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PARABOLIC DIFFERENTIAL EQUATIONS AND VECTOR-VALUED FOURIER ANALYSIS

BY

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0. Introduction. Consider the parabolic differential equation

$$\frac{\partial u}{\partial t}-(-1)^{m/2}P(D)u=f, \tag{1}$$

where $P(\xi)=P(\xi_1,\ldots,\xi_n)$ is a homogeneous polynomial of even degree m such that $P(\xi)$ has a negative real part for real ξ and $D=(\partial/\partial x_1,\ldots,\partial/\partial x_n)$.

When f is a nice function, a particular solution of (1) may be expressed in the form

$$u(x,t)=\int_0^t\int_{R^n}\Gamma(x-\xi,t-\tau)f(\xi,\tau)\,d\xi\,d\tau,\qquad t>0, \tag{2}$$

where

$$\Gamma(x,t)=(2\pi)^{-n}\int_{R^n}\exp\bigl(ix\cdot\xi+tP(\xi)\bigr)\,d\xi,\qquad t>0,$$

is a fundamental solution of the equation

$$\partial u/\partial t=(-1)^{m/2}P(D)u.$$

If $\varrho=(\varrho_1,\ldots,\varrho_m)$ with $\varrho_1+\ldots+\varrho_n=m$, then

$$S(x,t)=D_x^\varrho\Gamma(x,t)=(2\pi)^{-n}\int_{R^n}(i\xi)^\varrho\exp\bigl(ix\cdot\xi+tP(\xi)\bigr)\,d\xi, \tag{3}$$

and the following relation holds:

$$S(x,t)=t^{-1-n/m}S(t^{-1/m}x,1),\qquad t>0. \tag{4}$$

The function $S(x,t)$ fails to be integrable over the set $X=R^n\times(0,\infty)$ and the problem raised by B. F. Jones is to give a sense to the following limit:

$$D_x^\varrho u(x,t)=\lim_{\varepsilon\to 0}\int_0^{t-\varepsilon}\int_{R^n}S(x-\xi,t-\tau)f(\xi,\tau)\,d\xi\,d\tau. \tag{5}$$

Jones [6] showed that this limit can be taken in the L^p-sense, $1<p<\infty$.

Later on Fabes and Sadosky [5] showed that the limit exists almost everywhere for $f\in L^p(X)$, $1<p<\infty$.

In the present note we show that all these results can be obtained as a consequence of the (vector-valued) Calderón–Zygmund theory developed in spaces of homogeneous nature.

The technique allows us to obtain almost everywhere convergence for $f \in L^1(X)$ and also vector-valued almost everywhere convergence. The following result is obtained in Section 2:

If $f \in L^p(X)$, $1 \leqslant p < \infty$, the limit in (5) exists a.e. Moreover, if we call this limit $D_x^\varrho u(x, t)$, then for (f_j) in $L^p_{l^q}(X)$, $1 \leqslant p < \infty$, $1 < q < \infty$,

$$\lim_{\varepsilon \to 0} \sum_{j=1}^{\infty} \left| \int_0^{t-\varepsilon} \int_{R^n} S(x-\xi, t-\tau) f_j(\xi, \tau) d\xi d\tau - D_x^\varrho u_j(x, t) \right|^q = 0, \quad \text{a.e. } (x, t) \in X.$$

Mixed norm estimates are also considered for "Riesz potentials" in spaces of homogeneous nature and applications of those estimates to the Navier–Stokes equations are given.

The organization of this paper is as follows: in Section 1 we state the general results in the context of spaces of homogeneous nature, and Sections 2 and 3 are devoted to apply the theory to parabolic differential equations and Navier–Stokes equations, respectively.

Throughout this paper the letter C will be used to denote a positive constant, not necessarily the same at each occurrence.

1. Calderón–Zygmund operators and Riesz potentials in spaces of homogeneous nature. We shall say that (X, d, μ) is a *space of homogeneous nature* if X is a metric space with distance d and μ is a doubling positive Borel measure, i.e., there exists a constant $C > 0$ such that

$$\mu(B(x, 2r)) \leqslant C\mu(B(x, r)), \quad x \in X, \ r > 0,$$

where $B(x, r)$ denotes the ball of center x and radius r.

Given a set A, we shall denote by $|A|$ the measure $\mu(A)$ and by $L^p_E(X)$ (where E is a Banach space) the Bochner–Lebesgue space of E-valued strongly measurable functions f such that

$$\int_X \|f(x)\|_E^p d\mu(x) < +\infty.$$

If $E = C$, then we shall write simply $L^p(X)$.

In a space of homogeneous nature it is well known (see [3]) that the Hardy–Littlewood maximal operator defined by

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x, r)|} \int_{B(x,r)} |f(y)| d\mu(y), \quad x \in X,$$

is a sublinear operator bounded from $L^p(X)$ into $L^p(X)$, $1 < p \leqslant \infty$, and from $L^1(X)$ into weak-$L^1(X)$.

Moreover, if we assume the existence of a nice dense set (the boundedly supported continuous functions) in $L^p(X)$, then the classical Lebesgue differentiation theorem holds.

Under this condition (which we shall assume to hold in the sequel) we have one of the most beautiful tools in real analysis, that is, the *Calderón–Zygmund decomposition* (see [3]):

There exists a constant C such that for $f \in L^1(X)$, $f \geqslant 0$, and for $\alpha > C(\mu(X))^{-1}\|f\|_{L^1(X)}$ ($\alpha > 0$ if $\mu(X) = \infty$) the following decomposition holds:

$$f(x) = g(x) + \sum_{i=1}^{\infty} b_i(x)$$

with

(6) $|g(x)| < C\alpha$ for a.e. x and $\|g\|_{L^1(X)} \leqslant C\|f\|_{L^1(X)}$,

(7) there exists a family of balls $\{B(x_i, r_i)\}_{i=1}^{\infty}$ such that the support of b_i is contained in $B(x_i, r_i)$,

$$\int b_i(x)d\mu(x) = 0 \quad \text{and} \quad \sum_{i=1}^{\infty} \|b_i\|_{L^1(X)} \leqslant C\|f\|_{L^1(X)}.$$

Calderón–Zygmund operators. Let E, F be Banach spaces, $\mathscr{L}(E, F)$ the Banach space of bounded linear operators from E into F, and (X, d, μ) a space of homogeneous nature.

