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A DUAL SPACE CHARACTERIZATION OF P_1- AND P_2-LATTICES OF ORDER ω^+

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It is known that prime ideals of a Post algebra (more generally, of a P_0-lattice [6]) lie in disjoint finite maximal chains. This result was applied by Epstein and Horn [2] to characterize certain classes of chain based lattices. An analogous characterization has been given before (see [1]) in the case of generalized Post algebras with infinite chain base (but a finite monotonic representation of each element). However, in generalized Post algebras of order ω^+ as defined in [4], with infinite representations admissible, the poset of prime ideals is a disjoint union of posets, each of the form $C+D$, where C is an ω^+-chain and D is some poset (possibly empty). Very little is known about the structure of the posets D. However, it turns out that the just-stated weak condition imposed on the order structure of a dual space of a Heyting algebra, together with certain natural conditions concerning a topology of this space, is sufficient for a characterization of P_1- and P_2-lattices of order ω^+. Such a characterization is the aim of the paper.

1. Preliminaries. A poset $\langle P, \leqslant\rangle$ is said to be a *disjoint union* of its subposets P_t $(t \in T)$ if

$$P = \bigcup_{t\in T} P_t \quad \text{and} \quad x \not\leqslant y \text{ for any } x \in P_t,\ y \in P_{t'} \text{ with } t \neq t'.$$

Let $\langle A, \cup, \cap, 0, 1\rangle$ be a bounded distributive lattice (with notation xy for $x \cap y$). $B(A)$ denotes the set of all complemented elements of A (with complement of b denoted by $\bar{b}$). A dual space of A (see [3]) is the set $\varrho(A)$ of all prime ideals of A, partially ordered by inclusion and equipped with a topology given by a subbase

$$\{h(a): a \in A\} \cup \{\varrho(A) - h(a): a \in A\},$$

where $h(a) = \{J \in \varrho(A): a \notin J\}$. The mapping h is an isomorphism of A onto the lattice of all clopen decreasing subsets of $\varrho(A)$ (a set $X \subseteq \varrho(A)$ is *decreasing* if $J \in \varrho(A)$, $J \subseteq K \in X$ imply $J \in X$; an *increasing* set is defined dually). It is easy to see that if $a \in A$ and $h(a)$ is increasing, then $a \in B(A)$.

Now, let A be a *Heyting algebra*, i.e., a bounded distributive lattice with operation $\rightarrow$ satisfying the following condition: $z \leqslant x \rightarrow y$ iff $xz \leqslant y$. A chain $0 = e_0 \leqslant e_1 \leqslant \ldots \leqslant e_\omega = 1$ in A is called a *chain base* provided every element $x \in A$ has a monotonic representation, i.e.,

$$x = \bigcup_{i=1}^{\infty} x_i e_i \text{ (infinite lattice join),}$$

where $x_i \in B(A)$ and $x_1 \geqslant x_2 \geqslant \ldots$ The lattice A is called a *P_1-lattice of order ω^+* (shortly, *P_1-lattice*) if it has a chain base $(e_i)_{0 \leqslant i \leqslant \omega}$ satisfying $e_{i+1} \rightarrow e_i = e_i$ $(i = 0, 1, \ldots)$; such a chain base is, in fact, unique. If, moreover, for any $x \in A$ and $i = 1, 2, \ldots$ there exists $D_i(x) \in B(A)$ such that $D_i(x)$ is the greatest element x_i satisfying $x_i e_i \leqslant x$, then A is called a *P_2-lattice of order ω^+*. These definitions can be found in [8].

2. Prime ideals in P_1-lattices. Let A be a P_1-lattice with a chain base $(e_i)_{0 \leqslant i \leqslant \omega}$ and let $B = B(A)$. Obviously, for any $J \in \varrho(A)$ we have $J \cap B \in \varrho(B)$. Given $I \in \varrho(B)$, denote by $J_k(I)$ the ideal in A generated by $I \cup \{e_{k-1}\}$ $(k = 1, 2, \ldots)$, and by X_I the set of all prime ideals $J \in \varrho(A)$ such that $J \cap B = I$. Any ideal $I \in \varrho(B)$ being maximal, it is easily seen that the poset $\varrho(A)$ is a disjoint union of X_I $(I \in \varrho(B))$. The structure of X_I is partially described by the following

LEMMA 2.1. *If $J_k(I)$ is proper, then it is a unique member of X_I containing e_{k-1} but not e_k. It follows that X_I is either a finite chain*

$$J_1(I) \subset J_2(I) \subset \ldots \subset J_l(I),$$

where $l = \max\{k \colon J_k(I) \text{ is proper}\} \geqslant 1$, or an order sum $C_I + (X_I - C_I)$, where C_I is an ω^+-type chain, viz.

$$J_1(I) \subset J_2(I) \subset \ldots \subset J_\omega(I) = \bigcup_{k=1}^{\infty} J_k(I)$$

(i.e., in this case, any member of $X_I - C_I$ contains $J_\omega(I)$).

