) - \varphi_a(q_a) = \psi_a(p) - \psi_a(q_a) = 1$$

for $a < \Omega$. Therefore the point $\varphi g'h(\Omega, 0) = \varphi(p)$ does not belong to the closure of the set

$$\varphi g'h(A'\times\{0\}) = \{\varphi(q_a)\colon a < \Omega\}$$

in $R$, which contradicts the fact that the point $\Omega$ does belong to the closure of the set $A'$ in $A$.

The following problem has been suggested by B. A. Pasynkov.

**P 558.** *Let $X$ and $Y$ be compact Hausdorff spaces, and $g\colon X \to Y$, $f\colon Y \to S$, where $g$ is confluent. Does $fg \sim 1$ imply $f \sim 1$?*

Finally, we propose an analogue of confluent mappings in higher dimensions. Denote by $S^n$ the $n$-dimensional sphere ($n = 0, 1, \ldots$). In particular, $S^0$ is a two-point set and $S^1 = S$. Furthermore, in order that a mapping $g\colon X \to Y$ of $X$ onto $Y$ be confluent it is necessary and sufficient that if $C \subset Y$ is a closed subset contractible relative to $S^0$, $y \in C$ is a point, and $h\colon g^{-1}(C) \to S^0$ is a mapping, then $h|g^{-1}(y) \sim 1$ implies $h \sim 1$. For a given integer $n = 1, 2, \ldots$ let us say a mapping $g\colon X \to Y$ of $X$ onto $Y$ to be *$n$-confluent* provided that if $C \subset Y$ is a closed subset contractible relative to $S^m$ (where $0 \leqslant m < n$), $y \in C$ is a point, and $h$: $g^{-1}(C) \to S^m$ (where $0 \leqslant m < n$) is a mapping such that $h|g^{-1}(y)$ is null-homotopic, then $h$ is null-homotopic. Thus "1-confluent" means "confluent".

**P 559.** *Let $X$ and $Y$ be compacta, and $g\colon X \to Y$, $f\colon Y \to S^n$, where $g$ is $n$-confluent, for a given integer $n = 2, 3, \ldots$ Is it true that if $fg$ is null-homotopic, then $f$ is null-homotopic?*

For $n = 1$ an answer to this question follows from our theorem. Note, however, that the analogue of Corollary 2 for $n = 2$ would not be true, as is indicated by the Hopf fibering $f\colon S^3 \to S^2$.

Added in proof. A negative solution of P 559 for $n = 2$ has been found by J. W. Jaworowski. Namely, take the projective plane $P^2$ and the mapping $g\colon S^2 \to P^2$ which identifies the antipodal points in $S^2$. Then $g$ is 2-confluent, and there exists a mapping $f\colon P^2 \to S^2$ which is not null-homotopic. Since

$$g^*\colon H^2(P^2) \to H^2(S^2)$$

is zero, so is $(fg)^* = g^*f^*$, and thus $fg$ is null-homotopic.

## REFERENCES

[1] J. J. Charatonik, *Confluent mappings and unicoherence of continua*, Fundamenta Mathematicae 56 (1964), p. 213-220.

[2] E. Čech, *Topological spaces*, Prague 1959 (in Czech).

[3] C. H. Dowker, *Mapping theorems for non-compact spaces*, American Journal of Mathematics 69 (1947), p. 200-242.

[4] C. Kuratowski, *Topologie II*, Warszawa 1961.

[5] S. Mardešić, *Equivalence of singular and Čech homology for ANR-s. Application to unicoherence*, Fundamenta Mathematicae 46 (1958), p. 29-45.

[6] A. D. Wallace, *Quasi-monotone transformations*, Duke Mathematical Journal 7 (1940), p. 136-145.

[7] G. T. Whyburn, *Analytic topology*, New York 1942.

*Reçu par la Rédaction le 29. 1. 1965*