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PRIME NUMBERS SUCH THAT THE SUMS OF THE DIVISORS OF THEIR POWERS ARE PERFECT POWER NUMBERS

BY

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Introduction. Let n be a positive integer and denote by $\sigma(n)$ the sum of the divisors of n. Let a and b be integers and $a \geqslant 1$, $b \geqslant 3$. We consider the primes p such that

$$\sigma(p^a) = \text{perfect } b\text{-th power number.} \tag{1}$$

Gerono (see [2]), Moreau (see also [2]), Schinzel [3], Thébault [6], and Takaku [5] treat equation (1) for $b = 2$. By the celebrated general theorems of Siegel [4] and Baker [1] it is known that only a finite number of primes p satisfy (1) and

$$p < \exp\exp\{(5b)^{10} a^{10a^3}\}.$$

We prove the following

THEOREM. *Suppose that a is a positive integer, b is an odd prime, $b \nmid l$, $l \geqslant 1$, $b^l \| a+1$, and p is a prime. If equation* (1) *holds, then*

$$p < ab^2(2b)^{(a-1)b^a}.$$

1. Let a and b be integers and $a \geqslant 1$, $b \geqslant 3$, let X be a positive integer, and p be a prime. Suppose that

$$\sigma(p^a) = 1+p+p^2+ \ldots +p^a = X^b.$$

We have $p | X-1$ or $p | X^{b-1}+X^{b-2}+ \ldots +X+1$. If $p | X-1$, then the following lemmas hold.

LEMMA 1. *If $X \equiv 1 \pmod p$, $a \geqslant 2$ and $2 \leqslant k \leqslant a$, then*

$$b^{\beta(k)}(1+p+ \ldots +p^{a-k}) = p^{(b-1)k} x_k^b + \sum_{i=0}^{(b-1)k-b} p^i \sum_{j=0}^{b-1} \gamma_{i,j}^{(k)} (px_k)^j + b^{\mu(k)} x_k,$$

where $\beta(k)$, $\mu(k)$, and $\gamma_{i,j}^{(k)}$ are the integers that depend on b, k and are *independent of a, p, X and*

$$\mu(k) = b^{k-1}, \quad \beta(k) = b(1+b+\dots+b^{k-2}) < b^k,$$
$$|\gamma_{i,j}^{(k)}| < (2b)^{(k-1)b^k} \quad \textit{for any } i, j,$$

and x_k is an integer.

Proof. If we write $X-1 = px_1$, then $x_1 \geqslant 1$ and

$$X^{b-1} + X^{b-2} + \dots + X + 1 = \sum_{i=0}^{b-1} \binom{b}{i} (px_1)^{b-i-1}.$$

Since

$$1 + p + \dots + p^{a-1} = x_1 \sum_{i=0}^{b-1} \binom{b}{i} (px_1)^{b-1-i},$$

we have $p|bx_1 - 1$. If we write $bx_1 - 1 = px_2$, then $x_2 \geqslant 1$ and

$$b^b(1+p+\dots+p^{a-2}) = \sum_{i=0}^{b-2} \binom{b}{i+2} b^{b-2-i} p^i (px_2+1)^{i+2} + b^b x_2.$$

Hence $\beta(2) = \mu(2) = b$ and

$$\gamma_{i,j}^{(2)} = \varepsilon_{i,j} \binom{b}{i+2} b^{b-2-i} \binom{i+2}{j}, \quad \varepsilon_{i,j} = 0 \text{ or } 1.$$

Since

$$\binom{l}{k} \leqslant \binom{l+1}{k+1} \quad \text{for } k \leqslant l,$$

we obtain

$$0 < \binom{b}{j+1} = \binom{b-1}{j} + \binom{b-2}{j} + \dots + \binom{j}{j} \leqslant 2^{b-1}$$

and

$$0 \leqslant \gamma_{i,j}^{(2)} < 2^{b-1} b^{b-2-i} \cdot 2^{i+2} < (2b)^b.$$

Let $2 \leqslant k < a$ and we assume that Lemma 1 holds for k. Since

$$b^{\beta(k)} p(1+p+\dots+p^{a-k-1}) = p^{(b-1)k} x_k^b + \sum_{i=1}^{(b-1)k-b} p^i \sum_{j=0}^{b-1} \gamma_{i,j}^{(k)} (px_k)^j +$$
$$+ \gamma_{0,b-1}^{(k)} (px_k)^{b-1} + \dots + \gamma_{0,1}^{(k)} (px_k) + \gamma_{0,0}^{(k)} + b^{\mu(k)} x_k - b^{\beta(k)},$$