Proof. Let $J_k(I)$ be proper. Since $e_k \rightarrow e_{k-1} = e_{k-1}$, this condition is equivalent to $e_k \notin J_k(I)$. Given $x, y \in A$ with monotonic representations $\bigcup_{i=1}^{\infty} x_i e_i$ and $\bigcup_{i=1}^{\infty} y_i e_i$, respectively (in particular, $x \leqslant e_{k-1} \cup x_k$, $y \leqslant e_{k-1} \cup y_k$, and $x_k y_k e_k \leqslant xy$), the conditions $x, y \notin J_k(I)$ imply $x_k, y_k \notin I$, and hence $z_k = x_k y_k \notin I$, i.e., $\bar{z}_k \in I$, I being prime. Thus, the condition $xy \in J_k(I)$ would imply

$$e_k = z_k e_k \cup \bar{z}_k e_k \leqslant xy \cup \bar{z}_k \in J_k(I),$$

a contradiction, since $e_k \notin J_k(I)$. Therefore, $J_k(I)$ is prime, and it is easy to see that $J_k(I) \cap B = I$. On the other hand, if $J \in X_I$, $e_{k-1} \in J$, and $e_k \notin J$, then obviously $J_k(I) \subseteq J$. For the converse inclusion, let $x \in J$ with the monotonic representation as above. Since $e_k \notin J$ and J is prime, we have $x_k \in J$ $(x_k e_k \leqslant x \in J)$, i.e., $x_k \in J \cap B = I$ and $x \in J_k(I)$ $(x \leqslant x_k \cup e_{k-1})$. The remaining part of the proof is immediate.

Let us put

$$T_k = \{I \in \varrho(B)\colon X_I \text{ is a } k\text{-element chain}\}, \qquad Q_k(A) = \bigcup_{I \in T_k} X_I.$$

If A is a P_2-lattice, then it is easy to see that the set $Q_1(A) \cup \ldots \cup Q_k(A)$ is the value of the isomorphism h assumed on $D_{k+1}(e_k)$, and hence is clopen $(k = 1, 2, \ldots)$. (Indeed, $I \in Q_1(A) \cup \ldots \cup Q_k(A)$ iff $J_{k+1}(I)$ is not proper, i.e., $e_{k+1} \in (I \cup \{e_k\}]$, which, in turn, means that there exists $b \in I$ such that $e_{k+1} \leqslant b \cup e_k$. The last condition is equivalent to $\bar{b} \leqslant D_{k+1}(e_k)$, i.e., $D_{k+1}(e_k) \notin I$ and, finally, $I \in h(D_{k+1}(e_k))$.) More generally, we have the following

LEMMA 2.2. *Let A be a P_1-lattice. Then*

(i) *A is a P_2-lattice iff the sets $Q_k(A)$ are clopen* $(k = 1, 2, \ldots)$;

(ii) *A is a generalized Post algebra of order ω^+* (see [4]) *iff $Q_k(A) = \emptyset$ for $k = 1, 2, \ldots$, i.e., $\varrho(A)$ does not contain finite maximal chains.*

Proof. Let $Q_k(A)$ be clopen $(k = 1, 2, \ldots)$ and let $j > i$. It is easily seen that the clopen set $Q_1(A) \cup \ldots \cup Q_i(A)$ is the greatest simultaneously decreasing and increasing set Y satisfying $Y \cap h(e_j) \subseteq h(e_i)$. Thus there exists $D_j(e_i)$ (which is equal to $e_j \Rightarrow e_i$ in the notation of [7]). Using Lemma 3.2 (ii) of [7] we infer that $D_j(y) = (e_j \Rightarrow y)$ exists for any $y \in A$. Thus A is a P_2-lattice. (ii) follows immediately from Theorem 5.1 in [8].

The condition of Lemma 2.1, though containing no information concerning the structure of $X_I - C_I$, allows us to give a certain characterization of P_1-lattices. We need two lemmas which — because of their technical character — are given in the next section.

3. Auxiliary lemmas. The following lemma is a generalization of Lemma 7.10 from [2], where $\varrho(A)$ was assumed to be a disjoint union of chains.

