COLLOQUIUM MATHEMATICUM

VOL. XLVIII 1984 FASC. 1

AN INCOMPATIBILITY THEOREM

BY

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1. Introduction. For a non-empty set A of real numbers, let $D(A)$ denote the set of all numbers of the form $|x-y|$, where $x, y \in A$. In 1920, Steinhaus [11] proved that $D(A)$ contains a non-empty open interval if A has positive Lebesgue measure. Various authors ([2], [4], and [5]) have generalized this result by showing that $f(A, B) = \{f(a, b): a \in A, b \in B\}$ must contain a non-empty interval if both A and B have positive Lebesgue measure provided that the function $f: R \times R \to R$ satisfies appropriate conditions. Kuczma [3] and Sander [9] have proved analogues of the above-mentioned results in topological spaces. Miller [6] and Sander [10] obtained similar results in the case where A and B are both Baire sets of the second category.

For $A \subset R$, the condition $m(A) > 0$ (m is the Lebesgue measure) is sufficient for $D(A)$ to contain an interval. This condition is not necessary since $D(C) = [0, 1]$ (see [1]), where C is the Cantor set, although $m(C) = 0$. However, $\mu(C) = 1$, where μ is the measure on R induced by the function

$$F(x) = \begin{cases} 0 & \text{for } x < 0, \\ G(x) & \text{for } 0 \leqslant x \leqslant 1, \\ 1 & \text{for } x > 1, \end{cases}$$

G being the Cantor function [8]. A set $A \subset R$ is called a *universal null set* in case $\alpha(A) = 0$ for every measure α on R induced by a non-decreasing continuous function. By the above, the Cantor set is not a universal null set. In [7], using transfinite induction and assuming the continuum hypothesis, we have constructed a universal null set N such that $D(N) = [0, \infty)$. We have recently generalized this result by showing that if f is any function on $R \times R$ into R, satisfying appropriate conditions, then there exists a pair of universal null sets A and B such that $f(A, B)$ contains an interval.

In this note* we present an incompatibility result. We will show that if f and g are any pair of functions on $R \times R$ into R satisfying certain properties,

* This research was supported by the Republican Council for Scientific Work of Bosna and Hercegovina.

then there exist subsets A and B of the real line such that $f(A, B)$ contains a non-empty open interval and $g(A, B)$ does not contain a non-empty open interval. We use transfinite induction and assume the continuum hypothesis in our proof.

2. Results. We now prove the theorem mentioned in the Introduction.

THEOREM. *Assume that f and g are functions on $R \times R$ into R such that*

(a) *f_x, f_y, g_x, and g_y (partial derivatives) exist and are continuous on an open neighborhood of the origin;*

(b) $f_x(0, 0) \neq 0$, $f_y(0, 0) \neq 0$, $g_x(0, 0) \neq 0$, $g_y(0, 0) \neq 0$;

(c) $f(0, 0) = g(0, 0) = 0$;

(d) *the numbers $f_x(0, 0)/f_y(0, 0)$ and $g_x(0, 0)/g_y(0, 0)$ have opposite signs.*

Then there exist sets A, B such that $A, B \subset R$ and $f(A, B)$ contains an interval, but $g(A, B)$ does not.

Proof. Denote $-g_x(0, 0)/g_y(0, 0)$ by δ, $g_x(0, 0)$ by α, and $g_y(0, 0)$ by β. Let t_1 and t_2 be numbers such that $t_1 > 1 > t_2 > 0$. Then there exists $\varrho > 0$ such that

(i) g_x and g_y exist and are continuous in the set

$$N_\varrho(0) \times N_{6\varrho|\delta|}(0) = N, \quad \text{where } N_r(x) = \{y: |y-x| < r\};$$

(ii) $t_1|\delta| > |g_x(x, y)/g_y(x, y)| > t_2|\delta|$ for every $(x, y) \in N$;

(iii) $|g_x(x, y)| < 2|\alpha|$ and $|g_y(x, y)| > (2/3)|\beta|$ if $(x, y) \in N$.

The set $g(N_{\varrho/2}(0), \{0\})$ is a non-empty open interval; call it I. Let $t \in I$; then there exists $x_t \in N_{\varrho/2}(0)$ such that $g(x_t, 0) = t$. If $t \in I$, then for each $x \in N_\varrho(0)$, there exists a unique $y \in N_{6\varrho|\delta|}(0)$ such that $g(x, y) = t$. We denote this correspondence by the function $y = h_t(x)$. Then $h_t'(x)$ exists and is continuous for every $x \in N_\varrho(0)$ and

$$h_t'(x) = -g_x(x, h_t(x))/g_y(x, h_t(x)).$$

To see this suppose that $\bar{x} \in N_\varrho(0)$. Then from (iii) and the Mean Value Theorem we obtain

$$|g(\bar{x}, 0) - t| < 2|\alpha|(3/2)\varrho = 3\varrho|\alpha|.$$

The continuity of $g_y(x, y)$ and (ii) imply that $g_y(x, y)$ has the same sign throughout N. Hence the function $m(y) = g(\bar{x}, y) - t$ is strictly monotonic throughout $N_{6\varrho|\delta|}(0)$. Therefore, $m(y)$ has at most one zero in $N_{6\varrho|\delta|}(0)$. The inequality $|g_y(x, y)| > 2|\beta|/3$ in N and the Mean Value Theorem imply

$$|[g(\bar{x}, y) - t] - [g(\bar{x}, 0) - t]| > (2/3)|\beta|\,|y - 0| \quad \text{for every } y \in N_{6\varrho|\delta|}(0),\ y \neq 0.$$