We say that a linear operator T is a *Calderón–Zygmund operator* if T satisfies the following two conditions:

(A) There exists some p_0, $1 < p_0 \leqslant \infty$, such that T is bounded from $L_E^{p_0}(X)$ into $L_F^{p_0}(X)$.

(B) There exists an $\mathscr{L}(E, F)$-valued function K in $X \times X \setminus \{(x, x) \colon x \in X\}$ such that, for any $x \in X$, the functions $y \to K(x, y)$ and $y \to K(y, x)$ are integrable in balls not containing x, and for f in $L_E^{\infty}(X)$ with support in a ball we have

$$Tf(x) = \int_X K(x, y)f(y)d\mu(y) \quad \text{for } x \notin \operatorname{supp} f.$$

THEOREM 1. *Let T be a Calderón–Zygmund operator with a kernel K satisfying the following conditions:*

(K1) $$\int_{d(x,y') > 2d(x,x')} \|K(x, y) - K(x, y')\|\, d\mu(x) \leqslant C \quad (y, y' \in X);$$

(K2) *for* $d(x', y) > 2d(x, x')$,

$$\|K(x,y) - K(x',y)\| \leqslant C\frac{d(x,x')}{d(x',y)|B(x',d(x',y))|};$$

(K3)

$$\int_{a < d(x,y) < 2a} \|K(x,y)\|\, d\mu(y) + \int_{a < d(x,y) < 2a} \|K(x,y)\|\, d\mu(x) \leqslant C \quad (x, y \in X,\ a > 0).$$

Then

(i) *T maps $L^p_{l^q(E)}(X)$ into $L^p_{l^q(F)}(X)$ for $1 < p,\ q < \infty$;*

(ii) *T maps $L^1_{l^q(E)}(X)$ into weak-$L^1_{l^q(F)}(X)$ for $1 < q < \infty$.*

Moreover, if T^* is the maximum operator

$$T^*f(x) = \sup_{\varepsilon > 0} \left\| \int_{d(x,y) > \varepsilon} K(x, y)f(y)\,d\mu(y) \right\|_F,$$

then

(iii) *T^* maps $L^p_{l^q(E)}(X)$ into $L^p_{l^q}(X)$ for $1 < p,\ q < \infty$;*

(iv) *T^* maps $L^1_{l^q(E)}(X)$ into weak-$L^1_{l^q}(X)$ for $1 < q < \infty$.*

(Here, $\|K(x, y)\|$ denotes the norm of $K(x, y)$ in $\mathscr{L}(E, F)$; $l^q(E)$ is the space of E-valued sequences $\{a_n\}$ such that $\sum \|a_n\|_E^q < +\infty$ and $l^q(C) = l^q$.)

Note. When we say, for instance, that T maps $L^p_{l^q(E)}(X)$ into $L^p_{l^q(F)}(X)$, we mean that T has an l^q-valued bounded extension (see [8]), i.e.,

$$\left\| \left(\sum_{j=1}^{\infty} \|Tf_j(\cdot)\|_F^q \right)^{1/q} \right\|_{L^p(X)} \leqslant C_{p,q} \left\| \left(\sum_{j=1}^{\infty} \|f_j(\cdot)\|_E^q \right)^{1/q} \right\|_{L^p(X)}.$$

Remark 1. Hypothesis (K3) is not necessary for (i) and (ii) to hold. In fact, (i) and (ii) can be obtained under hypothesis (K1) plus

$$\text{(K2)}' \qquad \int_{d(x',y) > 2d(x,x')} \|K(x, y) - K(x', y)\|\,d\mu(y) \leqslant C \qquad (x,\ x' \in X)$$

(the fact that μ is doubling implies that (K2)$'$ is weaker than (K2)).

Proof of Theorem 1. The proof goes along the same lines as in the case $X = \mathbf{R}^n$ with Euclidean norm and Lebesgue measure (see [8]). However, for the sake of completeness we shall give the main steps of the proof in our case.

It can be seen in [3] that the Calderón–Zygmund decomposition, the previous boundedness on L^{p_0} and condition (K1) for the kernel imply that T is bounded from $L^1_E(X)$ into weak-$L^1_F(X)$, and then Marcinkiewicz's interpolation theorem gives us the boundedness of T from $L^p_E(X)$ into $L^p_F(X)$ for $1 < p \leqslant p_0$.

If $p_0 < \infty$, we shall prove the range $p_0 < p < \infty$ by a duality argument: it is enough to prove the inequality for functions $f \in L^p(X) \otimes E$, i.e.,

$$f(x) = \sum_{j=1}^{m} f_j(x)a_j \qquad (a_j \in E,\ f_j \in L^p(X)),$$

since these are dense in $L^p_E(X)$. Every such function takes values in a finite-dimensional subspace E_0 of E. Now, we fix E_0 and define $K_0(x, y) \in \mathscr{L}(E_0, F)$ as the restriction of T to $L^p_{E_0}(X)$.