LEMMA 3.1. *Let A be a bounded distributive lattice. Suppose the poset $\varrho(A)$ is a disjoint union of a family $(X_t)_{t \in T}$ such that, for each $t \in T$, X_t has the least element — say P_t. Assume further that there exists an element $e \in A$ such that $h(e) = \{P_t\colon t \in T\}$ and let us denote the interval $[e, 1]$ by A_1. Then:*

(i) *For every $x \in A$ there is an element $b \in B(A)$ such that $x = b(e \cup x)$.*

(ii) *If, moreover, for any $t \in T$ the set $X_t - \{P_t\}$, if not empty, has also the least element (denoted by R_t), then*

$$B(A_1) = \{b \cup e\colon b \in B(A)\}.$$

Proof. We proceed the proof in several steps.

(A) *If $t \in T$, $x \notin P_t$, and $J \in X_t$, then there exists $y \notin J$ such that $ye \leqslant x$.*

Indeed, let $V = \{v \in T\colon x \in P_v\}$. Since the ideals $P_v \in X_v$ $(v \in V)$ and $J \in X_t$ are in comparable by assumption, we can choose an element $y_v \in P_v - J$ for any $v \in V$. Suppose that x does not belong to the filter F generated by the set $\{e\} \cup \{y_v\colon v \in V\}$. By the prime ideal theorem, there exists $K \in \varrho(A)$ such that $x \in K$ and $K \cap F = \emptyset$. In particular, $e \notin K$, i.e., $K \in h(e) = \{P_v\colon v \in T\}$. Thus $K = P_v$ for a certain $v \in T$, and $v \in V$ since $x \in K = P_v$. Now $P_v \cap F = \emptyset$ implies $y_v \notin P_v$, which, however, contradicts the choice of y_v. Thus $x \in F$, i.e., $x \geqslant ey$, where y is a finite meet of the y_v. Also $y \notin J$, since $y_v \notin J$ and J is prime.

(B) *If* $t \in T$, $x \in P_t$, *and* $K \in X_t$, *then there exists* $z \notin K$ *such that* $xz = 0$.

Let $U = \{u \in T\colon x \notin P_u\}$. For each $u \in U$ we can choose an element $z_u \in P_u - K$. Suppose the filter F generated by $\{x\} \cup \{z_u\colon u \in U\}$ is proper. Then there exists $J \in \varrho(A)$ such that $J \cap F = \emptyset$. Since P_u is the least element of X_u, we can assume that $J = P_u$ for a certain $u \in T$; here $u \in U$ since $x \in F$. Now $P_u \cap F = \emptyset$ implies $z_u \notin P_u$, a contradiction with the choice of z_u. Thus $0 \in F$ and the desired z is a finite meet of the z_u.

(i) follows from (A) and (B) in the following way. Set

$$U = \{t \in T\colon x \notin P_t\}, \qquad Y = \bigcup_{u \in U} X_u,$$
$$V = T - U, \qquad Z = \varrho(A) - Y.$$

For any $J \in Y$ there exists, in virtue of (A), an element $y_J \notin J$ such that $y_J e \leqslant x$. Similarly, by (B), for any $K \in Z$ there exists $z_K \notin K$ such that $xz_K = 0$. Obviously, the ideal generated by $\{y_J\colon J \in Y\} \cup \{z_K\colon K \in Z\}$ is not contained in any prime ideal, i.e., it is not proper. Thus $1 = y \cup z$, where $ye \leqslant x$ and $xz = 0$. We have $yze \leqslant x0 = 0$, i.e., $h(yz) \cap h(e) = \emptyset$, and since $h(yz)$ is a decreasing set, we obtain $yz = 0$. Thus

$$y \in B(A) \quad \text{and} \quad x = x \cup ye = x(y \cup z) \cup ye = xy \cup ye = y(x \cup e),$$

which completes the proof of (i).

(ii) Let $x \in B(A_1)$, i.e., there is $\tilde{x} \in A_1$ such that $x\tilde{x} = e$, $x \cup \tilde{x} = 1$. We need to show that there is an element $y \in B(A)$ such that $x = y \cup e$. Set

$$T_1 = \big\{t \in T\colon X_t = \{P_t\}\big\}, \qquad U' = \{t \in T - T_1\colon x \notin R_t\},$$
$$Y' = \bigcup_{u \in U'} X_u, \qquad V' = \{t \in T - T_1\colon x \in R_t\}.$$

(C) *For any* $u \in U'$ *we have* $X_u \subseteq h(x)$.