we have $p|b^{\mu(k)} x_k - b^{\beta(k)} + \gamma_{0,0}^{(k)}$. If we write

$$b^{\mu(k)} x_k - b^{\beta(k)} + \gamma_{0,0}^{(k)} = px_{k+1},$$

then

$$b^{\beta(k)+b\mu(k)}(1+p+\ldots+p^{a-(k+1)})$$

$$= p^{(b-1)k-1}\left\{(px_{k+1})^b+\sum_{j=0}^{b-1}\binom{b}{b-j}(b^{\beta(k)}-\gamma_{0,0}^{(k)})^{b-j}(px_{k+1})^j\right\}+$$

$$+\sum_{l=0}^{(b-1)k-2} p^l \sum_{j=0}^{b-1}\gamma_{l-j+1,j}^{(k)} b^{(b-j)\mu(k)}\sum_{m=0}^{j}\binom{j}{m}(b^{\beta(k)}-\gamma_{0,0}^{(k)})^{j-m}(px_{k+1})^m+$$

$$+p^{b-2}\gamma_{0,b-1}^{(k)} b^{\mu(k)}\sum_{j=0}^{b-1}\binom{b-1}{j}(b^{\beta(k)}-\gamma_{0,0}^{(k)})^{b-1-j}(px_{k+1})^j+\ldots+$$

$$+\gamma_{0,1}^{(k)} b^{(b-1)\mu(k)} px_{k+1}+\gamma_{0,1}^{(k)} b^{(b-1)\mu(k)}(b^{\beta(k)}-\gamma_{0,0}^{(k)})+b^{b\mu(k)}x_{k+1}.$$

Hence, for $m = 0, 1, 2, \ldots, b-1$ we have

$$\gamma_{(b-1)k-1,m}^{(k+1)} = \binom{b}{b-m}(b^{\beta(k)}-\gamma_{0,0}^{(k)})^{b-m}, \tag{2}$$

$$\gamma_{l,m}^{(k+1)} = \sum_{j=m}^{b-1}\gamma_{l-j+1,j}^{(k)} b^{(b-j)\mu(k)}\binom{j}{m}(b^{\beta(k)}-\gamma_{0,0}^{(k)})^{j-m}+$$

$$+\varepsilon'_{l,j}\gamma_{0,l+1}^{(k)} b^{(b-1-l)\mu(k)}\binom{b-1}{b-1-m}(b^{\beta(k)}-\gamma_{0,0}^{(k)})^{l-m+1},$$

where $l<(b-1)k-1$, $\varepsilon'_{l,j}=0$ if $l\geqslant b-1$ and $\varepsilon'_{l,j}=1$ if $l\leqslant b-2$; moreover,

$$\mu(k+1) = b\mu(k) = b^k,$$

$$\beta(k+1) = \beta(k)+b\mu(k) = b(1+b+\ldots+b^{k-1}) < b^{k+1}.$$

Hence

$$|\gamma_{(b-1)k-1,m}^{(k+1)}| < 2^b(2(2b)^{(k-1)b^k})^b < (2b)^{kb^{k+1}},$$

$$|\gamma_{l,m}^{(k+1)}| < 2b(2b)^{(k-1)b^k} b^{b\mu(k)}\cdot 2^{b-1}(2(2b)^{(k-1)b^k})^{b-1}$$

$$= (2b)^{kb^{k+1}}(b\cdot 2^{2b-1-b^{k+1}}) < (2b)^{kb^{k+1}} \qquad (l<(b-1)k-1).$$

LEMMA 2. *Let* $2\leqslant a$ *and* $3\leqslant b$. *If* $b\nmid a$ *and*

$$p > 2(2b)^{(a-1)b^a},$$

then $x_k \neq 0$ $(2\leqslant k\leqslant a)$.

Proof. Let k be the smallest r such that $x_r = 0$. By the proof of Lemma 1, we have $k>2$ and

$$b^{\beta(k)}(1+p+\ldots+p^{a-k}) = \sum p^i\gamma_{i,0}^{(k)}. \tag{3}$$

Since $b\nmid a$, we have $a-k\neq(b-1)k-b$.

1. If $a-k>(b-1)k-b$, then, by the p-adic expressions of the both sides

of equation (3), the leading coefficient $b^{\beta(k)}$ of the left-hand side is zero; a contradiction.