Since $L^{p'}_{F^*}(X)$ is isometrically contained in $(L^p_F(X))^*$ and $L^{p'}_{E_0^*}(X) = (L^p_{E_0}(X))^*$, we can consider the adjoint of T_0 as a bounded operator from $L^{p_0'}_{F^*}(X)$ into $L^{p_0'}_{E_0^*}(X)$, and computing

$$\int \langle f(x),\ T_0^* g(x)\rangle\, d\mu(x) = \int \langle T_0 f(x),\ g(x)\rangle\, d\mu(x)$$

for functions $f \in L^{\infty}_{E_0}(X)$, $g \in L^{\infty}_{F^*}(X)$ with disjoint supports, one easily finds that T_0^* is a Calderón–Zygmund operator with kernel $\tilde{K}(x, y) = K_0(y, x)^*$ (where $K_0(y, x)^*$ denotes the adjoint of the operator $K_0(y, x)$).

Now, hypothesis (K2) (or even the weaker one (K2)′) implies that the kernel of T_0^* verifies (K1), and so T_0^* is bounded from $L^q_{F^*}(X)$ into $L^q_{E_0^*}(X)$, $1 < q \leqslant p_0$. Thus, T_0 is bounded from $L^p_{E_0}(X)$ into $L^p_F(X)$, $p_0 \leqslant p < \infty$, with constants independent of E_0, and this implies that T is bounded from $L^p_E(X)$ into $L^p_F(X)$, $p_0 \leqslant p < \infty$.

In order to obtain (i) and (ii) we can define a new operator $\tilde{T}$ mapping $l^q(E)$-valued functions into $l^q(F)$-valued ones (where q is fixed, $1 < q < \infty$) as

$$\tilde{T}(f_1, f_2, \ldots, f_j, \ldots) = (Tf_1,\ Tf_2, \ldots, Tf_j, \ldots).$$

It is clear that $\tilde{T}$ is bounded from $L^q_{l^q(E)}(X)$ into $L^q_{l^q(F)}(X)$. Moreover, $\tilde{T}$ is a Calderón–Zygmund operator again with $\mathscr{L}(l^q(E),\ l^q(F))$-valued kernel $\tilde{K}$ given by

$$\tilde{K}(x, y)[(\alpha_j)_{j=1}^{\infty}] = (K(x, y)[\alpha_j])_{j=1}^{\infty}, \qquad (\alpha_j) \subset E.$$

This kernel has a norm

$$\|\tilde{K}(x,\ y)\|_{\mathscr{L}(l^q(E), l^q(F))} = \|K(x,\ y)\|_{\mathscr{L}(E,F)},$$

and then the argument above can be applied to the operator $\tilde{T}$, so we obtain (i) and (ii).

For the proof of (iii) we state the following well-known lemma of Cotlar, whose proof can be deduced from (ii), (K2) and (K3) as in the Euclidean case (see [7]):

LEMMA 1. *There exists a constant C such that for any function $f \in L^{\infty}_E(X)$ with support in a ball we have*

$$T^* f(x) \leqslant C\big(M(\|Tf(\cdot)\|_F)(x) + M(\|f(\cdot)\|_E)(x)\big), \qquad x \in X,$$

where M is the Hardy–Littlewood maximal operator.

Now, the fact that the Hardy–Littlewood maximal function is bounded from $L^p_{l^q}(X)$ into itself, $1 < p, q < \infty$ (see [11]), together with (i) and the lemma imply (iii).

Once we have (iii) and, by using the arguments above, (iv) would be true if we could regard T^* as a Calderón–Zygmund operator with $\mathscr{L}(E, l^{\infty}(F))$-valued kernel satisfying (K1). In fact, we can look at

$$T^* f(x) = \|\{T_\varepsilon f(x)\}_{\varepsilon > 0}\|_{l^{\infty}}$$

(where $T_\varepsilon f(x) = \int_{d(x,y)>\varepsilon} K(x,y)f(y)d\mu(y)$) as a Calderón–Zygmund operator $\bar{T}f = \{T_\varepsilon f\}_{\varepsilon>0}$ with kernel given by

$$\{K(x,y)\chi_{\{\varepsilon<d(x,y)\}}(x,y)\}_{\varepsilon>0}$$

and satisfying previous boundedness (due to (iii)) on L^{p_0} for any p_0, $1<p_0<\infty$. The problem is that its kernel does not verify condition (K1).

In order to avoid this problem we take a function φ: $\boldsymbol{R}\to\boldsymbol{R}$ such that $\chi_{[2,\infty)} \leqslant \varphi \leqslant \chi_{[1,\infty)}$ and $|\varphi'(t)| \leqslant C/t$, and consider the operator

$$Rf(x) = \{R_\varepsilon f(x)\}_{\varepsilon>0} = \left\{\int_X K(x,y)\varphi\left(\frac{d(x,y)}{\varepsilon}\right)f(y)d\mu(y)\right\}_{\varepsilon>0}$$

with an $\mathscr{L}(E, l^\infty(F))$-valued kernel given by

$$\{R_\varepsilon(x,y)\}_{\varepsilon>0} = \left\{K(x,y)\varphi\left(\frac{d(x,y)}{\varepsilon}\right)\right\}_{\varepsilon>0}.$$

Now, this kernel satisfies (K1) since

$$\|R_\varepsilon(x,y)-R_\varepsilon(x,y')\| \leqslant \|K(x,y)-K(x,y')\| + \|K(x,y')\|\left|\varphi\left(\frac{d(x,y)}{\varepsilon}\right)-\varphi\left(\frac{d(x,y')}{\varepsilon}\right)\right|,$$

and then, by applying the mean value property, the fact that $|\varphi'(t)| \leqslant C/t$ and hypotheses (K1) and (K3) to K, we have

$$\begin{aligned}\int_{d(x,y')>2d(y,y')} &\sup_{\varepsilon>0} \|R_\varepsilon(x,y)-R_\varepsilon(x,y')\|\,d\mu(x)\\ &\leqslant \int_{d(x,y')>2d(y,y')} \|K(x,y)-K(x,y')\|\,d\mu(x)\\ &\qquad + C\sum_{j=1}^{\infty} \int_{\substack{2^j d(y,y')<d(x,y')\\<2^{j+1}d(y,y')}} \|K(x,y')\|2^{-j}d\mu(x) \leqslant C.\end{aligned}$$