For otherwise there is $K \in X_u$ such that $x \in K$. But $x \notin P_u$ and $x \notin R_u$, so $R_u \subset K$. We have $\tilde{x} \notin K$ $(x \cup \tilde{x} = 1 \notin K)$, whence $\tilde{x} \notin R_u$. But $x\tilde{x} = e \in R_u$, and since R_u is prime, we obtain $x \in R_u$ — a contradiction.

(D) *If $u \in U'$, then there exists $z \in P_u$ such that $x \cup z = 1$.*

Indeed, for any $K \in \varrho(A)$ such that $x \in K$ we have $K \notin X_u$ by virtue of (C). Thus we can choose $z_K \in P_u - K$. A standard argument shows that the ideal generated by $\{x\} \cup \{z_K\colon x \in K\}$ is not proper, and so (D) is proved.

(E) *If $u \in U'$ and $J \in X_u$, then there exists $y \notin J$ such that $y \leqslant x$ and $h(y) \cap X_v = \emptyset$ for any $v \in V'$.*

By (D), there is $z \in P_u$ such that $x \cup z = 1$. We have $u \in T$, $z \in P_u$, $J \in X_u$, and applying (B) we get an element $y \notin J$ such that $zy = 0$. Now $y = y(x \cup z) = xy \leqslant x$. We have $h(y) \cap X_v = \emptyset$ $(v \in V')$, for otherwise $P_v \in h(y)$ ($h(y)$ being decreasing), i.e., $y \notin P_v$, whence $z \in P_v \subset R_v$ ($yz = 0 \in P_v$, P_v prime). Since $x \in R_v$, we would have $1 = x \cup z \in R_v$, which is impossible.

Now we complete the proof of (ii). For each $J \in Y'$ we can choose (by (E)) an element $y_J \notin J$ such that $y_J \leqslant x$ and $h(y_J) \cap X_v = \emptyset$ for $v \in V'$. Suppose $x \notin (\{e\} \cup \{y_J\colon J \in Y'\}]$. Then there is an ideal $K \in \varrho(A)$ such that $e \in K$, $y_J \in K$, and $x \notin K$. We have $K \in X_u$ for a certain $u \in T$. Thus $e \in K$ implies $R_u \subseteq K$ and we infer that $x \notin R_u$, i.e., $u \in U'$. This, however, yields $K \in Y'$, i.e., $y_K \notin K$ has been chosen, a contradiction. Consequently, $x \leqslant e \cup y$, where y is a finite join of the y_J. Also $e \leqslant x$ $(x \in A_1)$ and $y \leqslant x$ $(y_J \leqslant x)$; hence $x = e \cup y$.

It remains to show that $y \in B(A)$ or, equivalently, that $h(y)$ is increasing. This, however, is implied by the definition of the disjoint union of posets and by the following two facts:

$$h(y) \cap X_v = \emptyset \quad \text{for } v \in V'$$

(see the analogous condition for the elements y_J) and

$$X_u \subseteq h(y) \quad \text{for } u \in U'$$

(for otherwise there would exist $J \in X_u$ such that $y \in J$; but then $J \in Y'$ and $y \geqslant y_J \notin J$, a contradiction).

LEMMA 3.2. *Let A be a bounded distributive lattice and let $A_1 = [e, 1]$, where $e \in A$. Then $\varrho(A_1)$ is order isomorphic to $\varrho(A) - h(e)$.*

Proof (cf. [2], Lemma 7.9). Let us define the mappings

$$\varphi\colon \varrho(A) - h(e) \to \varrho(A_1) \quad \text{and} \quad \psi\colon \varrho(A_1) \to \varrho(A) - h(e)$$

by

$$\varphi(J) = J \cap A_1 \quad \text{and} \quad \psi(K) = (K]_A.$$

An easy computation shows that φ and ψ are well defined, order-preserving, and mutually inverse. Thus ψ is the desired isomorphism.

4. Characterization of P_1- and P_2-lattices in the class of Heyting algebras. Let A be a bounded distributive lattice. Let $\varrho_1(A)$ be the set of all minimal elements in $\varrho(A)$ and, inductively, let $\varrho_i(A)$ be the set of all minimal elements in $\varrho(A) - \bigcup_{j=1}^{i-1} \varrho_j(A)$.