2. If $a-k<(b-1)k-b$, then $\gamma^{(k)}_{(b-1)k-b,0}=0$. Hence, by (2), we have $b^{\beta(k-1)}=\gamma^{(k-1)}_{0,0}$. Since

$$(4)\quad b^{\beta(k-1)}(1+p+\ldots+p^{a-(k-1)}) = p^{(b-1)(k-1)}x_{k-1}^{b}+\sum_{\substack{i=0\\ i+j>0}}^{(b-1)(k-1)-b}\ \sum_{\substack{j=0\\ i+j>0}}^{b-1} p^{i}\gamma^{(k-1)}_{i,j}(px_{k-1})^{j}+ +\gamma^{(k-1)}_{0,0}+b^{\mu(k-1)}x_{k-1},$$

we have $p|b$ or $p|x_{k-1}$. Since $p>b$, we obtain $p|x_{k-1}$. If $x_{k-1}\neq 0$, then, by the p-adic expressions $\sum a_i p^i$ ($|a_i|<p/2$) of the both sides of equation (4), the exponent of the maximal power of p in the left-hand side equals $a-(k-1)<(b-1)k-b+1$ and the exponent of the maximal power of p in the right-hand side is greater than $(b-1)(k-1)+b-1=(b-1)k$; a contradiction. Hence $x_{k-1}=0$. This contradicts the choice of k.

LEMMA 3. *Suppose that $a\geqslant 1$, $b\geqslant 3$, $b\nmid a$, p is a prime, and X is a positive integer. If $\sigma(p^a)=X^b$ and $p|X-1$, then*

$$p<ab^2(2b)^{(a-1)b^a}.$$

Proof. We put $\lambda=ab^2(2b)^{(a-1)b^a}$. If $p>\lambda$, then $0<\lambda/p<1$. Define

$$f(y)=p^{(b-1)a}y^b+\sum_{i=0}^{(b-1)a-b} p^i\sum_{j=0}^{b-1}\gamma^{(a)}_{i,j}(py)^j+b^{\mu(a)}y-b^{\beta(a)}.$$

If $\max(1,\lambda/p)<|y|$, then

$$|f(y)|>p^{(b-1)a-1}|y|^{b-1}\Big|p|y|-\sum_{i=0}^{(b-1)a-b}\sum_{j=0}^{b-1}|\gamma^{(a)}_{i,j}|-b^{\mu(a)}-b^{\beta(a)}\Big|$$
$$>p^{(b-1)a-1}|y|^{b-1}\big(p|y|-ab^2(2b)^{(a-1)b^a}\big)>0,$$

and so $f(y)\neq 0$. Hence $|x_a|<1$. By Lemma 1, we have $x_a=0$. This contradicts Lemma 2. Therefore $p<\lambda$.

2. Let b be an odd prime, a be an integer not less than 1, and p be a prime. Suppose that $\sigma(p^a)=X^b$, where X is an integer not less than 1. Then we have the following lemmas.

LEMMA 4. *If $\sigma(p^a)=X^b$ and p divides $X^{b-1}+\ldots+X+1$, then $p\equiv 1 \pmod b$, provided both p and b are primes.*

Proof. Since $X^b\equiv 1 \pmod p$, the order of $X \pmod p$ divides b. If it would equal unity, then $X\equiv 1 \pmod p$, which in view of

$$p|1+X+\ldots+X^{b-1}$$

implies $p^2|X^b-1$; however $X^b-1\equiv p \pmod{p^2}$. Thus this order equals b, and $b|p-1$.

LEMMA 5. *Suppose that b is an odd prime, and a is an integer not less than* 1. *If $b \nmid l$, $l \geqslant 1$, and $b^l \| a+1$, then the primes p such that*

$$\sigma(p^a) = X^b \quad \text{and} \quad p \mid X^{b-1}+X^{b-2}+\ldots+X+1$$

do not exist.

Proof. Assume $\sigma(p^a) = X^b$ and $p \mid X^{b-1}+X^{b-2}+\ldots+X+1$. By Lemma 4, we have $p \equiv 1 \pmod{b}$. Hence

$$0 \equiv a+1 \equiv 1+p+\ldots+p^a = X^b \equiv X \pmod{b}.$$

If we put $p = by+1$, then

$$X^b = 1+p+\ldots+p^a$$
$$= (by)^a + \binom{a+1}{a}(by)^{a-1} + \ldots + \binom{a+1}{2} by + \binom{a+1}{1}.$$

In the right-hand side, the last term $\binom{a+1}{1}$ is divisible by b^l and is not divisible by b^{l+1}. Other terms are divisible by b^{l+1}. On the other hand, the exponent of b in the left-hand side is divisible by b, so b divides $\min(l+1, l) = l$; a contradiction.

Now we see that the Theorem follows from Lemmas 3 and 5.

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