Observe that the difference operator $J = R-\bar{T}$ satisfies

$$\|Jf(x)\|_{l^\infty(F)} \leqslant \mathscr{M}f(x) = \sup_{\varepsilon>0}\int_{\varepsilon<d(x,y)<2\varepsilon} \|K(x,y)\|\,|f(y)|d\mu(y).$$

Now, due to (K3), $\mathscr{M}$ is bounded from $L^\infty_E(X)$ into $L^\infty(X)$ and, moreover, $\mathscr{M}$ can be majorized by a Calderón–Zygmund operator with kernel satisfying (K1) (see [8] for details in the Euclidean case).

This implies that $R = J+\bar{T}$ is bounded from $L^p_E(X)$ into $L^p_{l^\infty(F)}(X)$, $1<p<\infty$, and since its kernel satisfies (K1), we see that R, and then $\bar{T}$, satisfies (iv).

This completes the proof of Theorem 1.

Riesz potentials. Given α with $0<\alpha<1$, the maximal fractional operator of order α in a space of homogeneous nature (X, d, μ) will be

$$M_\alpha f(x) = \sup_{r>0}\frac{1}{|B(x,r)|^\alpha}\int_{B(x,r)} |f(y)|d\mu(y), \qquad x\in X.$$

LEMMA 2. (i) *M_α is bounded from $L^1(X)$ into weak-$L^{1/\alpha}(X)$.*

(ii) *M_α is bounded from $L^p(X)$ into $L^q(X)$ for $1/p = 1-\alpha+(1/q)$, $1/\alpha < q \leqslant \infty$.*

Proof. Hölder's inequality says that M_α is bounded from $L^{1/(1-\alpha)}(X)$ into $L^\infty(X)$. On the other hand, it is very easy to obtain the weak type boundedness by means of a classical covering lemma (see [3], [11]), and the remainder follows from Marcinkiewicz's interpolation theorem.

Parallel to M_α we can define the fractional integral operator

$$I_\beta f(x) = \int_X \frac{f(y)}{d(x, y)^\beta} d\mu(y), \qquad \beta > 0.$$

If X is an additive group, and d and μ are translation invariant and, moreover, there exists $\lambda > 0$ such that $\mu(B(0, r)) \sim r^\lambda$, then we have

PROPOSITION 1. *Let α be such that $0 < \alpha < 1$. Then $I_{\lambda\alpha}$ is bounded from $L^p(X)$ into $L^q(X)$ for $1/p = 1-\alpha+(1/q)$, $1/\alpha < q < \infty$.*

Proof. The proof follows from Lemma 2 and Hölder's inequality (as in the Euclidean case, see [10] and [12]) since $I_{\lambda\alpha}$ is controlled by M_α in the following sense:

Let α' be such that $1-(\alpha'/\lambda) = \alpha$. Then for any $\varepsilon > 0$ with $0 < \alpha'-\varepsilon < \alpha'+\varepsilon < \lambda$ there exists a constant C_ε such that

$$|I_{\lambda\alpha} f(x)| \leqslant C_\varepsilon \big(M_{1-(\alpha'-\varepsilon)/\lambda} f(x) \cdot M_{1-(\alpha'+\varepsilon)/\lambda} f(x)\big)^{1/2}, \qquad x \in X.$$

The following mixed norm result is an easy consequence of the last proposition (see [2] for a proof for the classical Riesz potentials which can be perfectly adapted to our case):

PROPOSITION 2. *Let (X_i, d_i, μ_i), $i = 1,\dots,n$, be a family of spaces of homogeneous nature such that X_i are additive groups, d_i and μ_i are translation invariant and there exists $\lambda_i > 0$ such that $\mu_i(B_i(0, r)) \sim r^{\lambda_i}$, $i = 1,\dots,n$. Suppose*

$$0 < \alpha_i < 1 \quad \text{and} \quad \sum_{i=1}^n \lambda_i \alpha_i = \beta.$$

Then

$$\|I_\beta f\|_{L^{q_1,\dots,q_n}(X_1\times\dots\times X_n)} \leqslant C\|f\|_{L^{p_1,\dots,p_n}(X_1\times\dots\times X_n)}$$

for $1/p_i = 1-\alpha_i+(1/q_i)$, $1/\alpha_i < q < \infty$, $i = 1,\dots,n$.

2. Parabolic differential equations. In this section, we shall see that the questions raised in the Introduction fall under the scope of Theorem 1. For that, consider the space of homogeneous nature $X = \mathbf{R}^n \times (0, \infty)$ endowed with the distance

$$d(\bar{x}, \bar{y}) = |x-y|+|t-s|^{1/m}, \qquad \bar{x} = (x, t),\ \bar{y} = (y, s),$$

and the Lebesgue measure $d\mu(x, t) = dx\otimes dt$.

For Hölder continuous functions f the iterated integral

$$\int_0^t \left(\int_{R^n} S(x-\xi, t-\tau) f(\xi, \tau) d\xi \right) d\tau$$

exists in the usual sense and the limit (5) exists for any $(x,t) \in X$ (see [6]).