THEOREM 4.1. *A Heyting algebra A is a P_1-lattice if and only if the following conditions are satisfied:*

(i) *The dual space of A can be represented as a disjoint union of a family $(X_t)_{t\in T}$, where each X_t is either a finite chain or an order sum $C_t + D_t$, where C_t is an ω^+-type chain.*

(ii) *The sets $\varrho_i(A)$ are clopen in $\varrho(A)$ $(i = 1, 2, \ldots)$ and $\varrho(A)$ is the only clopen decreasing set containing their union.*

Proof. Note first that (ii) means simply that there exist elements $e_i \in A$ such that

$$h(e_i) = \varrho_1(A) \cup \ldots \cup \varrho_i(A) \quad \text{and} \quad \bigcup_{i=1}^{\infty} e_i = 1.$$

Thus the necessity of conditions (i) and (ii) follows from Lemma 2.1. We prove that these conditions are sufficient. Set $B = B(A)$, $A_i = [e_i, 1]$, and $B_i = B(A_i)$. By Lemma 3.2, each of the intervals A_i satisfies the condition analogous to (i). Applying Lemma 3.1 we obtain inductively

$$B_i = \{b \cup e_i\colon\ b \in B\}. \tag{4.1}$$

Let $x \in A$. By Lemma 3.1 (i), $x = b_1(e_1 \cup x)$, where $b_1 \in B$. Applying this lemma again to the lattice A_1 and the element $e_1 \cup x$, we obtain

$$e_1 \cup x = c_2\big(e_2 \cup (e_1 \cup x)\big) = c_2(e_2 \cup x), \quad \text{where } c_2 \in B_1,$$

so there exists $b_2 \in B$ such that $c_2 = e_1 \cup b_2$ (see (4.1)). Hence

$$e_1 \cup x = (e_1 \cup b_2)(e_2 \cup x), \quad \text{where } b_2 \in B.$$

In general, by induction we get

$$e_i \cup x = (e_i \cup b_{i+1})(e_{i+1} \cup x), \quad \text{where } b_{i+1} \in B \ (i = 0, 1, \ldots).$$

Thus, for any k,

$$\begin{aligned} x &= b_1(e_1 \cup x) = b_1(e_1 \cup b_2)(e_2 \cup x) = \ldots \\ &= b_1(e_1 \cup b_2)(e_2 \cup b_3) \ldots (e_{k-1} \cup b_k)(e_k \cup x) \end{aligned}$$

and, in consequence,

$$\begin{aligned} xe_k &= b_1(e_1 \cup b_2)(e_2 \cup b_3) \ldots (e_{k-1} \cup b_k)e_k \\ &= b_1(e_1 \cup b_2)(e_2 \cup b_3) \ldots (e_{k-1} \cup b_k e_k) \\ &= b_1(e_1 \cup b_2)(e_2 \cup b_3) \ldots (e_{k-2} \cup b_{k-1}e_{k-1} \cup b_{k-1}b_k e_k) = \ldots \\ &= b_1 e_1 \cup b_1 b_2 e_2 \cup \ldots \cup b_1 b_2 \ldots b_k e_k. \end{aligned}$$

Applying the infinite $\cup$-distributivity of Heyting algebras (see, e.g., 11.2 of [5]) we obtain

$$x = x\cap 1 = x\cap \bigcup_{k=1}^{\infty} e_k = \bigcup_{k=1}^{\infty} xe_k = \bigcup_{k=1}^{\infty} b_1 \ldots b_k e_k,$$

i.e., x has a monotonic representation. Finally, $e_{i+1} \to e_i = e_i$ since the set

$$h(e_i) = \bigcup_{j=1}^{i} \varrho_j(A)$$

is obviously the greatest decreasing set contained in $\big(\varrho(A)-h(e_{i+1})\big)\cup h(e_i) = \varrho(A)-\varrho_{i+1}(A)$. This completes the proof.

Following the notation of Theorem 4.1 we put

$$T_k = \{t\in T\colon\ X_t \text{ is a } k\text{-element chain}\}, \qquad Q_k(A) = \bigcup_{t\in T_k} X_t.$$

Applying Lemma 2.2 we obtain immediately

THEOREM 4.2. *A Heyting algebra A is a P_2-lattice of order ω^+ if and only if conditions* (i) *and* (ii) *of Theorem* 4.1 *are satisfied and*

(iii) *$Q_k(A)$ are clopen in $\varrho(A)$ for $k = 1, 2, \ldots$*

THEOREM 4.3. *A Heyting algebra A is a generalized Post algebra of order ω^+* (see [4]) *if and only if* (i) *and* (ii) *are satisfied and*

(iii′) *$Q_k(A) = \emptyset$ for $k = 1, 2, \ldots$, i.e., $\varrho(A)$ does not contain finite maximal chains.*

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