Therefore, there exist y_1 and y_2 in $N_{6\varrho|\delta|}(0)$ such that $y_1 < 0 < y_2$ and

$$|[g(\bar{x}, y_1) - t] - [g(\bar{x}, 0) - t]| > 3\varrho|\alpha|, \quad |[g(\bar{x}, y_2) - t] - [g(\bar{x}, 0) - t]| > 3\varrho|\alpha|.$$

This follows from the fact that

$$(2/3)|\beta|\,6\varrho|\delta| = 4\varrho|\beta|\,|\delta| = 4\varrho|\alpha|.$$

Since $|g(\bar{x}, 0)-t| < 3\varrho|\alpha|$ and $m(y)$ is strictly monotonic in $N_{6\varrho|\delta|}(0)$, we infer that $g(\bar{x}, y_1)-t$ and $g(\bar{x}, y_2)-t$ have opposite signs, and therefore there exists y_0 $(y_1 < y_0 < y_2)$ such that $m(y_0) = g(\bar{x}, y_0)-t = 0$. This completes the proof of the fact that for each $x \in N_\varrho(0)$ there exists a unique $y \in N_{6\varrho|\delta|}(0)$ such that $g(x, y) = t$.

Denote $-f_x(0, 0)/f_y(0, 0)$ by d, $f_x(0, 0)$ by a, and $f_y(0, 0)$ by b. Then there exists $r > 0$ such that

(1) f_x and f_y exist and are continuous in the set $N_r(0) \times N_{6r|d|}(0) = M$;

(2) $t_1|d| > |f_x(x, y)/f_y(x, y)| > t_2|d|$ for every $(x, y) \in M$;

(3) $|f_x(x, y)| < 2|a|$ and $f_y(x, y) > (2/3)|b|$ for every $(x, y) \in M$;

(4) $f(x, y)$, $g(x, y) \in I$ for every $(x, y) \in M$;

(5) $M \subset N$.

The set $f(N_{r/2}(0), \{0\})$ is a non-empty open interval; call it J. If $t \in J$, then for each $x \in N_r(0)$ there exists a unique $y \in N_{6r|d|}(0)$ such that $f(x, y) = t$. We denote this correspondence by the function $y = k_t(x)$. Then $k_t'(x)$ exists and is continuous for every $x \in N_r(0)$ and

$$k_t'(x) = -f_x(x, k_t(x))/f_y(x, k_t(x)).$$

We now proceed to construct the required sets A and B. Using the well-ordering principle and the continuum hypothesis, we can write the interval J in the form $\{t_\alpha\}_{\alpha<\Omega}$, where Ω is the first uncountable ordinal. There exists $(x_1, y_1) \in M$ such that $f(x_1, y_1) = t_1$ and $g(x_1, y_1)$ is an irrational number. This follows from the fact that each k_t $(t \in J)$ is strictly monotonic on $N_r(0)$ (increasing if $d > 0$ and decreasing if $d < 0$) and each h_t (t rational and $t \in I$) is strictly monotonic, but in the opposite sense (because of the hypothesis that δ and d have opposite signs) as the functions k_t, so that the graph of a function k_t $(t \in J)$ can intersect the graph of a function h_t ($t \in I$ and t rational) in at most one point. Since the rationals are countable and the graph of k_t is an uncountable set, the required pair x_1, y_1 exists.

Suppose that (x_β, y_β) has been picked for each $\beta < \alpha$ $(\alpha < \Omega)$ in such a way that

$(x_\beta, y_\beta) \in M$ for every β $(\beta < \alpha)$;

$f(x_\beta, y_\beta) = t_\beta$ for every β $(\beta < \alpha)$;

$g(x, y)$ is irrational for every $x \in \{x_\beta : \beta < \alpha\}$ and for $y \in \{y_\beta : \beta < \alpha\}$.

Pick $(x_\alpha, y_\alpha) \in M$ such that $f(x_\alpha, y_\alpha) = t_\alpha$ and $g(x, y)$ is irrational for every $x \in \{x_\beta : \beta \leqslant \alpha\}$ and for every $y \in \{y_\beta : \beta \leqslant \alpha\}$. This is always possible by an argument similar to that used in the case $\alpha = 1$ and by the fact that α is a countable ordinal (as $\alpha < \Omega$).

Therefore, by transfinite induction we get two sequences $\{x_\alpha\}_{\alpha<\Omega}$ and $\{y_\alpha\}_{\alpha<\Omega}$ such that

$(x_\alpha, y_\alpha) \in M$ for every $\alpha < \Omega$;

$f(x_\alpha, y_\alpha) = t_\alpha$ for every $\alpha < \Omega$;

$g(x, y)$ is irrational if $x \in \{x_\alpha : \alpha < \Omega\}$ and $y \in \{y_\alpha : \alpha < \Omega\}$. Consequently, if we put

$$A = \{x_\alpha : \alpha < \Omega\} \quad \text{and} \quad B = \{y_\alpha : \alpha < \Omega\},$$

then $f(A, B) \supset J$ and $g(A, B)$ contains no interval as it contains no rational number.

Remark 1. In particular, there exist subsets A and B of the real line such that the set $\{x-y: x \in A, y \in B\}$ contains an interval while the set $\{x+y: x \in A, y \in B\}$ does not contain an interval.

Remark 2. Our results can be improved on the following: if f satisfies the conditions of the function "f" in our theorem and each of the functions $(g_n)_{n=1}^{\infty}$ satisfies the conditions of the function "g" in our theorem, then there exists a pair of sets of real numbers A and B such that $f(A, B)$ contains an interval, while $g_n(A, B)$ does not contain an interval for each $n = 1, 2, \ldots$, provided $\{g_n\}_x(0, 0) = \alpha$ for every $n = 1, 2, \ldots$ and $\{g_n\}_y(0, 0) = \beta$ for every $n = 1, 2, \ldots$

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Reçu par la Rédaction le 20. 4. 1979