In particular, the operator

$$Df(\bar{x}) = \int_X K(\bar{x}, \bar{y}) f(\bar{y}) d\mu(\bar{y}), \qquad \bar{x} \in X,$$

defined by means of the kernel

$$K\colon X \times X \setminus \{(\bar{x}, \bar{x})\colon \bar{x} \in X\} \to \mathscr{L}(C, C) \cong C,$$
$$(\bar{x}, \bar{y}) \to K(\bar{x}, \bar{y}) = S(x-y, t-s)$$

(where we define $S(x, t) = 0$ if $t \leqslant 0$) is a well-defined operator for functions (say) in $\mathscr{C}_0^\infty(X)$. More precisely, D is the convolution with the function $S(x, t)$. The $(n+1)$-dimensional Fourier transform of S (see [6]) is the bounded rational function

$$\hat{S}(\xi, \tau) = \frac{-(i\xi)^\varrho}{i\tau + P(\xi)}.$$

Then, the Plancherel theorem assures that

$$\|Df\|_{L^2(X)} \leqslant C \|f\|_{L^2(X)}, \qquad f \in \mathscr{C}_0^\infty(X),$$

and since $\mathscr{C}_0^\infty(X)$ is dense in $L^2(X)$, we conclude that D has a bounded extension to $L^2(X)$. So, D is a Calderón-Zygmund operator in the sense of Section 1.

On the other hand, the translation invariant properties of the distance and the measure and also the fact that $\mu(B(\bar{x}, r)) \sim r^{n+m}$ establish that in order to verify (K1) and (K2) for the kernel K it is enough to check that

$$|S(x-y, t-s) - S(x, t)| \leqslant C \frac{|y| + s^{1/m}}{(|x| + t^{1/m})^{n+m+1}} \tag{8}$$

for $|x| + t^{1/m} > 2(|y| + s^{1/m})$.

LEMMA 3. *Let* $S(x, t)$ *be a function satisfying*

(i) $S(x, t) = 0$ *if* $t = 0$;

(ii) $S(x, t) = t^{-1-n/m} S(t^{-1/m}x, 1)$, $t > 0$;

(iii) $|S(x, 1)| \leqslant C(1+|x|)^{-n-m-1}$ *and*

$$\left| \frac{\partial S}{\partial x_i}(x, 1) \right| \leqslant C(1+|x|)^{-n-m-2}, \qquad i = 1, \ldots, n.$$

Then condition (8) *is fulfilled.*

(Note that the function $S(x, t)$ defined by (3) satisfies hypothesis (iii).)

Proof. We shall distinguish between three cases.

(1) $t \leqslant s$. Then $S(x-y, t-s) = 0$ and

$$|S(x, t)| \leqslant C\frac{t^{-1-n/m}}{(1+|t^{-1/m}x|)^{n+m+1}} = C\frac{t^{1/m}}{(t^{1/m}+|x|)^{n+m+1}}$$
$$\leqslant C\frac{s^{1/m}+|y|}{(t^{1/m}+|x|)^{n+m+1}}.$$

(2) $t/2 < s < t$. Here

$$|S(x, t)| \leqslant C\frac{t^{1/m}}{(t^{1/m}+|x|)^{n+m+1}} \leqslant C\cdot 2^{1/m}\frac{s^{1/m}+|y|}{(t^{1/m}+|x|)^{n+m+1}}$$

and

$$|S(x-y, t-s)| \leqslant C\frac{(t-s)^{1/m}}{((t-s)^{1/m}+|x-y|)^{n+m+1}} \leqslant C\frac{s^{1/m}+|y|}{(t^{1/m}+|x|)^{n+m+1}}$$

since $(t-s)^{1/m} \leqslant s^{1/m} \leqslant s^{1/m}+|y|$ and

$$(t-s)^{1/m}+|x-y| \geqslant \tfrac{1}{2}(|x|+t^{1/m}) \quad \text{for } |x|+t^{1/m} > 2(|y|+s^{1/m}).$$

(3) $s \leqslant t/2$. This case can be straightforwardly deduced from the mean value property and the estimates in (iii).

Again, because of the translation invariance of the measure and the distance the following computation is enough to verify (K3) for our kernel:

$$(9) \qquad \int_{a<|x|+t^{1/m}<2a} |S(x, t)|\,dxdt$$
$$= \int_{R^n} |S(u, 1)|\Bigg(\int_{\substack{a(1+|u|)^{-1}<t^{1/m}\\ <2a(1+|u|)^{-1}}} dt/t\Bigg)du \leqslant C\int_{R^n} |S(u, 1)|\,du \leqslant C.$$

Thus we have proved the following result:

PROPOSITION 3. *The operator D satisfies all the conclusions of Theorem* 1.

In particular, we have obtained some boundedness for the maximal operator

$$D^*f(x, t) = \sup_{\varepsilon>0}\left|\int_X (S\chi_{G(\varepsilon)})(x-\xi, t-\tau)f(\xi, \tau)\,d\xi d\tau\right|,$$

where $G(\varepsilon) = \{(x, t)\colon\ |x|+t^{1/m} > \varepsilon\}$.

But in order to answer the problem proposed in the Introduction we must consider the operator

$$f^*(x, t) = \sup_{\varepsilon>0}\left|\int_X (S\chi_{H(\varepsilon)})(x-\xi, t-\tau)f(\xi, \tau)\,d\xi d\tau\right|,$$

where $H(\varepsilon) = \{(x, t)\colon\ t > \varepsilon\}$.

The following proposition says that, for all practical purposes, it is equivalent to estimate D^*f or f^*.

PROPOSITION 4. *The maximal operator*

$$U^*f(x, t) = \sup_{\varepsilon>0} \int_X |(S\chi_{G(\varepsilon)\setminus H(\varepsilon^m)})(x-\xi, t-\tau)f(\xi, \tau)|\, d\xi d\tau$$

is bounded from $L^p_{l^q}(X)$ *into* $L^p_{l^q}(X)$, $1 < p,\ q < \infty$, *and from* $L^1_{l^1}(X)$ *into weak-*$L^1_{l^q}(X)$, $1 < q < \infty$.

Before giving the proof, which involves again vector-valued singular integral techniques, we state the following consequence of the last two propositions:

THEOREM 2. *The mapping* $f \to f^*$, *defined a priori for* f *in* $\mathscr{C}_0^\infty(X)$ (see [6]), *can be boundedly extended from* $L^p(X)$ *into* $L^p(X)$, $1 < p < \infty$, *and from* $L^1(X)$ *into weak-*$L^1(X)$.

Moreover, the following vector-valued inequalities hold:

(i) $$\left\| \left(\sum_{j=1}^{\infty} (f_j^*)^q \right)^{1/q} \right\|_{L^p(X)} \leqslant C_{p,q} \left\| \left(\sum_{j=1}^{\infty} |f_j|^q \right)^{1/q} \right\|_{L^p(X)} \qquad (1 < p,\ q < \infty),$$

(ii) $$\left| \left\{ (x, t) \in X\colon \sum_{j=1}^{\infty} (f_j^*(x, t))^q > \lambda^q \right\} \right|$$

$$\leqslant \frac{C_q}{\lambda} \int_X \left(\sum_{j=1}^{\infty} |f_j(x, t)|^q \right)^{1/q} dx dt \qquad (\lambda > 0,\ 1 < q < \infty).$$

Remark 2. For $f \in \mathscr{C}_0^\infty(X)$ it is known (see [6]) that the limit in (5) exists in the L^p-sense and almost everywhere. Then the last theorem has the following consequence:

THEOREM 3. *The limit*, $D_x^\varrho u(x, t)$, *in* (5) *exists almost everywhere for functions* f *in* $L^p(X)$, $1 \leqslant p < \infty$. *Moreover, for* $(f_j) \in L^p_{l^q}(X)$, $1 \leqslant p < \infty$, $1 < q < \infty$, *we have*

(10) $$\lim_{\varepsilon \to 0} \left\| \left\{ \int_0^{t-\varepsilon} \int_{R^n} S(x-\xi, t-\tau) f_j(\xi, \tau) d\xi d\tau - D_x^\varrho u_j(x, t) \right\}_j \right\|_{l^q} = 0$$

for almost every $(x, t) \in X$.

Proof. Theorem 2 and the existence of a nice dense set ($\mathscr{C}_0^\infty(X)$), where the pointwise convergence holds, imply by a well-known standard argument the almost everywhere convergence for functions in $L^p(X)$, $1 \leqslant p < \infty$.

Then (10) will be true for the dense subspace of $L^p_{l^q}(X)$ formed by all $g(x) = (g_j(x))$ with a finite number of nonvanishing components $g_j(x)$. Then, denoting by $D_x^\varrho v_j$ the limit corresponding to $g_j(x)$, we have

$$\lim_{\varepsilon\to 0}\left\|\left\{\int_0^{t-\varepsilon}\int_{R^n} S(x-\xi, t-\tau)f_j(\xi, \tau)d\xi d\tau - D_x^\varrho u_j(x, t)\right\}_j\right\|_{l^q}$$

$$\leqslant \lim_{\varepsilon\to 0}\left\|\left\{\int_0^{t-\varepsilon}\int_{R^n} S(x-\xi, t-\tau)(f_j-g_j)(\xi, \tau)d\xi d\tau\right\}_j\right\|_{l^q}$$

$$+\|\{D_x^\varrho u_j(x, t)-D_x^\varrho v_j(x, t)\}_j\|_{l^q} \leqslant 2\|\{(f_j-g_j)^*(x, t)\}_j\|_{l^q}.$$

Then, given $\lambda > 0$, by Theorem 2 we have

$$\left|\left\{(x, t)\in X\colon \lim_{\varepsilon\to 0}\left\|\left\{\int_0^{t-\varepsilon}\int_{R^n} S(x-\xi, t-\tau)f_j(\xi, \tau)d\xi d\tau - D_x^\varrho u_j(x, t)\right\}_j\right\|_{l^q} > \lambda\right\}\right|$$

$$\leqslant |\{(x, t)\in X\colon \|\{(f_j-g_j)^*(x, t)\}_j\|_{l^q} > \lambda/2\}|$$

$$\leqslant \frac{C}{\lambda^p}\int_X \|\{(f_j-g_j)(x, t)\}_j\|_{l^q}^p dxdt.$$

The proof is completed by choosing (g_j) such that

$$\frac{C}{\lambda^p}\int_X \|\{(f_j-g_j)(x, t)\}_j\|_{l^q}^p dxdt < \varepsilon.$$

Proof of Proposition 4. First of all, we state the following fact whose proof will be given later:

(11) *There exists a constant C independent of ε such that*

$$\int_X |S\chi_{G(\varepsilon)\setminus H(\varepsilon^m)}(y, s)|dyds \leqslant C.$$

This implies that

$$\|U^*f\|_{L^\infty(X)} \leqslant C\|f\|_{L^\infty(X)}. \tag{12}$$

Now, given a function $f\geqslant 0$, $f\in L^1(X)$, and $\alpha > 0$, we decompose $f = g+b = g+\sum b_i$ as in Section 1.

From (12) it is clear that

$$|\{\bar{x}\in X\colon |U^*f(\bar{x})| > 2C\alpha\}| \leqslant |\{\bar{x}\in X\colon |U^*b(\bar{x})| > C\alpha\}|.$$

If $\{B(\bar{x}_i, r_i)\}$ are the balls containing the supports of b_i and we consider the set $D_\alpha = \bigcup B(\bar{x}_i, Cr_i)$ (with C a constant depending only on the dimension), the properties of the kernel S and geometrical arguments show that if $\bar{x}\notin D_\alpha$, then

$$U^*b(\bar{x}) \leqslant \sum_i \int_{B(\bar{x}_i,r_i)} |K(\bar{x}, \bar{y})-K(\bar{x}, \bar{x}_i)|\,|b(\bar{y})|d\mu(\bar{y})+C\cdot Mf(\bar{x}).$$

In particular,

$$|\{\bar{x}\in X:\ U^*b(\bar{x}) > C\alpha\}| \leqslant |D_\alpha| + \Big|\{\bar{x}\notin D_\alpha: \sum_i \int_{B(\bar{x}_i,r_i)} \ldots > C\alpha/2\}\Big|$$

$$+|\{x\notin D_\alpha:\ Mf(\bar{x}) > C\alpha/2\}| \leqslant \frac{C}{\alpha}\int_X |f(\bar{x})|d\mu(\bar{x}).$$

The last inequality is given by the (1, 1)-weak boundedness of the Hardy–Littlewood maximal operator and the standard techniques with Calderón–Zygmund kernels.

Now we observe that this procedure can be applied to the vector-valued operator

$$\tilde{U}f(x, t) = \left\{\int_X S\chi_{G(\varepsilon)\setminus H(\varepsilon^m)}(x-y, t-s)f(y, s)dyds\right\}_\varepsilon,$$

and then using the ideas in the proof of Theorem 1 we can show that $\tilde{U}$ maps $L^p_{l^q}(X)$ into $L^p_{l^q(l^\infty)}(X)$, $1 < p,\ q < \infty$, and $L^1_{l^q}(X)$ into weak-$L^1_{l^q(l^\infty)}(X)$, $1 < q < \infty$.

Finally, using the fact that

$$\|\tilde{U}f(x, t)\|_{l^\infty} = U^*f(x, t),$$

the conclusion of Proposition 4 holds.

Proof of (11). We have

$$\int_X |S\chi_{G(\varepsilon)\setminus H(\varepsilon^m)}(x, t)|dxdt \leqslant \int_{\{|x|>\varepsilon/4,\, 0<t<(\varepsilon/2)^m\}} + \int_{\{(\varepsilon/4)^m<t<(2\varepsilon)^m\}} |S(x, t)|dxdt.$$

The first term of this sum is

$$\int_0^{(\varepsilon/2)^m}\int_{|x|>\varepsilon/4} t^{-n/m-1}|S(t^{-1/m}x, 1)|dxdt$$

$$= \int_0^{(\varepsilon/2)^m} t^{-1}\int_{|x|>\varepsilon/4t^{1/m}} |S(x, 1)|dxdt \leqslant \frac{4}{\varepsilon}\int_0^{(\varepsilon/2)^m} t^{(1/m)-1}\int_{R^n}|x|\ |S(x, 1)|dxdt$$

$$\leqslant C\int_{R^n}|x|\ |S(x, 1)|dx \leqslant C,$$

and the second one is

$$\int_{(\varepsilon/4)^m}^{(2\varepsilon)^m} t^{-1}\int_{R^n}|S(x, 1)|dxdt \leqslant C\int_{R^n}|S(x, 1)|dx \leqslant C.$$

Remark 3. Since the operators considered in Theorem 2 are translation invariant and can be seen as linear operators, the l^q-valued bounded extensions (i) and (ii) in Theorem 2 imply, by a well-known procedure (see [1] and [9]), that these operators satisfy also mixed norm estimates of type $L^{p,q}$. In particular, we can obtain almost everywhere convergence results for functions such that

$$\big\|\, \|f(\ldots, x_{k+1}, \ldots, x_n, t)\|_{L^q(R^k)}\big\|_{L^p(R^{n-k}\times(0,\infty))} < +\infty.$$

Remark 4. The properties we have used of the kernel S are the following:

(13) $$S(x, t) = 0 \quad \text{if } t \leqslant 0;$$

(14) $$S(x, t) = t^{-1-n/m} S(t^{-1/m}x, 1), \quad x \in \boldsymbol{R}^n,\ t > 0;$$

(15) $$|S(x, 1)| \leqslant C(1+|x|)^{-n-m-1};$$

(16) $$\left|\frac{\partial}{\partial x_i} S(x, 1)\right| \leqslant C(1+|x|)^{-n-m-2}, \quad i = 1,\dots,n.$$

Fabes and Sadosky obtained in [5] the almost everywhere convergence for functions f in $L^p(X)$, $p > 1$, if S satisfies (13), (14) and

(17) $$|S(x, 1)| + \left|\frac{\partial}{\partial x_i} S(x, 1)\right| \leqslant C(1+|x|)^{-n-2}, \quad i = 1,\dots,n.$$

Their method is strongly based on the boundedness properties of the strong maximal Hardy–Littlewood function, and then it does not work for the (1, 1)-weak type maximal result. Using their result in $L^p(X)$, $p > 1$, we proved in [11] the almost everywhere convergence for functions in $L^1(X)$.

3. Navier–Stokes equations. Fabes et al. [4] considered the initial value problem for the Navier–Stokes equations in the infinite cylinder $S_T = \boldsymbol{R}^n \times [0, T)$. More precisely, given $g(x) = (g_1(x),\dots,g_n(x))$ satisfying

$$\operatorname{div}(g)(x) = \sum_{j=1}^{n} \left(\frac{\partial}{\partial x_j}\right) g_j(x) = 0, \quad x \in \boldsymbol{R}^n,$$

and a pressure function $P(x, t)$, they studied the solution vector

$$u(x, t) = (u_1(x, t),\dots,u_n(x, t)), \quad x \in \boldsymbol{R}^n,\ t \in (0, T),$$

such that

(18) $$\frac{\partial u_i}{\partial t} - \sum_{j=1}^{n} \frac{\partial^2 u_i}{\partial x_j^2} + \sum_{j=1}^{n} \frac{\partial u_i}{\partial x_j} u_j + \frac{\partial P}{\partial x_i} = 0, \quad i = 1,\dots,n,$$

(19) $$\sum_{j=1}^{n} \frac{\partial u_j}{\partial x_j} = 0,$$

(20) $$u(x, 0) = g(x).$$

They give conditions in order to have existence, regularity and uniqueness for weak solutions $u(x, t) \in L^{p,q}(S_T)$, where the exponents p and q always satisfy the relation $(n/p)+(2/q) \leqslant 1$, $n < p \leqslant \infty$. Here $L^{p,q}(S_T)$ is the space of functions f depending on x and on t such that

$$\int_0^T \left(\int_{\boldsymbol{R}^n} |f(x,t)|^p dx \right)^{q/p} dt < +\infty.$$

Their technique is to show that $u \in L^{p,q}(S_T)$, p, $q \geq 2$, $p < \infty$, is a weak solution of the Navier–Stokes equation if and only if u is a solution of a certain integral equation $u + B(u,u) = f$, where

$$(21)\qquad |B(u, v)(x, t)| \leqslant C \int_0^t \int_{R^n} \frac{|u(y, s)|\,|v(y, s)|}{\left(|x-y| + (t-s)^{1/2}\right)^{n+1}} dyds.$$

Looking for solutions of the integral equation the computational step is the following:

THEOREM 4 (see [4]). *For u, $v \in L^{p,q}(S_T)$ we have the following conclusions:*
(i) *If $(n/p)+(2/q) = 1$ with $n < p < \infty$, then*

$$\|B(u, v)\|_{L^{p,q}(S_T)} \leqslant C(n, p, q)\,\|u\|_{L^{p,q}(S_T)}\,\|v\|_{L^{p,q}(S_T)}.$$

(ii) *If $(n/p)+(2/q) < 1$ with $n < p \leqslant \infty$, then*

$$\|B(u, v)\|_{L^{p,q}(S_T)} \leqslant C(n, p, q)\,T^{(1/2)(1-(n/p)-(2/q))}\,\|u\|_{L^{p,q}(S_T)}\,\|v\|_{L^{p,q}(S_T)}.$$

This theorem and its proof can be very well understood from the point of view of mixed norm estimates for Riesz potentials in spaces of homogeneous nature obtained in Section 1.

If in Proposition 2 we put $X_1 = R^n$, $d_1(x, y) = |x-y|$, μ_1 Lebesgue measure on R^n, $X_2 = R$, $d_2(t, s) = |t-s|^{1/2}$, and μ_2 Lebesgue measure on R, we have $\lambda_1 = n$, $\lambda_2 = 2$, and we can state the following

PROPOSITION 5. *Let $0 < \alpha_1, \alpha_2 < 1$ be such that $n\alpha_1 + 2\alpha_2 = n+1$. Then the operator*

$$I_{n+1}f(x, t) = \int_{R^{n+1}} \frac{f(y, s)}{\left(|x-y| + |t-s|^{1/2}\right)^{n+1}} dyds$$

is bounded from $L^{p_1,p_2}(R^{n+1})$ into $L^{q_1,q_2}(R^{n+1})$ for $1/p_i = 1-\alpha_i+(1/q_i)$, $1/\alpha_i < q < \infty$, $i = 1, 2$.

Note. A typical dilation argument with $A_\sigma(x, t) = (\sigma x, \sigma^2 x)$ shows that the range of p_i, q_i is the best possible.

Now, (21), Proposition 5 and Hölder's inequality imply that B is continuous from $L^{p_1,p_2}(S_T) \times L^{p_1,p_2}(S_T)$ into $L^{q_1,q_2}(S_T)$ (uniformly in T) for the following range of p's and q's:

$$(22)\qquad \begin{aligned} 2/p_1 &= 1-\alpha_1+(1/q_1), & 1/\alpha_1 &< q_1 < \infty, \\ 2/p_2 &= 1-\alpha_2+(1/q_2), & 1/\alpha_2 &< q_2 < \infty, \end{aligned}$$

with α_1, α_2 such that $0 < \alpha_1, \alpha_2 < 1$ and $\alpha_1 n + 2\alpha_2 = n+1$.

Proof of Theorem 4. (i) If we put $p = p_1 = q_1$ and $q = p_2 = q_2$ in (22), then $\alpha_1 = 1/p'$ and $\alpha_2 = 1/q'$, and since $(n/p)+(2/q) = 1$, we have $\alpha_1 n + 2\alpha_2 = n+1$ indeed.

(ii) Fix p with $n < p < \infty$ and let q be such that $(n/p)+(2/q) < 1$ (the case

$p = \infty$ is more easy and it can be proved directly, see [4]).

If we put $p_1 = q_1 = p$ in (22), then $\alpha_1 = 1/p'$, and therefore $\alpha_2 = \frac{1}{2}+(n/2p)$. Since $(n/p)+(2/q) < 1$, we have $q > 1/\alpha_2$, and then it is possible to take $q_2 = q$ in (22).

Now, if p_2 is such that

$$2/p_2 = 1-\alpha_2+(1/q) = (1/q)+\tfrac{1}{2}-(n/2p),$$

then

$$\|B(u, v)\|_{L^{p,q}(S_T)} \leqslant C\|u\|_{L^{p,p_2}(S_T)}\|v\|_{L^{p,p_2}(S_T)},$$

and by Hölder's inequality this is less than

$$C\cdot T^{2((1/p_2)-(1/q))}\|u\|_{L^{p,q}(S_T)}\|v\|_{L^{p,q}(S_T)}.$$

Finally, observe that

$$2((1/p_2)-(1/q)) = 1-\alpha_2-(1/q) = \tfrac{1}{2}(1-(n/p)-(2/q)).$$

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